Spectral Analysis of Multi-variable Hankel Operators

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In memory of my father, Paschalis Tantalakis (1940-2015)

Contents

Abstract

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $a = \{a(j)\}_{j \in \mathbb{N}_0}$ be a complex valued sequence. A one-dimensional Hankel operator H_a is understood as an infinite matrix $H_a = [a(i+j)]_{i,j \geq 0}$ that acts on the elements of $\ell^2(\mathbb{N}_0)$. In the same spirit, for any $d \geq 2$, a d-dimensional Hankel operator H_a can be realised as the action of the d-dimensional matrix $H_a = [a(i+j)]_{i,j \in \mathbb{N}_0^d}$ on $\ell^2(\mathbb{N}_0^d)$, where $\mathbb{N}_0^d = \{(j_1, j_2, \dots, j_d) : \; j_i \in \mathbb{N}_0, \; i = 1, 2, \dots, d\}$. Let $\kappa \in \mathbb{R}_+^d$, where $\mathbb{R}_+^d = \{(x_1, x_2, \dots, x_d) : \; x_i > 0, \; i = 1, 2, \dots, d\}$ and denote by "$\cdot$" the usual inner product on $\mathbb{R}^d$. For $\gamma > 0$, we define a special class of d-dimensional compact Hankel matrices $H_a = [a(\kappa \cdot (i + j))]_{i,j \in \mathbb{N}_0^d}$, where $a : \mathbb{R}_+ \to \mathbb{R}$ is a one-dimensional function such that $a(j) \sim j^{-d}(\log j)^{-\gamma}$, when $j \to +\infty$. Then the positive and negative eigenvalues of H_a present power asymptotics of the form $\lambda_n^{\pm}(H_a) \sim C^{\pm} n^{-\gamma}$, when $n \to +\infty$. We also construct analogous results for d-variable integral Hankel operators that act on $L^2(\mathbb{R}_+^d)$. This work generalises earlier one-variable results by A. Pushnitski and D.Yafaev.

Preface

The purpose of this thesis is to briefly describe some recent results on the spectral theory of one-dimensional Hankel operators and ensuing, study some aspects of their multidimensional version. The main part of the thesis consists of constructing and presenting multidimensional analogues of the aforementioned one-dimensional spectral theory.

The theory of Hankel operators is a well studied branch of Mathematics that lies at the overlap of many fields such as complex analysis and operator theory. The first formulation of what is known today as a *Hankel matrix* was constructed by Herman Hankel in 1861 ([14]). More precisely, he constructed finite matrices $[a_{ij}]_{i,j=0}^{n}$, where the entries a_{ij} depend strictly on the sum of the i-th row and the j-th column, i.e. $a_{ij} = a_{i+j}$. The purpose of this construction was the investigation of determinants of such matrices, that today are known as *Hankel determinants*. The Hankel determinants appear in problems of moments (e.g. Hamburger and Stieltjes moment problems; for a comprehensive introduction see [29]). Besides, the Hamburger and Stieltjes moment problems and thus, Hankel determinants too, are interlinked with the theory of orthogonal polynomials. More precisely, the existence of a sequence of orthogonal polynomials is closely related to the positive definiteness of certain finite Hankel matrices; for more on this subject we refer to [6]. One of the first people who made the transition from finite Hankel matrices to infinite ones was Leopold Kronecker, who proved that the finite rank Hankel matrices are these whose entries are Taylor coefficients of rational functions (see [21, §1.3]). Many years later (1957), Zeev Nehari linked again the Taylor coefficients (of bounded function on the complex unit circle) with infinite Hankel matrices. More precisely, he gave a boundedness characterisation for Hankel matrices defined on the space of square summable sequences ([18]). Since then, many contributions have been made to the theory of Hankel operators with applications ranging from purely theoretical branches of mathematical analysis to more applied fields such as control theory and probabilities. For an integrated approach to the theory of Hankel operators, we refer to [19], [20] and [21].

We start our presentation with the small introductory Chapter 0. There the reader can find a brief description of our survey together with some results of A. Pushnitski and D. Yafaev (cf. [23] and [24]) that served as motivation for this work. By clarifying the motivation that prompts our multi-dimensional analysis, we gradually develop the articulation of the main results and explain the most important techniques of our proofs. Finally, Chapter 0 includes a quick presentation of the thesis' structure, where the reader may find a short synopsis of the next chapters.

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Chapter 0

Introduction

0.1 Background theory

0.1.1 One-dimensional operators

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and consider an arbitrary complex valued sequence $a = \{a(j)\}_{j\in\mathbb{N}_0}$. Then a Hankel operator H_a is formally defined as the infinite matrix $H_a = [a(i+j)]_{i,j\geq 0}$, whose action on the space of square summable sequences $\ell^2(\mathbb{N}_0)$ is described by

$$\big(H_a x\big)(i) = \sum_{j\in\mathbb{N}_0} a(i+j)x(j),\ \forall i \in \mathbb{N}_0,\ \forall x = \{x(j)\}_{j\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0).$$

In complete analogy, a function $\mathbf{a} : \mathbb{R}_+ \to \mathbb{C}$, where $\mathbb{R}_+ := (0, +\infty)$, formally defines an integral Hankel operator $\mathbf{H_a}$, described by

$$\big(\mathbf{H_a} f\big)(x) = \int_0^{+\infty} \mathbf{a}(x+y)f(y)\,\mathrm{d}y\,,\ \forall x \in \mathbb{R}_+,\ \forall f \in L^2(\mathbb{R}_+).$$

The sequence a and the function $\mathbf{a}$ will be called the *kernel* of the Hankel operator H_a and $\mathbf{H_a}$, respectively. From now on, we denote with boldface letters the integral Hankel operators and their kernels, so that we distinguish them from the discrete case.

In the context of this thesis we mostly deal with bounded, compact Hankel operators. To begin with, let us focus on the case of Hankel matrices H_a. The boundedness can be obtained if $a(j) = O(j^{-1})$, when $j \to +\infty$ ([33, Theorem 3.1]), and compactness if $a(j) = o(j^{-1})$, for $j \to +\infty$ ([33, Theorem 3.2]). In fact, the cited theorems prove that these conditions are also necessary when H_a is positive. It is essential to notice that a boundedness-compactness threshold, in terms of the rate of decay of a, is the exponent -1. In order to illustrate this more, we consider the following special class of kernels (for discrete Hankel operators):

$$a(j) = \frac{1}{(j+1)^\beta},\ \forall j \in \mathbb{N}_0,\quad \text{for}\ \ \beta > 0. \tag{1}$$

Then it can be verified that when $\beta \in (0, 1)$, the respective Hankel operator H_a is unbounded. For $\beta \geq 1$, H_a will be bounded and, for $\beta > 1$, will be compact. As a typical paradigm of the importance of the exponent $\beta = 1$, we mention the Hilbert matrix $\mathcal{H} = [\frac{1}{i+j+1}]_{i,j\geq 0}$, which is a bounded, non-compact Hankel operator.

H. Widom in [33] derived eigenvalue asymptotics for a special class of compact, positive Hankel matrices. More precisely, by considering sequences of the form (1), for $\beta > 1$, he proved that the eigenvalues of H_a, $\lambda_n(H_a)$, decay stretched exponentially:

$$\lambda_n(H_a) = \exp\left(-\pi\sqrt{2\beta n} + o(\sqrt{n})\right), \ n \to +\infty.$$

After the contributions on the subject by K. Glover, J. Lam, and J. R. Partington who constructed a class of Hankel operators with power symptotics (cf. [10]), A. Pushnitski and D. Yafaev noticed that there exists another whole class of compact Hankel operators that gives power asymptotics, too. More precisely, they started with a class of Hankel matrices which lies between the Hilbert matrix and the class that H. Widom considered. In fact, in [23] and [24] they introduced Hankel matrices $H_a = [a(i+j)]_{i,j\geq 0}$, where

$$a(j) = \frac{1}{(j+1)\left(\log(j+2)\right)^{\gamma}}, \ \forall j \in \mathbb{N}, \quad \text{for} \quad \gamma > 0.$$

Furthermore, in [23] they derived explicit eigenvalue asymptotics for this class of compact Hankel operators. More precisely, they showed that the positive, $\lambda_n^+(H_a)$, and the negative, $\lambda_n^-(H_a)$, eigenvalues of H_a obey the asymptotic formula below:

$$\lambda_n^+(H_a) = 2^{-\gamma} C_{1,\gamma} n^{-\gamma} + o(n^{-\gamma}) \ \text{ and } \ \lambda_n^-(H_a) = o(n^{-\gamma}), \ \text{ as } \ n \to +\infty, \tag{2}$$

where

$$C_{1,\gamma} = \left[\frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{\pi\cosh(\pi x)}\right)^{\frac{1}{\gamma}} \mathrm{d}x\right]^{\gamma} = \pi^{1-2\gamma} B\left(\frac{1}{2\gamma}, \frac{1}{2}\right)^{\gamma} \tag{3}$$

and $B(\cdot,\cdot)$ is the Beta function. Their respective result for the continuous case states that if $\mathbf{H_a} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is an integral Hankel operator with kernel

$$\mathbf{a}(x) = x^{-1}\left(1 + (\log x)^2\right)^{-\frac{\gamma}{2}}, \ \forall x > 0 \ \text{ and for some } \ \gamma > 0,$$

then the eigenvalue asymptotics for $\mathbf{H_a}$ are given by

$$\lambda_n^+(\mathbf{H_a}) = C_{1,\gamma} n^{-\gamma} + o(n^{-\gamma}) \ \text{ and } \ \lambda_n^-(\mathbf{H_a}) = o(n^{-\gamma}), \ \text{ as } \ n \to +\infty, \tag{4}$$

where the constant $C_{1,\gamma}$ is given in (3).

