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Algebraic Global Gadgetry for Surjective Constraint Satisfaction

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Abstract

The constraint satisfaction problem (CSP) on a finite relational structure \mathbf{B} is to decide, given a set of constraints on variables where the relations come from \mathbf{B} , whether or not there is an assignment to the variables satisfying all of the constraints; the surjective CSP is the variant where one decides the existence of a surjective satisfying assignment onto the universe of \mathbf{B} .

We present an algebraic framework for proving hardness results on surjective CSPs; essentially, this framework computes global gadgetry that permits one to present a reduction from a classical CSP to a surjective CSP. We show how to derive a number of hardness results for surjective CSP in this framework, including the hardness of the disconnected cut problem, of the no-rainbow 3-coloring problem, and of the surjective CSP on all 2-element structures known to be intractable (in this setting). Our framework thus allows us to unify these hardness results, and reveal common structure among them; we believe that our hardness proof for the disconnected cut problem is more succinct than the original. In our view, the framework also makes very transparent a way in which classical CSPs can be reduced to surjective CSPs.

1 Introduction

1.1 Background

The *constraint satisfaction problem (CSP)* is a computational problem in which one is to decide, given a set of constraints on variables, whether or not there is an assignment to the variables satisfying all of the constraints. This problem appears in many guises throughout computer science, for instance, in database theory, artificial intelligence, and the study of graph homomorphisms. One obtains a rich and natural family of problems by defining, for each relational structure \mathbf{B} , the problem $\text{CSP}(\mathbf{B})$ to be the case of the CSP where the relations used to specify constraints must come from \mathbf{B} . Recall that a relational structure consists of a *universe*, a set—assumed to be finite throughout this article—and indexed relations over this universe; we will refer to a relational structure whose universe has size k as a *k-element structure*. Let us refer to a problem of the form $\text{CSP}(\mathbf{B})$ as a *classical CSP*. In a celebrated development, Bulatov [3] and Zhuk [32], independently, classified the complexity of all classical CSPs, showing each to be either polynomial-time tractable or NP-complete; the algorithmic and complexity aspects of this problem family, and their many links to numerous regions of theoretical computer science, had drawn significant attention over the preceding two decades or so [21].

A natural variant of the CSP is the problem where an instance is again a set of constraints, but the question is to decide whether or not there is a *surjective* satisfying assignment to the variables. For each relational structure \mathbf{B} , in analogy to the definition of the problem $\text{CSP}(\mathbf{B})$, one can define the *surjective CSP* over \mathbf{B} , denoted by $\text{SCSP}(\mathbf{B})$, as the case of this variant where the relations used to specify constraints must come from \mathbf{B} ; then, the question is to decide if there exists a satisfying assignment that is surjective onto the universe of \mathbf{B} . Let us refer to a problem of the form $\text{SCSP}(\mathbf{B})$ as a *surjective CSP*; each such problem is readily seen to be in NP. We remark that such a problem $\text{SCSP}(\mathbf{B})$ can be formulated as the problem of deciding, given as input a relational structure \mathbf{A} , whether or not \mathbf{A} admits a *homomorphism* to \mathbf{B} that is surjective onto the universe of \mathbf{B} . Surjective CSPs are studied and relevant in a number of contexts:

- In graph theory, there is a literature studying the problems $\text{SCSP}(\mathbf{H})$, over graphs \mathbf{H} (see for example [1, 16, 27, 15, 24] and the references therein). Typically, in this literature such a problem is formulated as a homomorphism problem, and referred to as the *surjective H -coloring* problem.

Here, we want to introduce a particular such problem, the *disconnected cut* problem: given a graph G , decide if it has a cut that is disconnected; that is, decide if there exists a non-empty, proper subset U of its vertex set V such that the induced subgraphs $G[U]$ and $G[V \setminus U]$ are disconnected. This problem arises in numerous contexts [27, Section 1], including the theory of vertex cuts and graph connectivity, graph partitions, and vertex covers of graphs; it is equivalent to the problem $\text{SCSP}(\mathbf{H})$ where \mathbf{H} is the reflexive 4-cycle. This problem was proved to be NP-complete by Martin and Paulusma [27].

- In the theory of hypergraphs, colorings of hypergraphs with varied edge types have been studied. For example, *mixed hypergraphs* [31, 19] consist of *edges* and *co-edges*; an assignment of colors to vertices is considered to properly color an edge when two vertices therein are mapped to different colors, and to properly color a co-edge when two vertices therein are mapped to the same color.

Here, we wish to introduce the *no-rainbow 3-coloring* problem: given a hypergraph consisting of size 3 co-edges, decide if there is a coloring with 3 colors in its image. Equivalently, one is given a 3-uniform hypergraph, and the question is whether or not there is a vertex coloring, with image size 3, such that for each edge, it is not the case that the three vertices therein are mapped to three different colors—that is, no edge yields a rainbow. This problem is equivalent to the problem $\text{SCSP}(\mathbf{N})$ where \mathbf{N} is the structure with universe $N = \{0, 1, 2\}$ and a single ternary relation $\{(a, b, c) \in N^3 \mid \{a, b, c\} \neq N\}$; it was highlighted as an open problem in a survey on surjective CSPs [1], and this relation arises naturally in universal algebra within the theory of maximal clones [28]. This problem was recently proved to be NP-complete by Zhuk [33].

- Surjective homomorphisms appear naturally in the linear-algebraic theory of homomorphism-related combinatorial quantities that was pioneered by Lovász [25, 26, 12, 6].
- In the context of constraint satisfaction, surjectivity is a form of *global cardinality constraint*; such constraints are studied heavily in constraint processing and in constraint programming, and have been investigated from the complexity-theoretic viewpoint [30, 4].

The natural research program in this area is to work towards the obtention of a full classification of all surjective CSPs, akin to the classification of all classical CSPs. In addition to providing information on the complexity of individual surjective CSPs of interest, such a classification could provide a rich source of problems from which one could try to reduce in developing NP-hardness proofs.¹ Moreover, the classification of classical CSPs is appealing in that the criterion delineating the two complexity behaviors [3, 32, 9] is a relatively simple algebraic one which robustly admits multiple natural characterizations (and which possesses relatively low complexity!); one could hope for such a criterion in the context of surjective CSPs. Furthermore, studying the complexity of classical CSPs led to a rich general theory having interactions with many other areas of theoretical computer science, and one might hope for a similar situation with surjective CSPs. Regarding known classifications, a dichotomy theorem that classifies the complexity of each surjective CSP on a 2-element structure has been known for many years [11, Theorem 6.12], but a classification for all 3-element structures is currently open. Here, we can mention that optimization and counting flavors of the surjective CSP have also been studied [14, 13, 6].

A striking, frustrating, and somewhat mysterious aspect of the literature on surjective CSPs thus far is the apparent difficulty of establishing intractability results; this has been mentioned on a number of occasions (Sections 1 of [1, 15, 24]). In particular, the various techniques used for proving intractability seem disparate, ad hoc, and not obviously amenable to generalization. As barometers of this claim, consider that the intractability of the *disconnected cut* problem was the subject of an entire article [27], and that this article as well as the one [33] exhibiting the intractability of *no-rainbow 3-coloring* do not explicitly develop machinery nor technology that permits the derivation of other hardness results in the literature. The situation here can be contrasted with even the early work on classical CSPs, where results tended to address classes of relational structures [20], as opposed to individual relational structures. The present author did present an algebraic sufficient condition of hardness for surjective CSPs [5]; while this condition applies to each 2-element structure having a hard surjective CSP, it provably does not apply to prominent examples such as *disconnected cut* and *no-rainbow 3-coloring*.²

1.2 Contributions

What seems to be lacking in this research area is unified, general technology for proving hardness results. In this article, we make a contribution in this direction by presenting a framework wherein one can establish reductions from classical CSPs to surjective CSPs. We derive the following frontier results using our framework:

- The NP-hardness of the *disconnected cut* problem (Section 4.1).
- The NP-hardness of the *no-rainbow 3-coloring* problem (Section 4.2).
- The NP-hardness of each problem $\text{SCSP}(\mathbf{B})$ that is intractable according to the mentioned dichotomy theorem on 2-element structures \mathbf{B} (Section 4.3).

¹This has been done in the preceding literature: complexity hardness results on surjective CSPs were used crucially in a complexity classification of rooted phylogeny problems [2, Section 5]. Complexity hardness results on classical CSPs have long been used to derive further hardness results; indeed, we show hardness results on surjective CSPs in this article by reducing from classical CSPs!

²This inapplicability is proved in Section 4.3.

We believe that our hardness proof for the *disconnected cut* problem is more succinct than the original, not just literally, but also in a conceptual sense; consider that the original proof centrally involved reasoning about the *diameter* of produced instance graphs, whereas such reasoning is fully absent from our proof. The third hardness result named here is obtained by showing that the present author’s previous algebraic sufficient condition for hardness [5] can be obtained within the framework of the present article.

Our framework (presented in Section 3) can be described on a high-level as follows. Let \mathbf{B} be a structure. When the framework is applicable to the structure \mathbf{B} , it permits, for any finite set V of variables, the computation of an instance Φ_V of $\text{SCSP}(\mathbf{B})$ with the following properties: each *surjective* satisfying assignment of Φ_V encodes an assignment $g : V \rightarrow D$, where D is a finite set (which is uniform over all V); and, conversely, each assignment $g : V \rightarrow D$ is encoded by a surjective satisfying assignment of Φ_V . Thus, the instance Φ_V , in essence, encodes the space $G_{V,D}$ of all assignments $g : V \rightarrow D$. Then, adding constraints to Φ_V places constraints on the allowable assignments in this space $G_{V,D}$; in this way, the problem $\text{SCSP}(\mathbf{B})$ can simulate—formally, admit a reduction from—a problem $\text{CSP}(\mathbf{D})$, where \mathbf{D} is a structure with universe D and appropriately defined relations.

We conceive of the instances Φ_V as a form of *global gadgetry* or of scaffolding that, once configured, enables the use of *local gadgetry* encoding individual constraints. We believe that our framework makes very transparent *why* and *how* surjective CSPs permit reductions from classical CSPs; also, we believe that the *roles* of the various introduced constraints in these reductions is highly transparent. We hypothesize that progress in this area was hindered by the lack of systematic tools for computing and reasoning about such global gadgetry. We provide evidence for the well-behavedness of our framework, by showing that under a certain assumption, whether or not the framework is applicable is decidable (Section 5).

Our framework has implications beyond those described for the complexity of the decision problems $\text{SCSP}(\mathbf{B})$. We show (Section 6) that the framework can also be used to derive NP-hardness of cases of the problem of deciding if a given relational structure \mathbf{A} has a *condensation* to a second relational structure \mathbf{B} , where a condensation is a surjective satisfying assignment that also maps the tuples of each relation of \mathbf{A} surjectively onto the corresponding relation of \mathbf{B} . We also discuss implications for the *sparsifiability* of the problems $\text{SCSP}(\mathbf{B})$ (Section 7); this setting concerns polynomial-time reductions from a hard problem $\text{SCSP}(\mathbf{B})$ to a second problem, such that the output instance obeys a size guarantee.

Overall, we wish to emphasize our introduction of systematic machinery, the unifying nature of this machinery in terms of yielding a common understanding of previously disparate hardness proofs, and the clarifying and transparent nature of our approach. Our work also ties into a large body of work on CSPs and other problems that utilizes algebraic closure properties of solution spaces as a way of obtaining insight into the complexity of problems. In particular, we make use of such properties in the form of so-called *partial polymorphisms*, and our work contributes to a growing literature which uses partial polymorphisms as a tool to understand the complexity of various computational problems [18, 22, 7, 10, 23].

2 Preliminaries

When $f : A \rightarrow B$ and $g : B \rightarrow C$ are mappings, we use $g \circ f$ to denote their composition. We adhere to the convention that for any set S , there is a single element in S^0 ; this element is referred

to as the *empty tuple*, and is denoted by ϵ .

2.1 Structures, formulas, and problems

A *signature* is a set of *relation symbols*; each relation symbol R has an associated arity (a natural number), denoted by $\text{ar}(R)$. A *structure* \mathbf{B} over signature σ consists of a *universe* B which is a set, and an interpretation $R^{\mathbf{B}} \subseteq B^{\text{ar}(R)}$ for each relation symbol $R \in \sigma$. We tend to use the letters $\mathbf{A}, \mathbf{B}, \dots$ to denote structures, and the letters A, B, \dots to denote their respective universes. In this article, we assume that signatures under discussion are finite, and also assume that all structures under discussion are finite; a structure is finite when its universe is finite.

By an *atom* (over a signature σ), we refer to a formula of the form $R(v_1, \dots, v_k)$ where R is a relation symbol (in σ), $k = \text{ar}(R)$, and the v_i are variables. An \wedge -*formula* (over a signature σ) is a conjunction $\beta_1 \wedge \dots \wedge \beta_m$ where each conjunct β_i is an atom (over σ) or a variable equality $v = v'$. Here, we permit an empty conjunction. With respect to a structure \mathbf{B} , an \wedge -formula ϕ over the signature of \mathbf{B} is *satisfied* by a mapping f to B defined on the variables of ϕ when:

- for each atom $R(v_1, \dots, v_k)$ in ϕ , it holds that $(f(v_1), \dots, f(v_k)) \in R^{\mathbf{B}}$, and
- for each variable equality $v = v'$, it holds that $f(v) = f(v')$.

When this holds, we refer to f as a *satisfying assignment* of ϕ (over \mathbf{B}). Note that the empty conjunction is considered to be satisfied by any such mapping f .

We now define the computational problems to be studied. For each structure \mathbf{B} , define $\text{CSP}(\mathbf{B})$ to be the problem of deciding, given a \wedge -formula ϕ (over the signature of \mathbf{B}), whether or not there exists a satisfying assignment, that is, a map f to B , defined on the variables of ϕ , that satisfies ϕ over \mathbf{B} . For each structure \mathbf{B} , define $\text{SCSP}(\mathbf{B})$ to be the problem of deciding, given a pair (U, ϕ) where U is a set of variables and ϕ is a \wedge -formula (over the signature of \mathbf{B}) with variables from U , whether or not there exists a surjective satisfying assignment on U , that is, a surjective map $f : U \rightarrow B$ that satisfies ϕ over \mathbf{B} .³

Unless mentioned otherwise, when discussing NP-hardness and NP-completeness, we refer to these notions as defined with respect to polynomial-time many-one reductions.

2.2 Definability and algebra

Let \mathbf{B} be a relational structure. Let S be a finite set, and let F be a set of mappings, each of which is from S to B . We say that F is \wedge -*definable* over \mathbf{B} if there exists a \wedge -formula ϕ , whose variables are drawn from S , such that F is the set of satisfying assignments $f : S \rightarrow B$ of ϕ , with respect to \mathbf{B} . Let S be a finite set; when T is a set of mappings, each of which is from S to B , we use $\langle T \rangle_{\mathbf{B}}$ to denote the smallest \wedge -definable set of mappings from S to B (over \mathbf{B}) that contains T . (Such a smallest set exists: over a structure \mathbf{B} , the set of all mappings from a finite set S to B is \wedge -definable via the empty conjunction; and, one clearly has \wedge -definability of the intersection of two \wedge -definable sets of mappings all sharing the same type.)

³We remark that, given an instance of a problem $\text{SCSP}(\mathbf{B})$, variable equalities may be efficiently eliminated in a way that preserves the existence of a surjective satisfying assignment [5, Proposition 2.1]. Likewise, given an instance of a problem $\text{CSP}(\mathbf{B})$, variable equalities may be efficiently eliminated in a way that preserves the existence of a satisfying assignment. Thus, in these problems, whether or not one allows variable equalities in instances is a matter of presentation, for the complexity issues at hand.

Let T be a set of mappings from a finite set I to a finite set B . A *partial polymorphism* of T is a partial mapping $p : B^J \rightarrow B$ such that, for any selection $s : J \rightarrow T$ of maps, letting $c_i \in B^J$ denote the mapping taking each $j \in J$ to $(s(j))(i)$:

if the mapping from I to B sending each $i \in I$ to the value $p(c_i)$ is defined at each point $i \in I$,
then it is contained in T .

Conventionally, one speaks of a partial polymorphism of a relation $Q \subseteq B^k$; we here give a more general formulation, as it will be convenient for us to deal here with arbitrary index sets I . When $Q \subseteq B^k$ is a relation, we apply the just-given definition by viewing each element of Q as a set of mappings from the set $\{1, \dots, k\}$ to B ; likewise, when $p : B^k \rightarrow B$ is a partial mapping with $k \geq 0$ a natural number, we apply this definition by viewing each element of B^k as a mapping from $\{1, \dots, k\}$ to B . A partial mapping $p : B^J \rightarrow B$ is a partial polymorphism of a relational structure if it is a partial polymorphism of each of the relations of the structure.

The following is a known result; it connects \wedge -definability to closure under partial polymorphisms.

Theorem 2.1 [29] *Let \mathbf{B} be a structure. Let T be a set of non-empty mappings from a finite set I to B ; for each $i \in I$, let $\pi_i : T \rightarrow B$ denote the map defined by $\pi_i(t) = t(i)$. The set $\langle T \rangle_{\mathbf{B}}$ is equal to the set T' of maps from I to B having a definition of the form $i \mapsto p(\pi_i)$, where $p : B^T \rightarrow B$ is a partial polymorphism (of \mathbf{B}) with domain $\{\pi_i \mid i \in I\}$.*

We provide a proof of this theorem for the sake of completeness.

Proof. It is straightforward to verify that each partial polymorphism of \mathbf{B} is also a partial polymorphism of $\langle T \rangle_{\mathbf{B}}$. It thus holds that $T' \subseteq \langle T \rangle_{\mathbf{B}}$.