0.1.2 Multidimensional operators

For the discrete case, let $a : \mathbb{N}_0^d \to \mathbb{C}$ be a d-dimensional sequence. Then, the action of the corresponding Hankel operator on $\ell^2(\mathbb{N}_0^d)$ is formally described by

$$(H_a x)(i) = \sum_{j\in\mathbb{N}_0^d} a(i+j)x(j), \ \forall i \in \mathbb{N}_0^d, \ \forall x = \{x(j)\}_{j\in\mathbb{N}_0^d} \in \ell^2(\mathbb{N}_0^d). \tag{5}$$

Respectively, a function $\mathbf{a} : \mathbb{R}_+^d \to \mathbb{C}$, defines formally an integral Hankel operator $\mathbf{H_a}$, whose action on $L^2(\mathbb{R}_+^d)$ is described by

$$(\mathbf{H_a} f)(\mathbf{x}) = \int_{\mathbb{R}_+^d} \mathbf{a}(\mathbf{x}+\mathbf{y}) f(\mathbf{y})\,\mathrm{d}\mathbf{y}, \ \forall \mathbf{x} \in \mathbb{R}_+^d, \ \forall f \in L^2(\mathbb{R}_+^d). \tag{6}$$

Next we present the class of Hankel operators that we deal with throughout our survey. For let κ be an arbitrary constant in $\mathbb{R}_+^d$ and $a_0 : \mathbb{R}_+ \to \mathbb{R}$ be an arbitrary function. For the discrete case, we define kernels $a = \{a(j)\}_{j\in\mathbb{N}_0^d}$ of the type

$$a(j) = a_0(\kappa \cdot j), \ \forall j \in \mathbb{N}_0^d, \tag{7}$$

where "$\cdot$" is the usual inner product on $\mathbb{R}^d$. In the same spirit, the kernels $\mathbf{a} : \mathbb{R}_+^d \to \mathbb{R}$ of the continuous case are described by

$$\mathbf{a}(\mathbf{x}) = a_0(\kappa \cdot \mathbf{x}), \ \forall \mathbf{x} \in \mathbb{R}_+^d. \tag{8}$$

Relations (7) and (8) define a sufficiently rich and yet amenable to existing methods class of Hankel operators.

Now let $\beta > 0$,

$$a_0(x) = \frac{1}{(x+1)^\beta}, \ \forall x \in \mathbb{R}_+,$$

and a be as it is defined in (7). Then it can be verified that if $\beta \in (0, d)$, the generated Hankel operator H_a is unbounded. For $\beta \geq d$, H_a is bounded and, for $\beta > d$, compact. Thus, the threshold exponent that distinguishes boundedness from compactness is $\beta = d$. For some examples of bounded, non-compact Hankel operators of this type, we refer to §2.1.2.

0.2 Main results

The main results of our study comprise Theorems 3.3, 3.5 and 3.8. Here we display a more simplified version of these two results; see Theorems 0.1 and 0.2.

For choose an arbitrary $\gamma > 0$ and define the function $a_0 : \mathbb{R}_+ \to \mathbb{R}$, given by

$$a_0(t) = t^{-d}\left(1 + (\log t)^2\right)^{-\frac{\gamma}{2}}, \ \forall t > 0. \tag{9}$$

It is proved that for this class of operators we can obtain a multi-dimensional analogue of the asymptotics (2) and (4), for the discrete and the continuous case, respectively. In fact, we have the following theorems, for the discrete and the continuous case, respectively:

Theorem 0.1. *Let $\gamma > 0$ and $\kappa \in \mathbb{R}_+^d$. If a_0 is the function that is defined in (9), then the Hankel operator H_a, which is defined in (5), where $a(j) = a_0(\kappa \cdot j)$, $\forall j \in \mathbb{N}_0^d$, is compact and its eigenvalue asymptotics are given by*

$$\lambda_n^+(H_a) = \frac{C_{d,\gamma}}{\kappa_1\kappa_2\ldots\kappa_d} n^{-\gamma} + o(n^{-\gamma}) \ \text{ and } \ \lambda_n^-(H_a) = o(n^{-\gamma}), \ \text{ as } \ n \to +\infty, \tag{10}$$

where

$$C_{d,\gamma} = \frac{1}{2^d(d-1)!}\left(\int_{\mathbb{R}} \left[\left(\mathcal{F}^{-1}k_d\right)(x)\right]^{\frac{1}{\gamma}} \mathrm{d}x\right)^\gamma,$$

and $k_d(x) := \left(\cosh\left(\frac{x}{2}\right)\right)^{-d}$, for all $x \in \mathbb{R}$.

Notice that $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Theorem 0.2. *Let $\kappa \in \mathbb{R}_+^d$, $\gamma > 0$ and a_0 be the function that is defined in (9). Then the integral Hankel operator $\mathbf{H_a}$, defined by (6), for $\mathbf{a}(\mathbf{x}) = a_0(\kappa \cdot \mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}_+^d$, is a compact operator on $L^2(\mathbb{R}_+^d)$ and its eigenvalues obey the asymptotic law below:*

$$\lambda_n^+(H_a) = \frac{2^\gamma C_{d,\gamma}}{\kappa_1 \kappa_2 \ldots \kappa_d} n^{-\gamma} + o(n^{-\gamma}) \quad \text{and} \quad \lambda_n^-(H_a) = o(n^{-\gamma}), \quad \text{as} \quad n \to +\infty, \tag{11}$$

where the constant $C_{d,\gamma}$ is the same as in Theorem 0.1.

In Theorems 3.5 and 3.8 we obtain asymptotics similar to (10) and (11), for the discrete and the continuous case, respectively, for a more general class of functions a_0. More precisely, in the continuous case, we can assume that a_0 does not present the same behaviour at 0 and $+\infty$. Moreover, we can even perturb by an error term, under some smoothness conditions for it, and still obtain power asymptotics. The above generalisations of a_0 could be summarised as follows:

$$a_0(t) = b_0 t^{-d} \left(1 + (\log t)^2\right)^{-\frac{\gamma}{2}} + g_0(t), \quad \text{when } t \to 0^+,$$

and

$$a_0(t) = b_\infty t^{-d} \left(1 + (\log t)^2\right)^{-\frac{\gamma}{2}} + g_\infty(t), \quad \text{when } t \to +\infty,$$

for some positive constant b_0 and b_∞. As we have already mentioned, the error terms g_0 and g_∞ should satisfy certain smoothness conditions.

0.3 Key points of the proof

In contrast with the one-dimensional case, where the proving machinery is more or less the same for both the discrete and the continuous case, in the multi-dimensional one there is an essential underlying difficulty. In order to illustrate the situation, we mention that by a change of variables one can pass, in Theorem 0.2, from the general continuous case, namely from an arbitrary $\kappa \in \mathbb{R}_+^d$, to the special case of $\kappa = (1, 1, \ldots, 1)$. Unfortunately, the change of variables is obviously not possible in the discrete case. This obstacle is overcome by a series of some technical lemmas, that constitute a major part of our work. Despite these difficulties, eventually, we are able to obtain power spectral asymptotics for both the discrete and the continuous case. In both the two cases, the dependence of the leading term coefficient on κ has the same structure.

Let us cast some light on the main proof techniques that lead to the eigenvalue asymptotics. For sake of simplicity, we adopt the continuous case notation, but the main ideas below are applicable to the discrete case, too. Let us display the four principal techniques that are encountered:

- The construction of a model operator,
- reduction of the model operator to pseudo-differential operators,
- reduction to one-dimensional, weighted Hankel operators, and
- Schatten class inclusions.

The construction of the model operator (see §4.3.2) aims to give the leading term in the eigenvalue asymptotics. More precisely, the model operator will be a Hankel operator which behaves "similarly" to the initial Hankel operator but whose eigenvalue asymptotics are retrieved much easier. So let $\tilde{\mathbf{H}} := \mathbf{H}_{\tilde{\mathbf{a}}}$ denote the model operator, which is described by (6), for $\tilde{\mathbf{a}}(\mathbf{x}) = \tilde{\mathbf{a}}_{\mathbf{0}}(\kappa \cdot \mathbf{x})$, for every $\mathbf{x} \in \mathbb{R}_+^d$. Here $\tilde{\mathbf{a}}_{\mathbf{0}}$ will be the Laplace transform of a particular function σ. More precisely,

$$\tilde{\mathbf{a}}_{\mathbf{0}}(t) = \int_0^{+\infty} \sigma(\lambda) e^{-\lambda t} \, \mathrm{d}\lambda \,, \ \forall t > 0,$$

where σ is a smooth function, roughly described as follows:

$$\sigma(\lambda) = \begin{cases} \frac{1}{(d-1)!} \lambda^{d-1} \left(\log \frac{1}{\lambda}\right)^{-\gamma}, & \lambda \to 0^+ \\ \frac{1}{(d-1)!} \lambda^{d-1} \left(\log \lambda\right)^{-\gamma}, & \lambda \to +\infty \end{cases}.$$

By using Laplace transform asymptotics, we conclude that $\tilde{\mathbf{a}}_{\mathbf{0}}$ behaves similarly to a_0, as $t \to +\infty$ or 0. The latter yields the aforementioned "similarity" of $\mathbf{H}_{\mathbf{a}}$ and $\tilde{\mathbf{H}}$. For the discrete case, the model operator for $\kappa = (1, 1, \ldots, 1)$ is constructed in §4.2.1.2, and for general κ in §4.2.2.2.

The derivation of the eigenvalue asymptotics of the model operator is attained after a reduction to pseudo-differential operators; the same technique was also applied in [23] for the one-dimensional results. The key idea for this reduction is to express the model operator as a product of two operators, L^*L. More precisely, L maps $L^2(\mathbb{R}_+^d)$ to $L^2(\mathbb{R}_+)$ and is given by

$$\big(Lf\big)(\lambda) = \sqrt{\sigma(\lambda)} \int\limits_{\mathbb{R}_+} \cdots \int\limits_{\mathbb{R}_+} e^{-\lambda \sum_{i=1}^d x_i} f(x_1, \ldots, x_d) \, \mathrm{d}x_1 \ldots \mathrm{d}x_d \,, \ \forall f \in L^2(\mathbb{R}_+^d), \ \forall \lambda > 0.$$

Regarding its adjoint, L^* maps $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R}_+^d)$ and is given by

$$\big(L^*f\big)(x_1, \ldots, x_d) = \int_0^{+\infty} \sqrt{\sigma(\lambda)} f(\lambda) e^{-\lambda \sum_{i=1}^d x_i} \, \mathrm{d}\lambda \,, \ \forall f \in L^2(\mathbb{R}_+), \ \forall (x_1, \ldots, x_d) \in \mathbb{R}_+^d.$$

Then, by exploiting the unitary equivalence of the non-zero parts of L^*L and LL^*, we prove that the spectral investigation for $\tilde{\mathbf{H}}$ is equivalent to that one for LL^*; note that this technique was applied in the approach that H. Widom made in [33], too. In the sequel, we prove that LL^* is unitarily equivalent to a pseudo-differential operator, whose eigenvalue asymptotics are obtained by a Weyl-type formula.

Finally, the initial Hankel operator, $\mathbf{H}_{\mathbf{a}}$, can be expressed as a sum of operators, $\mathbf{H}_{\mathbf{a}} = \tilde{\mathbf{H}} + (\mathbf{H}_{\mathbf{a}} - \tilde{\mathbf{H}})$. Since we obtain the eigenvalue asymptotics for $\tilde{\mathbf{H}}$, we aim to prove that the spectral contribution of the operator $\mathbf{H}_{\mathbf{a}} - \tilde{\mathbf{H}}$ is negligible, compared to that one of $\tilde{\mathbf{H}}$. This will be achieved by proving certain Schatten-Lorentz class inclusions for $\mathbf{H}_{\mathbf{a}} - \tilde{\mathbf{H}}$. These inclusions depend on the range of the exponent γ in (9) and are obtained by a combination of interpolation and reduction to one-dimensional weighted Hankel operators. This reduction is the part of the proof in the discrete case that hides the most of the technicalities, so it is only applied for the special case of $\kappa = (1, 1, \ldots, 1)$.

Finally, in order to clarify this last step, we provide its key points for the continuous case, since the reduction in the discrete case is developed in a similar way. Moreover, notice that we also assume that $\kappa = (1, 1, \ldots, 1)$. This is possible, since, as we have already explained, it takes just a simple change of variables to reduce to this case. Observe that the action on $L^2(\mathbb{R}_+^d)$ of the integral Hankel operators $\mathbf{H_a}$, which we deal with, is described by

$$(\mathbf{H_a}f, g) = \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} \mathbf{a_0}\left(\sum_{i=1}^d (x_i + y_i)\right) f(y_1, \ldots, y_d)\overline{g(x_1, \ldots, x_d)}\, \mathrm{d}x_1 \ldots \mathrm{d}x_d\, \mathrm{d}y_1 \ldots \mathrm{d}y_d\,,$$

for any $L^2(\mathbb{R}_+^d)$ functions f and g. Besides, remember that we aim to reduce the expression above to something that resembles a one-dimensional Hankel operator. To this end, it is enough to notice that

$$\mathbf{a_0}\left(\sum_{i=1}^d (x_i + y_i)\right) = \mathbf{a_0}\left(\sum_{i=1}^d x_i + \sum_{i=1}^d y_i\right).$$

The latter prompts the change of variables $x = \sum_{i=1}^d x_i$ and $y = \sum_{i=1}^d y_i$, so that x and y lie in the one-dimensional space $\mathbb{R}_+$. By doing the algebra, we get that

$$(\mathbf{H_a}f, g) = (J^*\mathbf{\Gamma_0}Jf, g),\ \forall f, g \in L^2(\mathbb{R}_+^d),$$

where J is a partial isometry from $L^2(\mathbb{R}_+^d)$ to $L^2(\mathbb{R}_+)$, given by

$$(Jf)(x) = \sqrt{(d-1)!}\,x^{-\frac{d-1}{2}} \int_0^x \cdots \int_0^{x - \sum_{i=1}^{d-2} x_i} f\big(x_1, \ldots, x_{d-1}, x - \sum_{i=1}^{d-1} x_i\big)\, \mathrm{d}x_1 \ldots \mathrm{d}x_{d-1}\,,$$

for every $x \in \mathbb{R}_+$, and $\mathbf{\Gamma_0} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is the weighted Hankel operator defined by

$$\big(\mathbf{\Gamma_0}f\big)(x) = (d-1)! \int_0^{+\infty} x^{\frac{d-1}{2}}\mathbf{a_0}(x+y)y^{\frac{d-1}{2}} f(y)\, \mathrm{d}y\,,\ \forall x \in \mathbb{R}_+,\ \forall f \in L^2(\mathbb{R}_+).$$

So that we see that $\mathbf{H_a}$ and $\mathbf{\Gamma_0}$ are unitarily equivalent (modulo null-spaces). For a detailed presentation of the subject, we refer to §2.1.1.

0.4 Structure of the text

For the reader's convenience, we conclude this introductory chapter by giving the structure of the thesis. The thesis is divided into two main parts; **Part I - One-dimensional Hankel operators** and **Part II - Multidimensional Hankel operators**.

Part I is a brief introduction to the one-dimensional theory of Hankel operators, in order to familiarise the reader with the basic ideas. In paragraphs §1.1 and §1.2 are defined the Hankel operators for the discrete and the continuous case, respectively. Moreover, we present some boundedness-compactness conditions for both of the two cases. In section §1.3 we introduce the concept of weighted Hankel operators. In section §1.4 we introduce the notions of Schatten classes, as well as, of some other compact operator ideals. We also present the main ideas around Besov classes. This class of functions plays an important role in the theory of Hankel operators, since it gives extremely useful necessary and sufficient conditions for Schatten class inclusions. Many of these results are due to V. Peller and can be found in [21]. Finally, in the last section of this part, the reader will find some examples of one-dimensional Hankel operators that aim to make the transition to the multidimensional case smoother and more reasonable.

Part II is devoted, as its title reveals, to the multidimensional analysis of Hankel operators and constitutes the main part of our work. It starts with the introductory Chapter 2. There the reader will find the necessary definitions of discrete and integral multidimensional Hankel operators. In parallel with paragraph §1.5, in Section 2.1 we present the Hankel operators that constitute our main subject of research. In §2.1.1 we provide a concrete presentation of the reduction to one-dimensional, weighted Hankel operators, that was mentioned before, in the proving methods. The first chapter is concluded by the construction of some examples of bounded, non-compact Hankel operators, that aim to enlighten the importance of the dimension, in terms of compactness.

In Chapter 3 follows a presentation of the main results. Paragraph §3.1 is referred to the discrete case and §3.2 to the continuous.

Chapter 4 comprises all the proofs of the main results. It is divided into two main sections; Section 4.2 and Section 4.3, that consist of the proofs for the discrete and the continuous case, respectively. Moreover, because of the difficulties that occur in the discrete case, we furthermore split this section into two sub-sections, 4.2.1 and 4.2.2, for the case of $\kappa = (1, 1, \dots, 1)$ and for arbitrary $\kappa \in \mathbb{R}^d_+$, respectively.

Finally, Chapter 5 is the last one and complements the spectral analysis of the examined operators. Since the biggest part of the thesis is devoted into finding (non-zero) eigenvalue asymptotics for Hankel operators, the last chapter deals with the "zero" eigenvalues or better, with the null-spaces of Hankel operators.

In the **Appendices** the reader will find all the necessary theory that lies beyond the scope of the thesis but it is highly interlinked with the proving methods that we use.

Part I

One-dimensional Hankel operators

Chapter 1

One-dimensional Hankel operators

1.1 Hankel matrices

Let J be a countable set and A be an arbitrary set. We define the space of A-valued sequences $\{x_j\}_{j\in J}$ as the space of functions

$$A^J := \{x : J \to A \ \text{ s.t. } \ x_j = x(j),\ \forall j \in J\}.$$

Let $\mathbb{N}$ denote the set of natural numbers (i.e. $\mathbb{N} = \{1, 2, \dots\}$) and define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We recall that for any $p > 0$, the space of p-summable sequences over $\mathbb{N}_0$ is defined as

$$\ell^p(\mathbb{N}_0) := \left\{ x \in \mathbb{C}^{\mathbb{N}_0} : \ \|x\|_p := \left(\sum_{n\in\mathbb{N}_0} |x(n)|^p \right)^{\frac{1}{p}} < +\infty \right\}.$$

For $p = \infty$, we define the space of bounded sequences

$$\ell^\infty(\mathbb{N}_0) := \left\{ x \in \mathbb{C}^{\mathbb{N}_0} : \ \|x\|_\infty := \sup_{n\in\mathbb{N}_0} |x(n)| < +\infty \right\}.$$

Moreover, $\ell^2(\mathbb{N}_0)$ is a Hilbert space with inner product

$$(x, y) := \sum_{n\in\mathbb{N}_0} x(n)\overline{y(n)},\ \forall x, y \in \ell^2(\mathbb{N}_0).$$