In order to establish that $\langle T \rangle_{\mathbf{B}} \subseteq T'$, it suffices to show that T' is \wedge -definable over \mathbf{B} . Since equalities are permitted in \wedge -formulas, it suffices to prove the result in the case that the maps π_i are pairwise distinct. Let θ be the \wedge -formula where, for each relation symbol R , the atom $R(i_1, \dots, i_k)$ is included as a conjunct if and only if each $t \in T$ satisfies $(t(i_1), \dots, t(i_k)) \in R^{\mathbf{B}}$. By the definition of partial polymorphism, we have that each map in T' satisfies θ . On the other hand, when $u : I \rightarrow B$ is a map that satisfies the formula θ , consider the partial mapping $q : B^T \rightarrow B$ sending π_i to $u(i)$; this is well-defined since the π_i are pairwise distinct, and is a partial polymorphism, by the construction of θ . \square

A *polymorphism* is a partial polymorphism that is a total mapping. A total mapping $p : B^J \rightarrow B$ is *essentially unary* if there exists $j \in J$ and a unary operation $u : B \rightarrow B$ such that, for each mapping $h : J \rightarrow B$, it holds that $p(h) = u(h(j))$. An *automorphism* of a structure \mathbf{B} is a bijection $\sigma : B \rightarrow B$ such that, for each relation $R^{\mathbf{B}}$ of \mathbf{B} and for each tuple (b_1, \dots, b_k) whose arity k is that of $R^{\mathbf{B}}$, it holds that $(b_1, \dots, b_k) \in R^{\mathbf{B}}$ if and only if $(\sigma(b_1), \dots, \sigma(b_k)) \in R^{\mathbf{B}}$. It is well-known and straightforward to verify that, for each finite structure \mathbf{B} , a bijection $\sigma : B \rightarrow B$ is an automorphism if and only if it is a polymorphism.

Let $p : B^J \rightarrow B$ be a partial mapping. For each $b \in B$, let b^J denote the mapping from J to B that sends each element $j \in J$ to b . The *diagonal* of p , denoted by \hat{p} , is the partial unary mapping from B to B such that $\hat{p}(b) = p(b^J)$ for each $b \in B$. With respect to a structure \mathbf{B} , we say that a partial mapping $p : B^J \rightarrow B$ is *automorphism-like* when there exists $j \in J$ and an automorphism γ (of \mathbf{B}) such that, for each mapping $h : J \rightarrow B$, if $p(h)$ is defined, then it is equal to $\gamma(h(j))$.

3 Framework

Throughout this section, let \mathbf{B} be a finite relational structure, and let B be its universe. Let I be a finite set; let T be a set of mappings from I to B . Let us say that T is *surjectively closed* over \mathbf{B} if each surjective mapping in $\langle T \rangle_{\mathbf{B}}$ is contained in $\{\gamma \circ t \mid \gamma \text{ is an automorphism of } \mathbf{B}, t \in T\}$. The following is essentially a consequence of Theorem 2.1.

Proposition 3.1 *Let I, T be as described. For each $i \in I$, let $\pi_i : T \rightarrow B$ denote the mapping defined by $\pi_i(t) = t(i)$. The following are equivalent:*

- *The set T is surjectively closed over \mathbf{B} .*
- *Each surjective partial polymorphism $p : B^T \rightarrow B$ (of \mathbf{B}) with domain $\{\pi_i \mid i \in I\}$ is automorphism-like.*

We consider a partial polymorphism $p : B^T \rightarrow B$ to be *surjective* when, for each $b \in B$, there exists $h \in B^T$ such that $p(h)$ is defined and is equal to b .

Proof. Suppose that T is surjectively closed over \mathbf{B} . Let $p : B^T \rightarrow B$ be a surjective partial polymorphism with the described domain. By Theorem 2.1, the map t' defined by $i \mapsto p(\pi_i)$ is an element of $\langle T \rangle_{\mathbf{B}}$; note also that this map is surjective. By the definition of surjectively closed, it holds that t' has the form $\gamma(t)$, where γ is an automorphism of \mathbf{B} , and $t \in T$. It follows that p is automorphism-like.

Suppose that each surjective partial polymorphism from B^T to B with the described domain is automorphism-like. By Theorem 2.1, each surjective mapping t' in $\langle T \rangle_{\mathbf{B}}$ is defined by $i \mapsto p(\pi_i)$ where p is a partial polymorphism; note that p is surjective. By hypothesis, p is automorphism-like; it follows that t' has the form $\gamma(t)$ where γ is an automorphism of \mathbf{B} , and $t \in T$. We conclude that T is surjectively closed. \square

In what follows, we generally use D to denote a finite set, V to denote a finite set of variables, and $G_{V,D}$ to denote the set of all mappings from V to D ; we will sometimes refer to elements of $G_{V,D}$ as *assignments*.

Define an *encoding* (for \mathbf{B}) to be a finite set F of mappings, each of which is from a finite power D^k of a finite set D to B ; we refer to k as the *arity* of such a mapping. Formally, an *encoding* for \mathbf{B} is a finite set F such that there exists a finite set D where, for each $f \in F$, there exists $k \geq 0$ such that f is a mapping from D^k to B . In what follows, we will give a sufficient condition for an encoding to yield a reduction from a classical CSP over a structure with universe D to the problem SCSP(\mathbf{B}). Assume F to be an encoding; define a (V, F) -*application* to be a pair (\bar{v}, f) consisting of a tuple of variables from V and a mapping $f \in F$ such that the length of \bar{v} is equal to the arity of f . Let $A_{V,F}$ denote the set of all (V, F) -applications.

Example 3.2 Let $D = \{0, 1\}$, $B = \{0, 1, 2\}$, and let F be the encoding that contains the 3 mappings from D^0 to B as well as the 6 injective mappings from D^1 to B . We have $|F| = 9$. (This encoding will be used in Section 4.2.)

Let V be a set of size 4. As $|V| = 4$ and $|D| = 2$, we have $|G_{V,D}| = 2^4 = 16$. The arity 0 mappings in F give rise to 3 (V, F) -applications and the arity 1 mappings in F give rise to $6|V| = 24$ (V, F) -applications. Thus we have $|A_{V,F}| = 27$. \square

Let V be a finite set of variables; let F be an encoding. For each $g \in G_{V,D}$, when $\alpha = ((v_1, \dots, v_k), f)$ is an application in $A_{V,F}$, define $\alpha[g]$ to be the value $f(g(v_1), \dots, g(v_k))$; define $t[g]$ to be the map from $A_{V,F}$ to B where each application α is mapped to $\alpha[g]$. Define $T_{V,F} = \{t[g] \mid g \in G_{V,D}\}$.

Proposition 3.3 *Let F be an encoding. There exists a polynomial-time algorithm that, given a finite set V , computes an \wedge -formula (over the signature of \mathbf{B}) defining $\langle T_{V,F} \rangle_{\mathbf{B}}$.*

Let us clarify how polynomial time is measured here: we assume that the input to the algorithm is simply a finite set V , and that such a set is encoded by a list where each element appears explicitly; so, in particular, the length of a set's encoding is larger than $|V|$.

Proof. The algorithm performs the following. For each relation $R^{\mathbf{B}}$ of \mathbf{B} , let k be its arity. For each tuple $(\alpha_1, \dots, \alpha_k) \in A_{V,F}^k$, the projection of T onto $(\alpha_1, \dots, \alpha_k)$ can be computed, by considering all possible assignments on the variables in V that appear in the α_i ; note that the number of such variables is bounded by a constant, since both the signature of \mathbf{B} and F are assumed to be finite. If this projection is a subset of $R^{\mathbf{B}}$, then include $R(\alpha_1, \dots, \alpha_k)$ in the formula; otherwise, do not. Perform the same process for the equality relation on B , with $k = 2$; whenever the projection is a subset of this relation, include $\alpha_1 = \alpha_2$ in the formula. \square

Definition 3.4 Let F be an encoding. Define a relational structure \mathbf{B} to be *F-stable* if, for each non-empty finite set V , it holds that each map in $T_{V,F}$ is surjective, and $T_{V,F}$ is surjectively closed (over \mathbf{B}).

Note that, relative to an encoding F , only the size of V matters in the definition of $T_{V,F}$ in Definition 3.4, in the sense that when V and V' are of the same size, $T_{V,F}$ and $T_{V',F}$ are equal up to relabelling of indices.

Definition 3.5 Let F be an encoding. An *F-induced relation* of \mathbf{B} is a relation $Q' \subseteq D^s$ (with $s \geq 1$) such that, letting (u_1, \dots, u_s) be a tuple of pairwise distinct variables, there exists:

- a relation $Q \subseteq B^r$ that is either a relation of \mathbf{B} or the equality relation on B , and
- a tuple $(\alpha_1, \dots, \alpha_r) \in A_{U,F}^r$

such that $Q' = \{(g(u_1), \dots, g(u_s)) \mid g \in G_{U,D}, ((t[g])(\alpha_1), \dots, (t[g])(\alpha_r)) \in Q\}$. We refer to Q as the relation that induces Q' , and to the pair $(Q, (\alpha_1, \dots, \alpha_r))$ as the *definition* of Q' .

Definition 3.6 Let F be an encoding. An *F-induced template* of \mathbf{B} is a relational structure \mathbf{D} with universe D and whose relations are all F -induced relations of \mathbf{B} .