Furthermore, we can define weighted ℓ^p spaces as follows. Let $\upsilon = \{\upsilon(n)\}_{n\in\mathbb{N}_0}$ be a (positive valued) sequence and, for any $p \in (0, +\infty)$, define the spaces

$$\ell^p_\upsilon(\mathbb{N}_0) := \left\{ x \in \mathbb{C}^{\mathbb{N}_0} : \ \|x\|_{\ell^p_\upsilon} := \left(\sum_{n\in\mathbb{N}_0} |x(n)|^p \,\upsilon(n) \right)^{\frac{1}{p}} < +\infty \right\}.$$

Ensuing, we proceed with the definition of a *Hankel matrix* and subsequently, that one of bounded a Hankel operator (matrix) on $\ell^2(\mathbb{N}_0)$. Let $a = \{a(n)\}_{n\in\mathbb{N}_0}$ be a complex valued sequence and define the infinite matrix

$$H_a = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Observe that H_a presents a special kind of symmetry, with respect to its main diagonal. This happens because every (i,j)-entry depends on the sum $i+j$. Thus, H_a could be also denoted as $H_a = [a(i+j)]_{i,j\geq 0}$. Now let $a \in \ell^2(\mathbb{N}_0)$ and H_a act on $\ell^2(\mathbb{N}_0)$ via the following action:

$$\big(H_a x\big)(i) := \sum_{j\in\mathbb{N}_0} a(i+j)x(j),\ \forall i \in \mathbb{N}_0,\ \forall x \in \ell^2(\mathbb{N}_0). \tag{1.1}$$

Then (1.1) defines a *Hankel operator* and the sequence a is called *kernel* of H_a. Notice that this definition makes sense, since H_a is well-defined on the dense subset of eventually vanishing sequences. Indeed, if $x \in \mathbb{C}^{\mathbb{N}_0}$, such that $x(n) = 0$, for every $n > n_0$, where n_0 is an arbitrary natural number, then x is obviously in $\ell^2(\mathbb{N}_0)$ and

$$\begin{aligned}
\|H_a x\|_2^2 &= \sum_{i\in\mathbb{N}_0}\left|\sum_{j=0}^{n_0} a(i+j)x(j)\right|^2 \\
&\leq \sum_{i\in\mathbb{N}_0}\left(\sum_{j=0}^{n_0} |a(i+j)|\,|x(j)|\right)^2 \\
&\leq \|x\|_2^2 \sum_{i\in\mathbb{N}_0}\sum_{j=0}^{n_0} |a(i+j)|^2 \\
&= \|x\|_2^2 \sum_{j=0}^{n_0}\sum_{i\in\mathbb{N}_0} |a(i+j)|^2 \\
&= \|x\|_2^2 \sum_{j=0}^{n_0} \|a_j\|_2^2 < +\infty,
\end{aligned}$$

where $a_j = \{a_j(n)\}_{n\in\mathbb{N}_0}$, for every $j = 0, 1, \ldots, n_0$, with $a_j(n) := a(j+n)$, for every $n \in \mathbb{N}_0$. Thus, if H_a can be extended to a bounded operator, we say that (1.1) defines a bounded Hankel operator on $\ell^2(\mathbb{N}_0)$.

1.1.1 Boundedness and compactness

The first question arises is when a Hankel matrix gives rise to a bounded operator. The answer is given by Nehari's Theorem ([21, Theorem 1.1.1]) with an if and only if condition for the kernel of H_a, a. If $\mathbb{D}$ is the complex open unit disk and $\mathbb{T} = \partial\mathbb{D}$, then we have the following theorem.

Theorem 1.1 (Nehari Theorem)**.** *Let $H_a = [a(i+j)]_{i,j\geq 0}$ be a Hankel matrix acting on $\ell^2(\mathbb{N}_0)$. H_a is bounded if and only if there exists a function $\phi \in L^\infty(\mathbb{T})$ such that $\hat{\phi}(n) = a(n)$, for every $n \in \mathbb{N}_0$.*

An equivalent characterisation of a Hankel operator is given by the following theorem ([21, Theorem 1.1.2]).

Theorem 1.2. *Let $a = \{a(n)\}_{n\in\mathbb{N}_0}$ be a complex valued sequence and consider, formally, the function*

$$\phi(z) := \sum_{n\geq 0} a(n)z^n,\ \forall z \in \mathbb{D}.$$

Then the sequence a defines a kernel of a Hankel operator H_a if and only if ϕ belongs to the function space BMO$(\mathbb{T})$.

For the definition of the BMO space, see Appendix C.2. It is worth to mention that in the case of positive bounded Hankel operators, there is a much simpler boundedness characterisation, given in terms of the operator's kernel ([33, Theorem 3.1]).

Theorem 1.3. *Let $H_a = [a(i+j)]_{i,j\geq 0}$ be a Hankel matrix on $\ell^2(\mathbb{N}_0)$.*

(i) If $a(j) = O(j^{-1})$, as $j \to +\infty$, then H_a is bounded.

(ii) If H_a is positive and bounded, then $a(j) = O(j^{-1})$, as $j \to +\infty$.

Having the boundedness question answered, it is natural to ask when further operator properties of H_a are satisfied, like compactness, Schatten class inclusions etc. As it happens for boundedness, we also have a complete characterisation of compact Hankel operators (matrices) ([21, Theorems 1.5.5 and 1.5.8]).

Theorem 1.4. *Let $H_a = [a(i+j)]_{i,j\geq 0}$ be a bounded Hankel matrix. Then the following are equivalent:*

(i) H_a is compact.

(ii) There exists a continuous function $\phi : \mathbb{T} \to \mathbb{C}$ such that $\hat{\phi}(n) = a(n)$, for all $n \in \mathbb{N}_0$.

(iii) The function

$$\phi(z) := \sum_{n\geq 0} a(n)z^n, \ \ \forall z \in \mathbb{D},$$

belongs to $\mathrm{VMO}(\mathbb{T})$.

For the definition of the VMO space, see Appendix C.2. An analogue of Theorem 1.3 exists for compactness, too (see [33, Theorem 3.2]).

Theorem 1.5. *Let $H_a = [a(i+j)]_{i,j\geq 0}$ be a bounded Hankel operator on $\ell^2(\mathbb{N}_0)$.*

(i) If $a(j) = o(j^{-1})$, as $j \to +\infty$, then H_a is compact.

(ii) If H_a is compact and positive, then $a(j) = o(j^{-1})$, as $j \to +\infty$.

We close the compactness paragraph by giving a very characteristic example of bounded, but non-compact Hankel operator, the *Hilbert matrix* $\mathcal{H}$. For let $a = \{a(j)\}_{j\in\mathbb{N}_0}$ such that

$$a(j) = \frac{1}{1+j}, \ \ \forall j \in \mathbb{N}_0.$$

Then a is the kernel of the following Hankel matrix:

$$\mathcal{H} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It can be proved (see [21, Corollary 1.5.19]) that $\mathcal{H}$ is a bounded operator on $\ell^2(\mathbb{N}_0)$ with $\|\mathcal{H}\| = \pi$. Moreover, $\mathcal{H}$ is not compact and its distance from compact operators (aka essential norm) equals π, too.

1.2 Integral Hankel operators

This section generalises the concept of Hankel matrices to integral operators. Whenever we refer to an integral Hankel operator or its kernel, we will use boldface notation. Since the Hankel matrices (or the discrete Hankel operators) act on $\ell^2(\mathbb{N}_0)$, it is natural to expect that a Hankel (integral) operator in the continuous case acts on $L^2(\mathbb{R}_+)$, where $\mathbb{R}_+ := (0, +\infty)$. We recall that, for any $p > 0$,

$$L^p(\mathbb{R}_+) : \left\{ f : \mathbb{R}_+ \to \mathbb{C} : \; \|f\|_p := \left(\int_{\mathbb{R}_+} |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{2}} < +\infty \right\}.$$

Moreover, for $p = \infty$, we define the space of essentially bounded functions on A,

$$L^\infty(\mathbb{R}_+) := \left\{ f : \mathbb{R}_+ \to \mathbb{C} : \; \|f\|_\infty := \operatorname*{ess\,sup}_{x>0} |f(x)| < +\infty \right\},$$

where

$$\operatorname*{ess\,sup}_{x>0} |f(x)| := \inf \left\{ C \geq 0 : \; |\{x > 0 : \; |f(x)| > C\}| = 0 \right\}.$$

Furthermore, we remind that the space $L^2(\mathbb{R}_+)$ is a Hilbert space with inner product

$$(f, g) = \int_{\mathbb{R}_+} f(x)\overline{g(x)} \, \mathrm{d}x \,, \; \forall f, g \in L^2(\mathbb{R}_+).$$

For completeness, we mention that there are more L^p spaces defined on some measure space (X, ν), like the so called weak L^p. These spaces can be considered as a sub-class of the Lorentz spaces and they occur often in interpolation methods. For this reason, their definition is presented in the Appendix A.