Theorem 3.7 *Let F be an encoding. Suppose that \mathbf{B} is F -stable, and that \mathbf{D} is an F -induced template of \mathbf{B} . Then, the problem $\text{CSP}(\mathbf{D})$ polynomial-time many-one reduces to $\text{SCSP}(\mathbf{B})$.*

Proof. Let ϕ be an instance of $\text{CSP}(\mathbf{D})$ with variables V . We may assume (up to polynomial-time computation) that ϕ does not include any variable equalities. We create an instance $(A_{V,F}, \psi)$ of $\text{SCSP}(\mathbf{B})$; this is done by computing two \wedge -formulas ψ_0 and ψ_1 , and setting $\psi = \psi_0 \wedge \psi_1$.

Compute ψ_0 to be an \wedge -formula defining $\langle T_{V,F} \rangle_{\mathbf{B}}$, where $T_{V,F} = \{t[g] \mid g \in G_{V,D}\}$; such a formula is polynomial-time computable by Proposition 3.3.

Compute ψ_1 as follows. For each atom $R'(v_1, \dots, v_s)$ of ϕ where $R'^{\mathbf{D}}$ is induced by a relation $R^{\mathbf{B}}$, let $c : \{u_1, \dots, u_s\} \rightarrow V$ be the mapping sending each u_i to v_i , and include the atom $R(c(\alpha_1), \dots, c(\alpha_r))$ in ψ_1 ; here, $(\alpha_1, \dots, \alpha_r)$ is the tuple from Definition 3.5, and c acts on an application α by being applied individually to each variable in the variable tuple of α , that is, when $\alpha = ((u'_1, \dots, u'_k), f)$, we have $c(\alpha) = ((c(u'_1), \dots, c(u'_k)), f)$. For each atom $R'(v_1, \dots, v_s)$ of ϕ where $R'^{\mathbf{D}}$ is induced by the equality relation on B , let c and (α_1, α_2) be as above, and include the atom $c(\alpha_1) = c(\alpha_2)$ in ψ_1 . We make the observation that, from Definition 3.5, a mapping $g \in G_{V,D}$ satisfies an atom $R'(v_1, \dots, v_s)$ of ϕ if and only if $t[g]$ satisfies the corresponding atom or equality in ψ_1 .

We argue that ϕ is a yes instance of $\text{CSP}(\mathbf{D})$ if and only if $(A_{V,F}, \psi)$ is a yes instance of $\text{SCSP}(\mathbf{B})$. Suppose that $g \in G_{V,D}$ is a satisfying assignment of ϕ . The assignment $t[g]$ satisfies ψ_0 since $t[g] \in T_{V,F}$. Since the assignment g satisfies each atom of ϕ , by the observation, the assignment $t[g]$ satisfies each atom and equality of ψ_1 , and so $t[g]$ is a satisfying assignment of ψ_1 . It also holds that $t[g]$ is surjective by the definition of F -stable. Thus, we have that $t[g]$ is a surjective satisfying assignment of ψ . Next, suppose that there exists a surjective satisfying assignment t' of ψ . Since t' satisfies ψ_0 , it holds that $t' \in \langle T_{V,F} \rangle_{\mathbf{B}}$. Since $T_{V,F}$ is surjectively closed (over \mathbf{B}) by F -stability, there exists $g \in G_{V,D}$ such that $t' = \gamma(t[g])$ for an automorphism γ of \mathbf{B} . Since t' is a satisfying assignment of ψ , so is $t[g]$; it then follows from the observation that g is a satisfying assignment of ϕ . \square

Inner symmetry

Each set of the form $\langle T_{V,F} \rangle_{\mathbf{B}}$ is closed under the automorphisms of \mathbf{B} , since each automorphism is a partial polymorphism (recall Theorem 2.1). We here present another form of symmetry that such a set $\langle T_{V,F} \rangle_{\mathbf{B}}$ may possess, which we dub *inner symmetry*. Relative to a structure \mathbf{B} and an encoding F , we define an *inner symmetry* to be a pair $\sigma = (\rho, \tau)$ where $\rho : D \rightarrow D$ is a bijection, and $\tau : B \rightarrow B$ is an automorphism of \mathbf{B} such that $F = \{\tilde{\sigma} \circ f \mid f \in F\}$; here, when $g : D^k \rightarrow B$ is a mapping, $\tilde{\sigma}(g) : D^k \rightarrow B$ is defined as the composition $\tau \circ g \circ (\rho, \dots, \rho)$, where (ρ, \dots, ρ) denotes the mapping from D^k to D^k that applies ρ to each entry of a tuple in D^k . When σ is an inner symmetry, we naturally extend the definition of $\tilde{\sigma}$ so that it is defined on each application: when $\alpha = (\bar{v}, f)$ is an application, define $\tilde{\sigma}(\alpha) = (\bar{v}, \tilde{\sigma}(f))$.

The following theorem describes the symmetry on $\langle T_{V,F} \rangle_{\mathbf{B}}$ induced by an inner symmetry.

Theorem 3.8 *Let \mathbf{B} be a structure, let F be an encoding, and let $\sigma = (\rho, \tau)$ be an inner symmetry thereof. Let V be a non-empty finite set. For any map $u : A_{V,F} \rightarrow B$, define $u' : A_{V,F} \rightarrow B$ by $u'(\alpha) = u(\tilde{\sigma}(\alpha))$; it holds that $u \in \langle T_{V,F} \rangle_{\mathbf{B}}$ if and only if $u' \in \langle T_{V,F} \rangle_{\mathbf{B}}$.*

Proof. For each $g \in G_{V,D}$, define $t'[g] : A_{V,F} \rightarrow B$ to map each $\alpha \in A_{V,F}$ to $(\tilde{\sigma}(\alpha))[g]$. Define $T'_{V,F}$ as $\{t'[g] \mid g \in G_{V,D}\}$. By definition, $T'_{V,F} = \{\tau(t[\rho(g)]) \mid g \in G_{V,D}\}$; since ρ is a bijection, we have that $T'_{V,F} = \{\tau(t[h]) \mid h \in G_{V,D}\}$. Since τ is an automorphism of \mathbf{B} , we obtain $\langle T_{V,F} \rangle_{\mathbf{B}} = \langle T'_{V,F} \rangle_{\mathbf{B}}$.

Since σ is an inner symmetry, we have $F = \{\tilde{\sigma} \circ f \mid f \in F\}$, from which it follows that the action of $\tilde{\sigma}$ on applications in $A_{V,F}$ is a bijection on $A_{V,F}$. We have $t'[g](\alpha) = t[g](\tilde{\sigma}(\alpha))$. For any map $u : A_{V,F} \rightarrow B$, define $u' : A_{V,F} \rightarrow B$ by $u'(\alpha) = u(\tilde{\sigma}(\alpha))$. For all $u : A_{V,F} \rightarrow B$, we have $u' \in T'_{V,F} \Leftrightarrow u \in T_{V,F}$, implying that $u' \in \langle T'_{V,F} \rangle_{\mathbf{B}} \Leftrightarrow u \in \langle T_{V,F} \rangle_{\mathbf{B}}$. Since $\langle T_{V,F} \rangle_{\mathbf{B}} = \langle T'_{V,F} \rangle_{\mathbf{B}}$, the theorem follows. \square

4 Hardness results

Throughout this section, we employ the following conventions. When B is a set and $b \in B$, we use the notation \bar{b} to denote the arity 0 function from D^0 to B sending the empty tuple to b . Let V be a set, and let F be an encoding. When $\bar{b} \in F$, we overload the notation \bar{b} and also use it to denote the unique (V, F) -application in which it appears. Relative to a structure \mathbf{B} (understood from the context), when $\alpha_1, \dots, \alpha_k \in A_{V,F}$ are applications and R is a relation symbol, we write $R(\alpha_1, \dots, \alpha_k)$ when, for each $g \in G_{V,D}$, it holds that $(\alpha_1[g], \dots, \alpha_k[g]) \in R^{\mathbf{B}}$; when R is a symmetric binary relation, we also say that α_1 and α_2 are adjacent. When α is an application in $A_{V,F}$, and $p : B^{T_{V,F}} \rightarrow B$ is a partial mapping, we simplify notation by using $p(\alpha)$ to denote the value $p(\pi_\alpha)$ (recall the definition of π_α from Proposition 3.1).

4.1 Disconnected cut: the reflexive 4-cycle

Let us use \mathbf{C} to denote the reflexive 4-cycle, that is, the structure with universe $C = \{0, 1, 2, 3\}$ and single binary relation $E^{\mathbf{C}} = C^2 \setminus \{(0, 2), (2, 0), (1, 3), (3, 1)\}$. The problem $\text{SCSP}(\mathbf{C})$ was shown to be NP-complete by [27]; we here give a proof using our framework. When discussing this structure, we will say that two values $c, c' \in C$ are adjacent when $(c, c') \in E^{\mathbf{C}}$. Set $D = \{0, 1, 3\}$. We use the notation $[abc]$ to denote the function $f : D \rightarrow C$ with $(f(0), f(1), f(3)) = (a, b, c)$, so, for example $[013]$ denotes the identity mapping from D to C . Define F as the encoding

$$\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, [013], [010], [323], [313], [112], [003], [113]\}.$$

We will prove the following.

Theorem 4.1 *The reflexive 4-cycle \mathbf{C} is F -stable.*

We begin by observing the following.

Proposition 4.2 *Define $\rho : D \rightarrow D$ as the bijection that swaps 1 and 3; define $\tau : B \rightarrow B$ as the bijection that swaps 1 and 3. The pair $\sigma = (\rho, \tau)$ is an inner symmetry of F and \mathbf{C} .*

Proof. Consider the action of $\tilde{\sigma}$ on F . This action $\tilde{\sigma}$ transposes $\bar{1}$ and $\bar{3}$; $[010]$ and $[003]$; $[323]$ and $[112]$; and, $[313]$ and $[113]$. It fixes each other element of F . \square

Proof. (Theorem 4.1) Let V be a non-empty finite set; we need to show that $T_{V,F}$ is surjectively closed. We make use of Proposition 3.1. Consider a partial polymorphism $p : C^{T_{V,F}} \rightarrow C$ whose domain is π_α over all (V, F) -applications $\alpha \in A_{V,F}$.

We first consider the situation where p has a surjective diagonal; we want to show that p is automorphism-like. Define $\beta : C \rightarrow C$ by $\beta(c) = p(\bar{c})$. We have that β is surjective and a polymorphism of \mathbf{C} , implying that it is an automorphism of \mathbf{C} . By considerations of symmetry (in particular, by replacing p with the translation $\beta^{-1} \circ p$, and then translating back post-argument), we may assume that $(p(\bar{0}), p(\bar{1}), p(\bar{2}), p(\bar{3})) = (0, 1, 2, 3)$. Under this assumption, it is straightforward to verify the following fact: for any application $(v, [abc])$, it holds that $p(v, [abc]) \in \{a, b, c\}$. For example, in the case that the set $\{a, b, c\}$ has size 2 and its two elements a', b' are adjacent in \mathbf{C} , $(v, [abc])$ is adjacent to $\bar{a'}$ and $\bar{b'}$, implying that $p(v, [abc])$ is adjacent to both $p(\bar{a'}) = a'$ and $p(\bar{b'}) = b'$, and is thus in $\{a', b'\}$. In the following reasoning, we implicitly use this fact, continually.

Fix v to be a variable. We show that there exists a map $g_v : \{v\} \rightarrow D$ such that, for each $f \in F$, $p(v, f) = f(g_v(v))$. This suffices, since then one can define a map $g \in G_{V,D}$ that extends all of the maps (g_v) to derive the map $t[g] \in T_{V,F}$ equal to the map sending each application α to $p(\alpha) = p(\pi_\alpha)$.

We have that $(v, [010])$, $(v, [323])$, and $(v, [313])$ are pairwise adjacent. When the value of $p(v, [010])$ is 0, we have, by the identified adjacencies, $p(v, [323]) = 3$ and $p(v, [313]) = 3$, which by the adjacency of $(v, [313])$ and $(v, [003])$, implies $p(v, [003])$ is 0 or 3. When the value of $p(v, [010])$ is 1, we have, by the identified adjacencies, $p(v, [323]) = 2$ and $p(v, [313]) = 1$, which by the adjacency of $(v, [313])$ and $(v, [003])$, implies $p(v, [003])$ is 0. We thus have 3 cases, depending on the value of the tuple $(p(v, [010]), p(v, [323]), p(v, [313]), p(v, [003]))$: this tuple is equal to either $(0, 3, 3, 0)$, $(1, 2, 1, 0)$, or $(0, 3, 3, 3)$. By using the pairwise adjacency of $(v, [112])$, $(v, [003])$, and $(v, [113])$, we can confirm that in the 3 cases, these applications are mapped by p to $(1, 0, 1)$, $(1, 0, 1)$, and $(2, 3, 3)$, respectively. Then, by using the adjacency of $(v, [013])$ with each of $(v, [010])$, $(v, [323])$, and $(v, [112])$, we can confirm that $p(v, [013])$ is, in the 3 cases, equal to 0, 1, and 3, respectively.

We have analyzed the situation where p has a surjective diagonal. To establish the result, it suffices to show that if p has a non-surjective diagonal, then it is not surjective; this is what we do in the rest of the proof.

By considerations of symmetry (namely, by the automorphisms and by the inner symmetries), it suffices to consider the following values for the diagonal values $(p(\bar{0}), p(\bar{1}), p(\bar{2}), p(\bar{3}))$: $(0, 0, 0, 0)$, $(0, 0, 1, 0)$, $(0, 1, 0, 0)$, $(1, 0, 0, 0)$, $(0, 0, 1, 1)$, $(0, 1, 0, 1)$, $(0, 1, 2, 1)$, $(0, 1, 0, 3)$. We consider each of these cases.

In each of the first 3 cases, we argue as follows. Consider an application (v, f) with $f : D \rightarrow B$ in F and $v \in V$; it is adjacent to $\bar{0}$, adjacent to $\bar{3}$, or adjacent to both $\bar{1}$ and $\bar{2}$; thus, for such an application, we have $p(v, f)$ is adjacent to 0. It follows that for no such application do we have $p(v, f) = 2$, and so p is not surjective.

Case: diagonal $(1, 0, 0, 0)$. Observe that any application (v, f) with $f : D \rightarrow C$ in $F \setminus \{[013]\}$ and $v \in V$; is adjacent to $\bar{1}$, $\bar{2}$, or $\bar{3}$, and thus, for any such application, $p(v, f)$ is adjacent to 0. Thus if 2 is in the image of p , there exists a variable $u \in V$ such that $p(u, [013]) = 2$. But then, for each $f : D \rightarrow C$ in $F \setminus \{[013]\}$, we have that $(u, [013])$ and (u, f) are adjacent. $(u, [010])$ and $(u, [003])$ are adjacent to $\bar{0}$, which p maps to 1, and to $(u, [013])$, which p maps to 2; this, by the observation, $p(u, [010]) = p(u, [003]) = 1$. $(u, [323])$ and $(u, [112])$ are adjacent to the just-mentioned applications, from which we obtain $p(u, [323]) = p(u, [112]) = 1$.

In order for p to be surjective, there exists a different variable $u' \in V$ and $f' \in F$ such that $p(u', f') = 3$. It must be that (u', f') is not adjacent to $\bar{0}$. But then f' is $[323]$ or $[112]$, and this contradicts that (u', f') is adjacent to (u, f) .

Case: diagonal $(0, 0, 1, 1)$. There must be an application (v, f) with $p(v, f) = 2$. Since (v, f) cannot be adjacent to $\bar{0}$ nor $\bar{1}$, it must be that $f = [323]$. There must also be an application (v', f') with $p(v', f') = 3$; this application cannot be adjacent to $\bar{2}$ nor $\bar{3}$, and so f' is $[013]$ or $[010]$. $(v', [323])$ is adjacent to (v', f') , to $\bar{3}$, and to $(v, [323])$, and so $p(v', [323])$ is 2. $(v', [313])$ is adjacent to $(v', [323])$, to f' , to $\bar{0}$, and to $\bar{2}$, and thus it cannot be mapped to any value.

Case: diagonal $(0, 1, 0, 1)$ or $(0, 1, 0, 3)$. For each variable $u \in F$ and each $f : D \rightarrow B$ in F , it holds that (u, f) is adjacent to either $\bar{0}$ or $\bar{2}$. It follows that no such pair (u, f) maps to 2 under p .

Case: diagonal $(0, 1, 2, 1)$. If p is surjective, there exists (v, f) such that $p(v, f) = 2$. (v, f) cannot be adjacent to $\bar{0}$, implying that $f = [323]$ or $[112]$. When $f = [323]$, we infer that $p(v, [010]) = 1$, and then that $p(v, [013]) = p(v, [313]) = p(v, [113]) = 1$. Similarly, when $f = [112]$, we infer that

$p(v, [003]) = 1$, and then that $p(v, [013]) = p(v, [313]) = p(v, [113]) = 1$. There exists (v', f') such that $p(v', f') = 3$. (v', f') cannot be adjacent to $\bar{1}$ nor $\bar{3}$, so f' is one of $[013]$, $[313]$, $[113]$. This contradicts that (v', f') and (v, f') are adjacent. \square

Theorem 4.3 *The problem SCSP(C) is NP-complete.*

Proof. Define \mathbf{D}' (following [27, Section 2]) to be the structure with universe $\{0, 1, 3\}$ and with relations

$$S_1^{\mathbf{D}'} = \{(0, 3), (1, 1), (3, 1), (3, 3)\}, S_2^{\mathbf{D}'} = \{(1, 0), (1, 1), (3, 1), (3, 3)\}, \\ S_3^{\mathbf{D}'} = \{(1, 3), (3, 1), (3, 3)\}, S_4^{\mathbf{D}'} = \{(1, 1), (1, 3), (3, 1)\}.$$