Since the action of a discrete Hankel operator is fully described by (1.1), it is natural to define the respective continuous action via integration, instead of summation. Summing up, for a (complex valued) function $\mathbf{a} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ we define the *(integral) Hankel operator* $\mathbf{H_a} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ by

$$\big(\mathbf{H_a} f\big)(x) = \int_{\mathbb{R}_+} \mathbf{a}(x + y) f(y) \, \mathrm{d}y \,, \; \forall x \in \mathbb{R}_+, \; \forall f \in L^2(\mathbb{R}_+). \tag{1.2}$$

The function $\mathbf{a}$ will be called the *kernel* of $\mathbf{H_a}$. Notice that (1.2) is well-defined on the dense subset of compactly supported smooth functions. Indeed, let $f \in C_c^\infty(\mathbb{R}_+)$. If $\alpha > 0$ such that $\operatorname{supp}(f) \subset [0, \alpha]$, and $M_f = \max_{x \in \operatorname{supp}(f)} |f(x)|$, then, for every fixed $x > 0$,

$$\big|\big(\mathbf{H_a} f\big)(x)\big| \leq M_f \int_{\mathbb{R}_+} |\mathbf{a}(x + y)| \, \mathrm{d}y < +\infty.$$

1.2.1 Boundedness and compactness

If (1.2) defines an operator that can be extended to a bounded operator on the whole $L^2(\mathbb{R}_+)$,we say that $\mathbf{H_a}$ is a bounded (integral) Hankel operator on $L^2(\mathbb{R}_+)$.

Similarly to the discrete case, we also have an analogue of Nehari's theorem ([21, Theorem 1.8.8]) as a boundedness equivalent, as well as some other boundedness and compactness characterisations.

Theorem 1.6 (Nehari's analogue). *Let $a : \mathbb{R}_+ \to \mathbb{C}$ be the kernel of the integral Hankel operator H_a. Then H_a is bounded on $L^2(\mathbb{R}_+)$ if and only if there exists a function $\phi \in L^\infty(\mathbb{R})$ such that $a = \mathcal{F}_{|\mathbb{R}_+}\phi$; where $\mathcal{F}$ is the Fourier transform.*

Moreover, boundedness of integral Hankel operators is also connected with the BMO class of functions, exactly as it happens in the discrete case.

Theorem 1.7 ([21], Theorem 1.8.8). *Let $a : \mathbb{R}_+ \to \mathbb{C}$ be the kernel of the integral Hankel operator H_a. Then H_a is bounded on $L^2(\mathbb{R}_+)$ if and only if there exists a function $\phi \in L^1(\mathbb{R}_+)$ such that $a = \mathcal{F}\phi$ and $\phi \in BMO(\mathbb{R})$.*

Moreover, another sufficient condition for boundedness is the kernel $\mathbf{a}$ to satisfy the boundedness relation below:

$$|\mathbf{a}(x)| \leq \frac{C}{|x|},\ \forall x > 0,$$

where C is a some positive constant. This is proved by using the Carleman operator (see the example at the end of the paragraph). Regarding the compactness characterisations, we have the following Theorem:

Theorem 1.8 ([21], Theorem 1.8.10). *Let $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ be the kernel of the integral Hankel operator H_a. Then the following are equivalent:*

(i) H_a is compact.

(ii) There exists a continuous function ϕ defined on $\mathbb{R}$ such that the limits $\lim_{x\to+\infty}\phi(x)$ and $\lim_{x\to-\infty}\phi(x)$ are equal, and $a = \mathcal{F}_{|\mathbb{R}_+}\phi$.

(iii) There exists a function $\phi \in L^1(\mathbb{R}_+)$ such that $a = \mathcal{F}\phi$ and $\phi \in VMO(\mathbb{R})$.

The reader may be able to notice that if $a \in L^1(\mathbb{R}_+)$, then $\mathcal{F}^*a \in C(\mathbb{R})$ and both the two limits $\lim_{x\to+\infty}\big(\mathcal{F}^*a\big)(x)$ $\lim_{x\to-\infty}\big(\mathcal{F}^*a\big)(x)$ are equal to zero. Therefore, from the previous theorem we infer the following corollary:

Corollary 1.9. *Let $a \in L^1(\mathbb{R}_+)$ be the kernel of the integral Hankel operator H_a. Then H_a is compact.*

We close this section by giving an analogue of the Hilbert operator; namely, a characteristic example of a bounded, non-compact integral Hankel operator. For define the function $a : \mathbb{R}_+ \to \mathbb{R}$

$$a(x) = \frac{1}{x},\ \forall x > 0.$$

Then a gives rise (as a kernel) to the Carleman operator $\mathcal{C} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, with

$$\big(\mathcal{C}f\big)(x) = \int_0^{+\infty} \frac{f(y)}{x+y}\,\mathrm{d}y\,,\ \forall x > 0,\ \forall f \in L^2(\mathbb{R}_+).$$

It can be shown (check [21, Theorem 1.8.14]), that the distance of $\mathcal{C}$ from compact operators equals its operator norm, which is equal to π; just like it happens with Hilbert operator.

1.3 Weighted Hankel operators

A generalisation of the concept of Hankel operators is that one of weighted Hankel operators. These operators are defined for both the discrete and the continuous case. For the discrete case, let ν and ξ be real constants and $\{a(j)\}_{j\in\mathbb{N}_0}$ be a sequence such that the operator $\Gamma_a^{\nu,\xi} : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$, with

$$\left(\Gamma_a^{\nu,\xi} x\right)(i) = \sum_{j\in\mathbb{N}_0} (1+i)^\nu a(i+j)(1+j)^\xi x(j),\ \forall x \in \ell^2(\mathbb{N}_0),\ \forall i \in \mathbb{N}, \tag{1.3}$$

is bounded. Then $\Gamma_a^{\nu,\xi}$ is a bounded weighted Hankel operator on $\ell^2(\mathbb{N}_0)$.

Respectively, in the continuous case, if ν and ξ are real constants and $\mathbf{a}$ is a function such that the operator $\mathbf{\Gamma_a}^{\nu,\xi} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, with

$$\left(\mathbf{\Gamma_a}^{\nu,\xi} f\right)(x) = \int_0^{+\infty} x^\nu \mathbf{a}(x+y) y^\xi f(y)\, \mathrm{d}y\ ,\ \forall f \in L^2(\mathbb{R}_+),\ \forall x \in \mathbb{R}_+, \tag{1.4}$$

is bounded, then $\mathbf{\Gamma_a}^{\nu,\xi}$ is a bounded weighted Hankel operator on $L^2(\mathbb{R}_+)$.

Certain boundedness and compactness conditions exist for this class of operators, as well. Nevertheless, a dive into this theory would be beyond the scope of our survey and therefore, is avoided. For a brief introduction to weighted (or generalised) Hankel operators we refer to [21, §6.8]. Finally, it is worth to mention that in [11] is obtained a certain class of weighted Hankel operators $\mathbf{\Gamma_a}^{-\frac{1}{4},-\frac{1}{4}}$ with power spectral asymptotics. Furthermore, in [1] one finds a more general construction of weighted Hankel operators on $L^2(\mathbb{R}_+)$, as well as some conditions for their boundedness, and Schatten class inclusions (more precisely, $\mathbf{S}_1$ and $\mathbf{S}_2$).

1.4 Hankel operators and compact operator ideals

The class of Hankel operators is closely related to a special class of functions, the so co called Besov class. Vladimir Peller has conducted a deep research, which results, roughly, that Schatten class inclusions for Hankel operators are equivalent to Besov class inclusions of their kernels. This paragraph aims to depict the most necessary aspects of this theory, which will also be used in our survey.

1.4.1 Compact operator ideals

In this section we briefly explain the construction and the importance of the so called Schatten classes, as well as of some other compact operators ideals.

When an operator is compact, one can ask further questions under this frame. A very important one is "how much compact" our operator is. A typical property of compact operators is that they are approximated (in operator norm) by finite rank operators. This suggests that, under this approximation, their distance from finite rank operators converges to zero. The faster this convergence is, the more compact the operator.

The distance of an operator from finite rank operators is measured by a quantity called singular value. More precisely, let T be a compact (bounded) linear operator (acting on a Hilbert space). If we denote by T^* its adjoint, then T^*T is a compact, positive operator. This implies that its eigenvalues, $\lambda_n(T^*T)$, where $n \in \mathbb{N}$, are also positive, which allows us to consider their square root. Thus, define the sequence of *singular values* of T, $\{s_n(T)\}_{n\in\mathbb{N}}$, by

$$s_n(T) := \sqrt{\lambda_n(T^*T)}, \ \forall n \in \mathbb{N}.$$

Notice that, due to compactness, the singular values of T are of finite multiplicity and converge to zero (see [26, Theorem VI.15 and VI.16]). Moreover, it can be proved (see [4, Chapter 11, §1.3]) that

$$s_n(T) = \min_{\operatorname{rank}(K)\le n-1} \|T-K\|, \ \forall n \in \mathbb{N}.$$

Now, depending on the properties of the singular values, we can define various compact operator ideals and we begin with Schatten classes. For let $p \in (0,+\infty]$. For $p=+\infty$, denote by $\mathbf{S}_\infty$ the class of all compact operators. Then, for every $p \in (0,+\infty)$, define the *Schatten class* $\mathbf{S}_p$ as follows:

$$\mathbf{S}_p := \left\{ T \in \mathbf{S}_\infty : \ \|T\|_{\mathbf{S}_p} := \left(\sum_{n\in\mathbb{N}} s_n(T)^p\right)^{\frac{1}{p}} < +\infty \right\}.$$

Furthermore, we define two more ideals of compact operators. More precisely, for any $p \in (0,+\infty)$,

$$\mathbf{S}_{p,\infty} := \left\{ T \in \mathbf{S}_\infty : \ \|T\|_{\mathbf{S}_{p,\infty}} := \sup_{n\in\mathbb{N}} n^{\frac{1}{p}} s_n(T) < +\infty \right\},$$

and

$$\mathbf{S}^0_{p,\infty} := \left\{ T \in \mathbf{S}_\infty : \ \lim_{n\to+\infty} n^{\frac{1}{p}} s_n(T) = 0 \right\}.$$

Moreover, for any operator T, we define the singular value counting function $\pi_T : \mathbb{R}_+ \to \mathbb{N}_0$ by

$$\pi_T(\lambda) := \sum_{\{n\in\mathbb{N}: \ s_n(T)>\lambda\}} 1.$$

We also mention that the following relation holds true

$$\|T\|_{\mathbf{S}_{p,\infty}} = \sup_{\lambda>0} \lambda \pi_T^{\frac{1}{p}}(\lambda).$$

We also remind that

$$\mathbf{S}_p \subset \mathbf{S}^0_{p,\infty} \subset \mathbf{S}_{p,\infty},$$

with

$$\|\cdot\|_{\mathbf{S}_{p,\infty}} \le \|\cdot\|_{\mathbf{S}_p}, \ \forall p > 0.$$

Finally in the Appendix A, the reader may find the definition of the Schatten-Lorentz classes. These spaces arise as a natural parallel of the Lorentz spaces and they usually occur in interpolation methods. For a complete discussion on compact operator ideals we refer to [4, Chapter 11].