Each of these relations is the intersection of binary F -induced relations: for $S_1^{\mathbf{D}'}$, use the definitions $(E^{\mathbf{C}}, (\bar{2}, (u_2, [013])))$, $(E^{\mathbf{C}}, ((u_1, [013]), (u_2, [323])))$; for $S_2^{\mathbf{D}'}$, the definitions $(E^{\mathbf{C}}, (\bar{2}, (u_1, [013])))$, $(E^{\mathbf{C}}, ((u_1, [112]), (u_2, [013])))$; for $S_3^{\mathbf{D}'}$, the definitions $(E^{\mathbf{C}}, (\bar{2}, (u_1, [013])))$, $(E^{\mathbf{C}}, (\bar{2}, (u_2, [013])))$, $(E^{\mathbf{C}}, ((u_1, [003]), (u_2, [323])))$; and, for $S_4^{\mathbf{D}'}$, the definitions $(E^{\mathbf{C}}, (\bar{2}, (u_1, [013])))$, $(E^{\mathbf{C}}, (\bar{2}, (u_2, [013])))$, $(E^{\mathbf{C}}, ((u_1, [112]), (u_2, [010])))$. Let \mathbf{D} be the F -induced template of \mathbf{B} whose relations are all of the mentioned F -induced relations. Then, we have $\text{CSP}(\mathbf{D}')$ reduces to $\text{CSP}(\mathbf{D})$, and that $\text{CSP}(\mathbf{D})$ reduces to $\text{SCSP}(\mathbf{C})$ by Theorem 3.7. The problem $\text{CSP}(\mathbf{D}')$ is NP-complete, as argued in [27, Section 2], and thus we conclude that $\text{SCSP}(\mathbf{C})$ is NP-complete. \square

4.2 No-rainbow 3-coloring

Let \mathbf{N} be the structure with universe $N = \{0, 1, 2\}$ and a single ternary relation

$$R^{\mathbf{N}} = \{(a, b, c) \in N^3 \mid \{a, b, c\} \neq N\}.$$

The problem $\text{SCSP}(\mathbf{N})$ was first shown to be NP-complete by Zhuk [33]; we give a proof which is akin to proofs given by Zhuk [33], using our framework.

Define $D = \{0, 1\}$, and define F as $\{\bar{0}, \bar{1}, \bar{2}\} \cup U$ where U is the set of all injective mappings from D to N . We use the notation $[ab]$ to denote the mapping $f : D \rightarrow N$ with $(f(0), f(1)) = (a, b)$. Let $\iota_D : D \rightarrow D$ denote the identity mapping on D ; it is straightforwardly verified that, for each bijection $\tau : N \rightarrow N$, the pair (ι_D, τ) is an inner symmetry of \mathbf{N} and F .

Theorem 4.4 *The structure \mathbf{N} is F -stable.*

Proof. Let V be a non-empty finite set; we need to show that $T_{V,F}$ is surjectively closed. We use Proposition 2.1. It is straightforward to verify that each surjective partial polymorphism of the described form having a surjective diagonal is automorphism-like. Consider a partial polymorphism $p : N^{T_{V,F}} \rightarrow N$ whose domain is π_α over all (V, F) -applications $\alpha \in A_{V,F}$. We show that if p has a non-surjective diagonal, then it is not surjective. By considerations of symmetry (namely, by the automorphisms and by the inner symmetries), we need only consider the following values for the diagonal $(\hat{p}(0), \hat{p}(1), \hat{p}(2))$: $(0, 0, 0)$, $(0, 1, 1)$.

Diagonal $(0, 0, 0)$. Assume p is surjective; there exist applications $(v, [ab])$, $(v', [a'b'])$ such that $p(v, [ab]) = 1$, $p(v', [a'b']) = 2$. In the case that $\{a, b\} = \{a', b'\}$, we have $R(\bar{a}, (v, [ab]), (v', [a'b']))$ but that these applications are, under p , equal to $(0, 1, 2)$, a contradiction. Otherwise, there is one value in $\{a, b\} \cap \{a', b'\}$; suppose this value is $b = b'$. Let c be the value in $N \setminus \{a, b\}$, and c' be the value in

$N \setminus \{a', b'\}$. We claim that $p(v, [ac]) = 1$: if it is 2, we get a contradiction via $R(\bar{b}, (v, [ab]), (v, [ac]))$, and if it is 0, we get a contradiction via $R((v, [ac]), (v, [ab]), (v', [a'b']))$. By analogous reasoning, we obtain that $p(v', [ac']) = 2$. But since $\{a, c\} = \{a', c'\}$, we may reason as in the previous case to obtain a contradiction.

Diagonal $(0, 1, 1)$. Assume p is surjective; there exists an application $(v, [ab])$ such that $p(v, [ab]) = 2$. We have that $\{a, b\} \neq \{0, 1\}$, for if not, we would have a contradiction via $R(\bar{0}, \bar{1}, (v, [ab]))$. Analogously, we have that $\{a, b\} \neq \{0, 2\}$. Thus, we have $\{a, b\} = \{1, 2\}$. Suppose that $[ab]$ is $[12]$ (the case where it is $[21]$ is analogous). Consider the value of $p(v, [02])$: if it is 0, we have a contradiction via $R((v, [02]), \bar{1}, (v, [12]))$, if it is 1, we have a contradiction via $R(\bar{0}, (v, [02]), (v, [12]))$; if it is 2, we have a contradiction via $R(\bar{0}, \bar{2}, (v, [02]))$. \square

Theorem 4.5 *The problem SCSP(\mathbf{N}) is NP-complete.*

Proof. The *not-all-equal* relation $\{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}$ is an F -induced relation of \mathbf{N} , via the definition $(R^{\mathbf{N}}, (u_1, [01]), (u_2, [12]), (u_3, [20]))$. It is well-known that the problem CSP(\cdot) on a structure having this relation is NP-complete via Schaefer's theorem, and thus we obtain the result by Theorem 3.7. \square

4.3 Diagonal-cautious clones

We show how the notion of stability can be used to derive the previous hardness result of the present author [5]. Let $B = \{b_1^*, \dots, b_n^*\}$ be a set of size n . When \mathbf{B} is a structure with universe B , we use \mathbf{B}^* to denote the structure obtained from \mathbf{B} by adding, for each $b_i^* \in B$, a relation $\{(b_i^*)\}$. A set C of operations on B is *diagonal-cautious* if there exists a map $G : B^n \rightarrow \wp(B)$ such that:

- for each operation $f \in C$, it holds that $\text{image}(f) \subseteq G(\hat{f}(b_1^*), \dots, \hat{f}(b_n^*))$, and
- for each tuple $(b_1, \dots, b_n) \in B^n$, if $\{b_1, \dots, b_n\} \neq B$, then $G(b_1, \dots, b_n) \neq B$.

The previous hardness result [5] that we rederive here is that when \mathbf{B} is a structure whose polymorphisms are *diagonal-cautious*, it holds that $\text{CSP}(\mathbf{B}^*)$ reduces to $\text{SCSP}(\mathbf{B})$.

Theorem 4.6 *Suppose that the set of polymorphisms of a relational structure \mathbf{B} is diagonal-cautious, and that the universe B of \mathbf{B} has size $n \geq 2$. There exists an encoding F , whose elements each have arity ≤ 1 , such that:*

- the structure \mathbf{B} is F -stable, and
- there is a surjective mapping $f_x : D \rightarrow B$ in F where, for each relation $Q \subseteq B^k$ of \mathbf{B} , the relation $\bigcup_{(b_1, \dots, b_k) \in Q} (f_x^{-1}(b_1) \times \dots \times f_x^{-1}(b_k))$ is an F -induced relation.

It consequently holds that $\text{CSP}(\mathbf{B}^)$ reduces to $\text{SCSP}(\mathbf{B})$.*

Proof. Suppose that the polymorphisms of \mathbf{B} are diagonal-cautious via $G : B^n \rightarrow \wp(B)$. By Lemma 3.3 of [5], there exists a relation $P \subseteq B^{(n^n)}$ with the following properties:

- (0) P is \wedge -definable.

- (1) For each tuple $(b_1, \dots, b_n, c, d_1, \dots, d_m) \in P$, it holds that each entry of this tuple is in $G(b_1, \dots, b_n)$.
- (2) For each $c \in B$, there exist values $d_1, \dots, d_m \in B$ such that $(b_1^*, \dots, b_n^*, c, d_1, \dots, d_m) \in P$.
- (3) For each tuple $(b_1, \dots, b_n, c, d_1, \dots, d_m) \in P$, there exists a polymorphism $p^+ : B \rightarrow B$ of \mathbf{B} such that $p^+(b_i^*) = b_i$.

We associate the coordinates of P with the variables $(v_1, \dots, v_n, x, y_1, \dots, y_m)$. In the scope of this proof, when z is one of these variables, we use π_z to denote the operator that projects a tuple onto the coordinate corresponding to z . Let P' be the subset of P that contains each tuple $q \in P$ such that, for each $i = 1, \dots, n$, it holds that $\pi_{v_i}(q) = b_i^*$. Let q_1, \dots, q_ℓ be a listing of the tuples in P' . Let $D = \{1, \dots, \ell\}$, and let F contain, for each $z \in \{v_1, \dots, v_n, x, y_1, \dots, y_m\}$, the map $f_z : D \rightarrow B$ defined by $f_z(i) = \pi_z(q_i)$. Observe that f_x is surjective, by (2).

We verify that the structure \mathbf{B} is F -stable, as follows. Let V be a finite non-empty set. Observe that for each $g \in G_{V,D}$ and each $u \in V$, the map $t[g]$ sends the applications $(u, f_{v_1}), \dots, (u, f_{v_n})$ to b_1^*, \dots, b_n^* , respectively; thus, each map in $T_{V,F}$ is surjective. To show that $T_{V,F}$ is surjectively closed, consider a mapping $t' \in \langle T_{V,F} \rangle_{\mathbf{B}}$. Observe first that because for any $u_1, u_2 \in V$ (and any i), (u_1, f_{v_i}) and (u_2, f_{v_i}) are sent to the same value by any map in $T_{V,F}$, the same holds for any map in $\langle T_{V,F} \rangle_{\mathbf{B}}$, and for t' in particular. Let $u \in V$; since P is \wedge -definable and the tuple of applications

$$s^1 = ((u, f_{v_1}), \dots, (u, f_{v_n}), (u, f_x), (u, f_{y_1}), \dots, (u, f_{y_m}))$$

is pointwise mapped by each $t \in T_{V,F}$ to a tuple in P' , it holds that this tuple is sent by t' to a tuple in P .

- Case: Suppose that the restriction of t' to $(u, f_{v_1}), \dots, (u, f_{v_n})$ is not surjective onto B . Then by (1), the restriction of t' to $(u, f_{v_1}), \dots, (u, f_{v_n}), (u, f_x), (u, f_{y_1}), \dots, (u, f_{y_m})$ has image contained in the set $G(t'(u, f_{v_1}), \dots, t'(u, f_{v_n}))$, which is not equal to B by the definition of diagonal-cautious. Since u was chosen arbitrarily, we obtain that t' is not surjective.
- Case: Suppose that the previous case's assumption does not hold. We have that the tuple s^1 is pointwise mapped by t' to a tuple $(b_1, \dots, b_n, c, d_1, \dots, d_m)$ in P . Thus, by (3), there exists a unary polymorphism p^+ such that $p^+(b_i^*) = b_i = t'(u, f_{v_i})$, for each i . By this case's assumption, we have $\{b_1, \dots, b_n\} = B$; thus, the operation p^+ is a bijection, and hence an automorphism of \mathbf{B} . As p^+ and each of its powers is thus a partial polymorphism of P , it follows that s^1 mapped under $(p^+)^{-1}(t')$ is inside P and, due to the choice of p^+ , inside P' . Since u was chosen arbitrarily, we obtain that $(p^+)^{-1}(t')$ is in $T_{V,F}$, and so t' has the desired form.

For any relation $Q \subseteq B^k$ of \mathbf{B} , consider the F -induced relation $Q' \subseteq D^k$ of \mathbf{B} defined by $(Q, (u_1, f_x), \dots, (u_k, f_x))$. We have that Q' is the union of $f_x^{-1}(b_1) \times \dots \times f_x^{-1}(b_k)$ over all tuples $(b_1, \dots, b_k) \in Q$. Moreover, for each $b \in B$, each set $f_x^{-1}(b)$ is an F -induced relation. We may conclude that \mathbf{B}^* , the expansion of \mathbf{B} by all constant relations (those relations $\{(b)\}$, over $b \in B$), has that $\text{CSP}(\mathbf{B}^*)$ reduces to $\text{SCSP}(\mathbf{B})$. \square

As discussed in the previous article [5, Proof of Corollary 3.5], when a structure \mathbf{B} (with non-trivial universe size) has only essentially unary polymorphisms, it holds that the polymorphisms of \mathbf{B} are diagonal-cautious, and that the problem $\text{CSP}(\mathbf{B}^*)$ is NP-complete. With these facts in hand, we obtain the following corollary of the just-given theorem.

Corollary 4.7 *Suppose that \mathbf{B} is a finite structure whose universe B has size strictly greater than 1. If each polymorphism of \mathbf{B} is essentially unary, the problem $\text{CSP}(\mathbf{B}^*)$, and hence the problem $\text{SCSP}(\mathbf{B})$, is NP-complete.*

It is known, under the assumption that P does not equal NP, that the problem $\text{SCSP}(\mathbf{B})$ for any 2-element structure \mathbf{B} is NP-complete if and only if each polymorphism of \mathbf{B} is essentially unary (this follows from [11, Theorem 6.12]). Hence, under this assumption, the hardness result of Corollary 4.7 covers all hardness results for the problems $\text{SCSP}(\mathbf{B})$ over each 2-element structure \mathbf{B} .

In the remainder of this section, we show that the previous hardness result [5] rederived here provably does not apply to the structures \mathbf{C} and \mathbf{N} of Sections 4.1 and 4.2, respectively.

Theorem 4.8 *The set of polymorphisms of the structure \mathbf{C} is not diagonal-cautious.*

Proof. Suppose that the stated set is diagonal-cautious via $G : C^4 \rightarrow \wp(C)$. We consider polymorphisms $f : C^2 \rightarrow C$ of \mathbf{C} such that the tuple of diagonal values $(\hat{f}(0), \hat{f}(1), \hat{f}(2), \hat{f}(3))$ is equal to $(0, 1, 0, 1)$. By the first condition in the definition of diagonal-cautious, it must hold for such a polymorphism that $\text{image}(f) \subseteq G(0, 1, 0, 1)$.

There exists such a polymorphism having 2 in its image, namely, the operation g_2 defined as follows: $g_2(a, b)$ is 2 when $(a, b) = (0, 2)$; $b \bmod 2$ when $a = 2$ or $b = 0$; and, 1 otherwise. Analogously, there exists such a polymorphism having 3 in its image, namely, the operation g_3 defined as follows: $g_3(a, b)$ is 3 when $(a, b) = (1, 3)$; $b \bmod 2$ when $a = 3$ or $b = 1$; and, 0 otherwise.

It follows that $\{0, 1, 2, 3\} = G(0, 1, 0, 1)$, which contradicts the second condition in the definition of diagonal-cautious. \square

Theorem 4.9 *The set of polymorphisms of the structure \mathbf{N} is not diagonal-cautious.*

Proof. Suppose that the stated set is diagonal-cautious via $G : N^3 \rightarrow \wp(N)$. It is straightforward to verify that each non-surjective operation $f : N^2 \rightarrow N$ is a polymorphism of \mathbf{N} (indeed, each non-surjective operation on N is a polymorphism of \mathbf{N}). Consider the non-surjective operations $f : N^2 \rightarrow N$ having diagonal 0, that is, such that $\hat{f}(0) = \hat{f}(1) = \hat{f}(2) = 0$. For each such operation, it holds that $\text{image}(f) \subseteq G(0, 0, 0)$, by the first condition in the definition of diagonal-cautious. But since both 1 and 2 fall into the images of such operations, it holds that $G(0, 0, 0) = \{0, 1, 2\}$. This contradicts the second condition in the definition of diagonal-cautious. \square

5 Establishing stability

In this section, we present a decidability result for stability in the case that F contains only maps of arity at most 1. For each $b \in B$, we use b_0 to denote the mapping from D^0 to B that sends the empty tuple to b ; we use B_0 to denote $\{b_0 \mid b \in B\}$.

Theorem 5.1 *Let \mathbf{B} , D , and F be as described. Suppose that F contains only maps of arity ≤ 1 , and that it contains B_0 as a subset. The structure \mathbf{B} is F -stable if and only if when W has size $\leq |B|$, the set $T_{W,F} = \{t[g] \mid g \in G_{W,D}\}$ is surjectively closed (over \mathbf{B}).*

Proof. The forward direction is immediate, so we prove the backward direction. We need to show that, for each non-empty finite set V , the set $T_{V,F} = \{t[g] \mid g \in G_{V,D}\}$ fulfills the conditions given in Definition 3.4. Each map in $T_{V,F}$ is surjective since $B_0 \subseteq F$. We thus argue that $T_{V,F}$ is surjectively closed. This follows from the assumptions when $|V| \leq |B|$.

When $|V| > |B|$, suppose (for a contradiction) that $T_{V,F}$ is not surjectively closed. There exists a surjective mapping t' in $\langle T_{V,F} \rangle_{\mathbf{B}}$ violating the definition of surjectively closed. For any variable $v \in V$, let A_v be the set of applications that involve either B_0 or the variable v . Let us say that v is *well-formed* if the map t' restricted to A_v has the form $\beta_v(t[h_v])$ where β_v is an automorphism (of \mathbf{B}) and h_v is a map in $G_{\{v\},D}$. We claim that there exists a variable in V that is not well-formed. Suppose, for a contradiction, that each variable $v \in V$ is well-formed; then, the automorphisms β_v over $v \in V$ are equal to each other, since each set A_v of applications includes each application α_b defined as (ϵ, b_0) for each $b \in B$ (where ϵ is the empty tuple), and this application is mapped by t' to the value $t'(\alpha_b)$ which by well-formedness is equal to $\beta_v(\alpha_b[h_v])$ (as α_b begins with the empty tuple, the value $\alpha_b[h_v]$ does not depend on h_v). Let β denote the common value of the automorphisms β_v over $v \in V$. Then, the tuple t' is equal to $\beta(t[h])$, where $h \in G_{V,D}$ is the unique map extending each of the maps $(h_v)_{v \in V}$; this implies that t' does not violate the definition of surjectively closed (for $T_{V,F}$), contradicting the choice of the map t' . The claim follows: there exists a variable $u' \in V$ that is not well-formed.