1.4.2 Besov classes of analytic functions

We distinguish two cases; Besov classes on the unit circle $\mathbb{T}$ and the real line $\mathbb{R}$. To this end, let v be a $C_c^\infty(\mathbb{R})$ non-negative valued function, such that $\operatorname{supp}(v) = [2^{-1}, 2]$.

1.4.2.1 Besov classes on $\mathbb{T}$

In order to define the Besov classes on $\mathbb{T}$, let us consider a function v in $C_c^\infty(\mathbb{R})$, such that $\mathrm{supp}(v) = [\frac{1}{2}, 2]$, $v(1) = 1$, and $v([\frac{1}{2}, 2]) = [0, 1]$; note that $v(\frac{1}{2}) = v(2) = 0$. Then consider the sequence of functions $\{v_n\}_{n\in\mathbb{N}_0}$, such that,

$$v_n(t) = v\left(\frac{t}{2^n}\right), \ \forall t \in \mathbb{R}, \ \forall n \in \mathbb{N}_0, \tag{1.5}$$

and

$$\sum_{n\geq 0} v_n(t) = 1, \ \forall t \geq 1.$$

Ensuing, we define the polynomials

$$V_0(z) = \overline{z} + 1 + z, \ \forall z \in \mathbb{T}, \tag{1.6}$$

and, for every $n \in \mathbb{N}$,

$$V_n(z) = \sum_{j\in\mathbb{N}} v_n(j) z^j = \sum_{j=2^{n-1}}^{2^{n+1}} v_n(j) z^j, \ \forall z \in \mathbb{T}. \tag{1.7}$$

Then we say that an analytic function f of $\mathbb{T}$ belongs to the Besov class $B^p_{q,r}(\mathbb{T})$ iff

$$\|f\|_{B^p_{q,r}} := \left(\sum_{n\in\mathbb{N}_0} 2^{npr} \|f * V_n\|_q^r\right)^{\frac{1}{r}} < +\infty.$$

1.4.2.2 Besov classes on $\mathbb{R}$

Respectively, we define the Besov class $B^p_{q,r}(\mathbb{R})$ in a quite similar way. More precisely, let v be the $C_c^\infty(\mathbb{R})$ function that was defined in the unit circle case and consider the sequence of $C_c^\infty(\mathbb{R})$ non-negative valued functions $\{v_n\}_{n\in\mathbb{Z}}$, such that, for any $t \in \mathbb{R}$,

$$v_n(t) = v\left(\frac{t}{2^n}\right), \ \forall n \in \mathbb{Z}, \tag{1.8}$$

and

$$\sum_{n\in\mathbb{Z}} v_n(t) = 1, \ \forall t \geq 0.$$

Finally, define the sequence of functions $\{V_n\}_{n\in\mathbb{Z}}$, such that, for any $x \in \mathbb{R}$,

$$V_n(x) = \big(\mathcal{F}^* v_n\big)(x), \ \forall n \in \mathbb{Z}. \tag{1.9}$$

Then we define the space $B^p_{q,r}(\mathbb{R})$ to be

$$B^p_{q,r}(\mathbb{R}) := \left\{\mathcal{F}^* f : \ f \in L^1_{\mathrm{loc}}(\mathbb{R}_+) \text{ s.t. } \|\mathcal{F}^* f\|_{B^p_{q,r}} := \left(\sum_{n\in\mathbb{Z}} 2^{npr} \|\mathcal{F}^*(f v_n)\|_q^r\right)^{\frac{1}{r}} < +\infty\right\}$$

Notice that

$$\|f\|_{B^p_{q,r}} := \left(\sum_{n\in\mathbb{Z}} 2^{npr} \|f * V_n\|_q^r\right)^{\frac{1}{r}} < +\infty, \ \forall f \in B^p_{q,r}(\mathbb{R}). \tag{1.10}$$

Having defined the Besov classes we can now display a useful theorem. Moreover, from now on and for sake of simplicity, whenever we want to indicate that one quantity is less than or equal to another one, modulo some multiplicative (positive) constant, we will use the notation $\lesssim$.

Theorem 1.10 (cf. [2]). *Let $p \in (0, +\infty)$ Then, we have the following results for functions defined on $\mathbb{T}$ and $\mathbb{R}$, respectively.*

(i) Let $a = \{a(j)\}_{j\in\mathbb{N}_0}$. Then

$$\|\phi\|_{B_p^{\frac{1}{p}+d-1}} \lesssim \left\|\Gamma_a^{\frac{d-1}{2},\frac{d-1}{2}}\right\|_{\mathbf{S}_p} \lesssim \|\phi\|_{B_p^{\frac{1}{p}+d-1}},$$

where

$$\phi(e^{it}) = \sum_{n\in\mathbb{N}_0} a(n)e^{2\pi int}, \ \forall t \in [0, 1),$$

and $\Gamma_a^{\frac{d-1}{2},\frac{d-1}{2}}$ is defined in (1.3).

(ii) Let $\mathbf{a} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. Then

$$\|\mathcal{F}^*\mathbf{a}\|_{B_p^{\frac{1}{p}+d-1}} \lesssim \left\|\mathbf{\Gamma_a}^{\frac{d-1}{2},\frac{d-1}{2}}\right\|_{\mathbf{S}_p} \lesssim \|\mathcal{F}^*\mathbf{a}\|_{B_p^{\frac{1}{p}+d-1}},$$

where $\mathbf{\Gamma_a}^{\frac{d-1}{2},\frac{d-1}{2}}$ is defined in (1.4).

1.5 A special class of kernels

We close our introduction to one-dimensional Hankel matrices by presenting some interesting cases of kernels for both the discrete (Hankel matrices) and the continuous (integral operators) case.

1.5.1 Discrete case

Let $\beta > 0$, $\gamma \geq 0$ and consider the sequence $a = \{a(j)\}_{j\in\mathbb{N}_0}$ by

$$a(j) = \frac{1}{(j+1)^\beta\big(\log(j+2)\big)^\gamma}, \ \forall j \in \mathbb{N}_0. \tag{1.11}$$

Depending on the values of β and γ we can deduce boundedness, compactness, Schatten class inclusions and spectral asymptotics.

We first start with the case where $\gamma = 0$; namely, for some $\beta > 0$,

$$a(j) = \frac{1}{(j+1)^\beta}, \ \forall j \in \mathbb{N}_0. \tag{1.12}$$

If $\beta \in (0, 1)$, then H_a is not bounded. To see this, consider the sequence of sequences $\{x_N\}_{N\in\mathbb{N}} \subset \ell^2(\mathbb{N}_0)$, with

$$x_N(n) = \begin{cases} \frac{1}{n+1}, & \text{if } n = 0, 1, \dots, N \\ 0, & \text{if } n > N \end{cases}.$$

Clearly, x_N converges to $x \in \ell^2(\mathbb{N}_0)$, in the $\|\cdot\|_2$-sense, when $N \to +\infty$, where

$$x(n) := \frac{1}{n+1}, \ \forall n \in \mathbb{N}_0.$$

If the Hankel matrix H_a is a bounded linear operator, then $\|H_a x_N\|_2 \to \|H_a x\|_2$, when $N \to +\infty$. Nonetheless, it can be checked that there exists a positive constant C_β such that

$$\|H_a x_N\|_2 \geq C_\beta N^{\frac{1-\beta}{2}}, \quad \text{for } N \text{ large enough.}$$

The latter shows that the norm $\|H_a x_N\|_2$ blows up, when $N \to +\infty$, and therefore, H_a cannot be bounded.

If $\beta \geq 1$, then (1.12) defines a kernel of a bounded, positive Hankel operator. In fact we have the following theorem (see the respective discussion in [33, §3], and [29, Theorem 1.2])

Theorem 1.11. *Let $a = \{a(j)\}_{j\in\mathbb{N}_0}$ be the kernel of the bounded Hankel matrix H_a. Then H_a is positive if and only if there exists a non-decreasing function $\mu : [-1, 1] \to \mathbb{R}$ such that*

$$a(j) = \int_{-1}^{1} x^j \, \mathrm{d}\mu(x) \,, \ \forall j \in \mathbb{N}_0.$$

The existence of such function μ is investigated by the so called Hamburger moment problem ([29]). At this point, observe that if a is given by (1.12), for $\beta \geq 1$, then

$$a(j) = \frac{1}{\Gamma(\beta)} \int_0^1 x^j \left(\log \frac{1}{x}\right)^{\beta-1} \mathrm{d}x \,, \ \forall j \in \mathbb{N}_0;$$

where Γ stands for the Gamma function. Thus, Theorems 1.3 and 1.11 imply that the Hankel matrix H_a is a bounded, positive operator. Moreover, in [33] is proved that the eigenvalues of these operators converge exponentially to 0. More precisely,

$$\lambda_n(H_a) = \exp\left(-\pi\sqrt{2\gamma n} + o(\sqrt{n})\right), \ n \to +\infty.$$

Apparently, this implies that H_a belongs to any Schatten class $\mathbf{S}_p$, for $p > 0$.

We proceed to the more general situation where $\gamma > 0$ in (1.11). It can be readily verified that for $\beta \in (0, 1)$ the generated Hankel operator is unbounded. Indeed, it is enough to observe that for any $\beta' \in (\beta, 1)$

$$\frac{1}{j^{\beta'}} = o\left(\frac{1}{j^\beta (\log j)^\gamma}\right), \quad \text{when } j \to +\infty,$$

and that, according to the discussion above, the Hankel operator that corresponds to the kernel $(j+1)^{-\beta'}$ is unbounded. For $\beta \geq 1$, it is easily seen that $a(j) = o(j^{-1})$, for $j \to +\infty$, and therefore, Theorems 1.3 and 1.5 imply the boundedness and compactness, respectively.

Perhaps the most interesting case of this class of Hankel operators is that one which arises from the choice $\beta = 1$ and $\gamma > 0$; namely, from kernels of the form

$$a(j) = \frac{1}{(j+1)\,(\log(j+2))^\gamma}, \ \forall j \in \mathbb{N}_0. \tag{1.13}$$

This choice of kernels shows that there is a whole class of compact operators between the Hilbert operator, which we already know that is not compact, and Hankel operators with kernels described by (1.12) for $\beta > 1$. This observation triggered a complete investigation of operators of this class which is presented in [23] and [24]. Before we present the respective results, we display some necessary notation. For any real number x, we define

$$x_\pm := \max\{0, \pm x\}, \quad [x] := \max\{n \in \mathbb{Z} : \ n \le x\}.$$

In addition, for any $\gamma > 0$, we define

$$M(\gamma) = \begin{cases} 0, & \text{if } \gamma < \frac{1}{2}, \\ [\gamma] + 1, & \text{if } \gamma \ge \frac{1}{2} \end{cases}. \tag{1.14}$$

Moreover, for any sequence $x = \{x(j)\}_{j \in \mathbb{N}_0}$, we define the sequence of *iterated differences* $x^{(m)} = \{x^{(m)}(j)\}_{j \in \mathbb{N}_0}$, by

$$x^{(0)}(j) := x(j), \ \forall j \in \mathbb{N}_0; \ x^{(m)}(j) := x^{(m-1)}(j+1) - x^{(m-1)}(j), \ \forall j \in \mathbb{N}_0, \ \forall m \in \mathbb{N}.$$

Theorem 1.12 ([24], Theorem 2.2). *Let $a = \{a(j)\}_{j \in \mathbb{N}_0}$ be a real valued sequence such that*

$$a^{(m)}(j) = O\left(j^{-1-m} (\log j)^{-\gamma}\right), \quad \text{when } j \to +\infty,$$

for all $m = 0, 1, \ldots, M(\gamma)$, where M is defined in (1.14). Then H_a is compact and its singular values satisfy

$$s_n(H_a) = O\left(n^{-\gamma}\right), \quad \text{for } n \to +\infty.$$

Theorem 1.13 ([24], Theorem 2.3). *Let $a = \{a(j)\}_{j \in \mathbb{N}_0}$ be a real valued sequence such that*

$$a^{(m)}(j) = o\left(j^{-1-m} (\log j)^{-\gamma}\right), \quad \text{when } j \to +\infty,$$

for all $m = 0, 1, \ldots, M(\gamma)$. Then H_a is compact and its singular values satisfy

$$s_n(H_a) = o\left(n^{-\gamma}\right), \quad \text{for } n \to +\infty.$$

Finally, for any self-adjoint operator T, we denote by $\{\lambda_n^+(T)\}_{n \in \mathbb{N}}$ the sequence of its positive eigenvalues and, for any $n \in \mathbb{N}$, $\lambda_n^-(T) := \lambda_n^+(-T)$. Then we have the following theorem:

Theorem 1.14 ([23], Theorem 4.1). *Let $\gamma > 0$ and $a = \{a(j)\}_{j \in \mathbb{N}_0}$ be a sequence described by (1.13) and define the respective Hankel operator H_a, with kernel a. Then H_a is a compact operator and its eigenvalues satisfy the following asymptotic law:*

$$\lambda_n^\pm(H_a) = C^\pm n^{-\gamma} + o(n^{-\gamma}), \quad \text{for } n \to +\infty,$$

where

$$C^+ = 2^{-\gamma} \pi^{1-2\gamma} B\left(\frac{1}{2\gamma}, \frac{1}{2}\right)^\gamma \quad \text{and} \quad C^- = 0;$$

here B denotes the Beta function.

By the discussion above, Theorem 1.12 shows that if H_a is a Hankel operator with kernel $a = \{a(j)\}_{j \in \mathbb{N}_0}$ described by (1.11), for $\beta = 1$ and $\gamma > 0$, then $H_a \in \mathbf{S}_p$, for any $p > \frac{1}{\gamma}$ and $H_a \in \mathbf{S}_{\frac{1}{\gamma}, \infty}$. In addition, if $\beta > 1$ and γ is positive again, then Theorem 1.13 implies that $H_a \in \mathbf{S}_p$, for any $p > 0$.

1.5.2 Continuous case

Motivated by the work that has been presented for the discrete case, we can define Hankel integral kernels of the following form:

$$\mathbf{a}(x) = x^{-\beta} \langle \log x \rangle^{-\gamma}, \ \forall x > 0, \tag{1.15}$$

where $\beta > 0$, $\gamma \geq 0$ and

$$\langle x \rangle := \sqrt{1 + x^2}, \ \forall x > 0. \tag{1.16}$$

Again it is not difficult to check that in order to achieve boundedness, β should be no less than one.

We close the continuous case of these special kernels by briefly mentioning the respective results from [23] and [24].

Theorem 1.15 ([24], Theorem 2.6)**.** *Let $\gamma > 0$ and consider an $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ function $\mathbf{a}$ such that, for $m = 0, 1, \ldots, M(\gamma)$,*

$$|\mathbf{a}^{(m)}(x)| \leq A_m x^{-1-m} \langle \log x \rangle^{-\gamma}, \ \forall x > 0,$$

where A_m is some positive constant and $M(\gamma)$ is defined in (1.14). Then the singular values of $H_{\mathbf{a}}$ present the asymptotic behaviour below:

$$s_n(H_{\mathbf{a}}) = O(n^{-\gamma}), \ n \to +\infty.$$

Theorem 1.16 ([24], Theorem 2.7)**.** *Let $\gamma > 0$ and consider an $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ function $\mathbf{a}$ such that, for $m = 0, 1, \ldots, M(\gamma)$,*

$$|\mathbf{a}^{(m)}(x)| = o\left(x^{-1-m} \langle \log x \rangle^{-\gamma}\right), \quad \textit{for} \ \ x \to 0^+ \ \ \textit{and} \ \ x \to +\infty.$$

Then the singular values of $H_{\mathbf{a}}$ present the asymptotic behaviour below:

$$s_n(H_{\mathbf{a}}) = o(n^{-\gamma}), \ n \to +\infty.$$

Theorem 1.17 ([23], Theorem 3.1)**.** *Let $\gamma > 0$ and consider the function $\mathbf{a}$ defined in (1.15) for $\beta = 1$ and $\gamma > 0$. Then the corresponding Hankel operator $H_{\mathbf{a}}$ is compact and its eigenvalues verify the asymptotic formula below:*

$$\lambda_n^{\pm}(H_{\mathbf{a}}) = C^{\pm} n^{-\gamma} + o\left(n^{-\gamma}\right), \quad \textit{when} \ \ n \to +\infty;$$

where

$$C^{+} = \pi^{1-2\gamma} B\left(\frac{1}{2\gamma}, \frac{1}{2}\right)^{\gamma} \quad \textit{and} \ \ C^{-} = 0.$$

Therefore, it is not difficult to see that if $H_{\mathbf{a}}$ is an integral Hankel operator with kernel $\mathbf{a}$ described by (1.15) for $\beta = 1$ and $\gamma > 0$, then Theorem 1.15 implies that $H_{\mathbf{a}} \in \mathbf{S}_p$, for any $p > \frac{1}{\gamma}$ and $H_{\mathbf{a}} \in \mathbf{S}_{\frac{1}{\gamma},\infty}$. In addition, if $\beta > 1$ and $\gamma \geq 0$, then Theorem 1.16 implies that $H_{\mathbf{a}} \in \mathbf{S}_p$, for any $p > 0$.

Finally, it is worth to mention that we are also aware of another class of kernels which yields a spectral behaviour similar to this of Theorem 1.15. More precisely, in [10], K. Glover, J. Lam, and J. R. Partington have constructed a special class of integral Hankel operators that admit spectral asymptotics of polynomial decay. These operators have kernel $\mathbf{a} \in C^M(\mathbb{R}_+)$ such that

$$\mathbf{a}^{(m)}(t) - \mathbf{a}^{(m)}(s) = \int_s^t \mathbf{a}^{(m+1)}(x)\,\mathrm{d}x\,,\ \forall m = 0, 1, \ldots, M,$$

and $(\cdot)^{\frac{1}{2}}\,\mathbf{a}^{(M+1)} \in L^2(\mathbb{R}_+)$, for some $M \in \mathbb{N}_0$. Moreover, the functions $\mathbf{a}^{(m)}(t)$, for every $m = 0, 1, \ldots, M$, decay to zero, when $t \to +\infty$, faster than any polynomial. Then the positive and negative eigenvalues of the integral Hankel operator $\mathbf{H_a}$ are symmetric and decay polynomially fast to 0.