Since u' is not well-formed, the map t' restricted to $A_{u'}$ does not have the form $\beta'(t[h'])$ for an automorphism β' and a map $h' \in G_{\{u'\},D}$. As $B_0 \subseteq F$, it is possible to pick a set A^- of applications using at most $|B| - 1$ variables such that the restriction of t' to A^- is surjective. Let U be the set of variables appearing in A^- ; we have $|U| \leq |B| - 1$. Set $U' = U \cup \{u'\}$.

It is an immediate consequence of Theorem 2.1 that the restriction of t' to $A_{U',F}$ is contained in $\langle T_{U',F} \rangle_{\mathbf{B}}$, where $T_{U',F} = \{t[g] \mid g \in G_{U',D}\}$. By the choice of u' , we have that the described restriction of t' is not in $\langle T_{U',F} \rangle_{\mathbf{B}}$; since this restriction, by the choice of U , is surjective, we obtain that $T_{U',F}$ is not surjectively closed, a contradiction to the assumption of the backward direction. \square

Theorem 5.2 *There exists an algorithm that, given a structure \mathbf{B} and a set F of maps each having arity ≤ 1 and where $B_0 \subseteq F$, decides whether or not \mathbf{B} is F -stable.*

Proof. The algorithm checks the condition of Theorem 5.1, in particular, it checks whether or not, for all $i = 1, \dots, |B|$, the set $T_{W,F}$ is surjectively closed for a set W having size i . \square

6 Condensations

Define a *condensation* from a relational structure \mathbf{A} to a relational structure \mathbf{B} to be a homomorphism $h : A \rightarrow B$ from \mathbf{A} to \mathbf{B} such that $h(A) = B$ and $h(R^{\mathbf{A}}) = R^{\mathbf{B}}$ for each relation symbol R . That is, a condensation is a homomorphism that maps the universe of the first structure surjectively onto the universe of the second, and in addition, maps each relation of the first structure surjectively onto the corresponding relation of the second structure. When \mathbf{B} is a structure, we define the problem $\text{COND}(\mathbf{B})$ of deciding whether or not a given structure admits a condensation to \mathbf{B} , using the formulation of the present paper, as follows. The instances of $\text{COND}(\mathbf{B})$ are the instances of the problem $\text{SCSP}(\mathbf{B})$; an instance (U, ϕ) is a *yes* instance if there exists a surjective map $f : U \rightarrow B$ that satisfies ϕ (over \mathbf{B}) such that, for each relation symbol R and each tuple $(b_1, \dots, b_k) \in R^{\mathbf{B}}$,

there exists an atom $R(u_1, \dots, u_k)$ of ϕ such that $(f(u_1), \dots, f(u_k)) = (b_1, \dots, b_k)$. Clearly, each *yes* instance of $\text{COND}(\mathbf{B})$ is also a *yes* instance of $\text{SCSP}(\mathbf{B})$. In graph-theoretic settings, a related notion has been studied: a *compaction* is defined similarly, but typically is not required to map onto self-loops of the target graph \mathbf{B} .

There exists an elementary argument that, for each structure \mathbf{B} , the problem $\text{SCSP}(\mathbf{B})$ polynomial-time Turing reduces to the problem $\text{COND}(\mathbf{B})$; see [1, Proof of Proposition 1].⁴ We observe here a condition that allows one to show NP-hardness for the problem $\text{COND}(\mathbf{B})$, under polynomial-time many-one reduction, using the developed framework.

Theorem 6.1 *Let F be an encoding that contains all constants in the sense that, for each $b \in B$, there exists a map $f_b \in F$ such that $f_b(e) = b$ for each e in the domain of f_b . Suppose that \mathbf{B} is F -stable, and that \mathbf{D} is an F -induced template of \mathbf{B} . Then, the problem $\text{CSP}(\mathbf{D})$ polynomial-time many-one reduces to $\text{COND}(\mathbf{B})$.*

Proof. The reduction is that of Theorem 3.7. We use the notation from that proof. It suffices to argue that when $(A_{V,F}, \psi)$ is a *yes* instance of $\text{SCSP}(\mathbf{B})$, it is also a *yes* instance of $\text{COND}(\mathbf{B})$. From that proof, when there exists a surjective satisfying assignment of ψ , there exists such a surjective satisfying assignment of the form $t[g]$. Since $t[g]$ maps each application of the form (\cdot, f_b) to b , we obtain that $(A_{V,F}, \psi)$ is a *yes* instance of $\text{COND}(\mathbf{B})$. \square

Each encoding presented in Section 4 contains all constants; thus, via Theorem 6.1, we obtain that, for each structure \mathbf{B} treated in that section, the problem $\text{COND}(\mathbf{B})$ is NP-complete.

7 Sparsifiability

The framework presented here also has implications for the sparsifiability of the problems $\text{SCSP}(\mathbf{B})$. We explain the main ideas here; our presentation is based on the framework and terminology of [8].

For a problem Q of the form $\text{SCSP}(\mathbf{B})$ or of the form $\text{CSP}(\mathbf{B})$, we define a *generalized kernel* as a polynomial-time many-one reduction f from Q to any other decision problem Q' . We say that a generalized kernel f has *bitsize* $h : \mathbb{N} \rightarrow \mathbb{N}$ when, for any instance x of Q having n variables, it holds that $|f(x)| \leq h(n)$. That is, we grade each generalized kernel according to the image size of each instance of Q , measured as a function of the number of variables of Q .

When one has an encoding F of maps each having arity ≤ 1 , and the hypotheses of Theorem 3.7 hold, this theorem's reduction from $\text{CSP}(\mathbf{D})$ to $\text{SCSP}(\mathbf{B})$ increases the number of variables at most linearly: for each instance of the first problem having $n \geq 1$ variables, the instance's image under the reduction has at most $|F|n$ variables. This reduction thus constitutes a *linear-parameter transformation*, in the parlance of [8, Definition 2.2]. Such transformations allow one to transfer lower bounds from the first problem (in our case, $\text{CSP}(\mathbf{D})$) to the second problem (in our case, $\text{SCSP}(\mathbf{B})$).

We establish the following new result.

Theorem 7.1 *The problem $\text{SCSP}(\mathbf{N})$ (studied in Section 4.2) has a generalized kernel of bitsize $O(n^2 \log n)$, but no generalized kernel of bitsize $O(n^{2-\epsilon})$, for any $\epsilon > 0$, under the assumption that NP is not in coNP/poly .*

⁴The proof there concerns the compaction problem, but is immediately adapted to the present setting.

Proof. Let \mathbf{B}_{NAE} be a structure having as its sole relation the not-all-equal relation of Theorem 4.5. The problem $\text{CSP}(\mathbf{B}_{\text{NAE}})$ does not have a generalized kernel of bitsize $O(n^{2-\epsilon})$ (for any $\epsilon > 0$), under the assumption that NP is not in coNP/poly; this follows from Theorem 3.10 of [8]. From the linear-parameter transformation from $\text{CSP}(\mathbf{B}_{\text{NAE}})$ to $\text{SCSP}(\mathbf{N})$, one obtains the lower bound of the theorem statement.

It remains to argue that the problem $\text{SCSP}(\mathbf{N})$ has a generalized kernel of bitsize $O(n^2 \log n)$. To show this, we use an adaptation of the polynomial method pioneered by Jansen and Pieterse [17]; see also [8]. Consider an instance (V, ϕ) of the problem $\text{SCSP}(\mathbf{N})$. Each constraint has the form $R(v_1, v_2, v_3)$, where $v_1, v_2, v_3 \in V$; for each constraint, we create an equation $e_{\{v_1, v_2\}} + e_{\{v_1, v_3\}} + e_{\{v_2, v_3\}} - 1 = 0$, to be interpreted over F_2 , the field with 2 elements. Here, the variables used in equations are those in the set $\{e_{\{v, v'\}} \mid v, v' \in V, v \neq v'\}$, which is a set of $\binom{|V|}{2}$ variables. For any assignment $f : V \rightarrow \{0, 1, 2\}$, set the variable $e_{\{v, v'\}}$ to be 1 if $f(v) = f(v')$, and to be 0 otherwise; then, the assignment satisfies $R(v_1, v_2, v_3)$ if and only if its corresponding equation is true over F_2 . The solution space of the created equations thus capture the set of satisfying assignments. However, since there are $O(n^2)$ variables used in these equations, by linear algebra, a polynomial-time algorithm may find a subset of these equations having size $O(n^2)$ that has the same solution space as the original set. The generalized kernel outputs the constraints corresponding to these equations; each constraint can be written in $O(\log n)$ space, and so the claimed upper bound follows. \square

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