Part II

Multidimensional Hankel operators

Chapter 2

Introduction

We begin our approach to multidimensional Hankel operators by giving the necessary definitions and in general, the frame of our work.

Since the discrete and continuous realisations of Hankel operators in the one-dimensional case have been clarified, it is not difficult to make multidimensional analogues. We begin with the definition of multidimensional sequences and function spaces and then, we proceed to those of multidimensional Hankel operators. For if $\mathbb{N}_0$ is as defined in the first Chapter, we define the set $\mathbb{N}_0^d$ as

$$\mathbb{N}_0^d := \big\{ j = (j_1, j_2, \ldots, j_d) : \; j_k \in \mathbb{N}_0, \; \forall k = 1, 2, \ldots, d \big\},$$

where $d \in \mathbb{N}$. Similarly, for any $d \in \mathbb{N}$, let $\mathbb{R}_+^d$ be defined as

$$\mathbb{R}_+^d := \big\{ x = (x_1, x_2, \ldots, x_d) : \; x_k \in \mathbb{R}_+, \; \forall k = 1, 2, \ldots, d \big\}.$$

Observe that, in complete analogy with the one-dimensional case, a multidimensional (complex valued) sequence $x = \{x(j)\}_{j \in \mathbb{N}_0^d}$ could be understood as a function $x : \mathbb{N}_0^d \to \mathbb{C}$. Then, for any $p > 0$, we can define the sequence space $\ell^p(\mathbb{N}_0^d)$ and the function space $L^p(\mathbb{R}_+^d)$ as follows:

$$\ell^p(\mathbb{N}_0^d) := \left\{ x = \{x(j)\}_{j \in \mathbb{N}_0^d} : \; \|x\|_p := \left(\sum_{j \in \mathbb{N}_0^d} |x(j)|^p \right)^{\frac{1}{p}} < +\infty \right\};$$

and

$$L^p(\mathbb{R}_+^d) := \left\{ f : \mathbb{R}_+^d \to \mathbb{C} : \; \|f\|_p := \left(\int_{\mathbb{R}_+^d} |f(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{p}} < +\infty \right\}.$$

Finally, for $p = +\infty$, we define

$$\ell^\infty(\mathbb{N}_0^d) := \left\{ x = \{x(j)\}_{j \in \mathbb{N}_0^d} : \; \|x\|_\infty := \sup_{j \in \mathbb{N}_0^d} |x(j)| < +\infty \right\},$$

and

$$L^\infty(\mathbb{R}_+^d) := \left\{ f : \mathbb{R}_+^d \to \mathbb{C} : \|f\|_\infty := \operatorname*{ess\,sup}_{\mathbf{x} \in \mathbb{R}_+^d} |f(x)| < +\infty \right\}.$$

From now on, all the vectors of $\mathbb{R}^d$ will be denoted by boldface symbols. Finally, we remind that the spaces $\ell^2(\mathbb{N}_0^d)$ and $L^2(\mathbb{R}_+^d)$ are Hilbert spaces, with inner product described by

$$(x, y) = \sum_{j \in \mathbb{N}_0^d} x(j)\overline{y(j)}, \ \forall x = \{x(j)\}_{j \in \mathbb{N}_0^d}, y = \{y(j)\}_{j \in \mathbb{N}_0^d} \in \ell^2(\mathbb{N}_0^d),$$

and

$$(f, g) = \int_{\mathbb{R}_+^d} f(\mathbf{x})\overline{g(\mathbf{x})}\, \mathrm{d}\mathbf{x}\,, \ \forall f, g \in L^2(\mathbb{R}_+^d),$$

respectively. Finally, similarly to the discrete case, we can define the following weighted versions. So, for a multidimensional (positive valued) sequence $\upsilon = \{\upsilon(j)\}_{j \in \mathbb{N}_0^d}$, and for any $p \in (0, +\infty)$, define the spaces

$$\ell^p_\upsilon(\mathbb{N}_0^d) := \left\{ x = \{x(j)\}_{j \in \mathbb{N}_0^d} : \ \|x\|_{\ell^p_\upsilon} := \left(\sum_{j \in \mathbb{N}_0^d} |x(j)|^p\, \upsilon(j) \right)^{\frac{1}{p}} < +\infty \right\}.$$

Now we are able to define the multidimensional Hankel operators. At first, notice that, for the discrete case, a matrix representation is a much more difficult task in the multidimensional framework. Though, relation (1.1) may give us the necessary intuition to reach the desired definition. For let $a = \{a(j)\}_{j \in \mathbb{N}_0^d}$ be a sequence such that enables us to define the (bounded) operator $H_a : \ell^2(\mathbb{N}_0^d) \to \ell^2(\mathbb{N}_0^d)$ with

$$\big(H_a x\big)(i) := \sum_{j \in \mathbb{N}_0^d} a(i + j)x(j), \ \forall i \in \mathbb{N}_0^d, \ \forall x \in \ell^2(\mathbb{N}_0^d).$$

Then H_a is a discrete (bounded), multidimensional *Hankel operator* and the sequence a will be called the *kernel* of H_a.

Similarly, one can define the multidimensional analogue of integral Hankel operators. For let $\mathbf{a} \in L^1_{\mathrm{loc}}(\mathbb{R}_+^d)$ and define the (bounded) operator $\mathbf{H_a} : L^2(\mathbb{R}_+^d) \to L^2(\mathbb{R}_+^d)$, with

$$\big(\mathbf{H_a} f\big)(\mathbf{x}) := \int_{\mathbb{R}_+^d} \mathbf{a}(\mathbf{x} + \mathbf{y}) f(\mathbf{y})\, \mathrm{d}\mathbf{y}\,, \ \forall \mathbf{x} \in \mathbb{R}_+^d, \ \forall f \in L^2(\mathbb{R}_+^d).$$

Then $\mathbf{H_a}$ will be a multidimensional (bounded) *integral Hankel operator* with *kernel* $\mathbf{a}$.

Unfortunately, in the multidimensional case, it is much more difficult to construct necessary and sufficient conditions for boundedness or compactness of Hankel operators. Despite the attempts of various mathematicians, the existence of such conditions is unknown to the author. Nonetheless, one may retrieve some sufficient boundedness or compactness conditions, in the spirit of Theorems 1.3 (i) and 1.5 (i), which will be clarified in the sequel.

2.1 A special class of kernels

The aim of this paragraph is to develop multidimensional analogues of kernel that have been already presented in §1.5.

Nevertheless, by a quick look one may realise that there exists a large variety of ways to define a multidimensional kernel. To enlighten a bit the situation, we start by presenting just a couple of examples for the discrete case, where analogous constructions are achievable for the continuous case, too.

We first examine kernels that lead to a tensor product of Hankel operators. For let $\{a_i\}_{i=1}^d$ be d one-dimensional sequences, with $a_i = \{a_i(j)\}_{j\in\mathbb{N}_0}$, for $i = 1, 2, \ldots, d$. Then define the multidimensional sequence $a = \{a(j)\}_{j\in\mathbb{N}_0^d}$, with

$$a(j) = a_1(j_1)a_2(j_2)\ldots a_d(j_d), \ \forall j = (j_1, j_2, \ldots, j_d) \in \mathbb{N}_0^d.$$

If each one of a_is defines a (bounded) Hankel operator H_{a_i} on $\ell^2(\mathbb{N}_0)$, then a gives rise to the *tensor product* $H_a = \otimes_{i=1}^d H_{a_i}$, which is a (bounded) Hankel operator on $\ell^2(\mathbb{N}_0^d)$.

After the first example, it becomes obvious how numerous examples of kernel we can construct. Another example, could be the following. Let $a_0 = \{a_0(j)\}_{j\in\mathbb{N}_0}$ be a one-dimensional sequence and define the d-dimensional sequence $a = \{a(j)\}_{j\in\mathbb{N}_0^d}$, with

$$a(j) = a_0(|j|), \quad \text{where} \quad |j| := \sum_{i=1}^d j_i, \ \forall j = (j_1, j_2, \ldots, j_d) \in \mathbb{N}_0^d. \tag{2.1}$$

Then a could be the kernel of a multidimensional Hankel operator H_a.

A natural question is whether there exist conditions similar to Theorems 1.3 (i), and 1.5 (i) which imply boundedness or compactness. To answer this question, it is not difficult to check that a sufficient boundedness (resp. compactness) condition for a tensor product is

$$a_i(j) = O\left(j^{-1}\right) \quad \left(\text{resp. } o\left(j^{-1}\right)\right), \ \forall i = 1, 2, \ldots, d, \quad \text{when} \quad j \to +\infty.$$

On the other hand, when it comes to kernels defined by (2.1), it can be checked again that a sufficient boundedness (resp. compactness) condition is

$$a_0(j) = O\left(j^{-d}\right) \quad \left(\text{resp. } o\left(j^{-d}\right)\right), \quad \text{when} \quad j \to +\infty.$$

It is now evident how the choice of kernel may affect the boundedness/compactness conditions in the multidimensional analysis. An important characteristic of kernels defined in (2.1) is that the dimension d is the "compactness threshold", exactly as it happens in the one-dimensional case. So, since we are working this paragraph in complete analogy to §1.5, we are prompted to consider kernels of type (2.1), where

$$a_0(j) = (j+1)^{-d}\big(\log(j+2)\big)^{-\gamma}, \ \forall j \in \mathbb{N}_0;$$

for some positive γ. Obviously, the continuous analogue will be integral Hankel operators $\mathbf{H_a}$, with kernel

$$\mathbf{a}(\mathbf{x}) = \mathbf{a_0}\left(\sum_{i=1}^d x_i\right), \ \forall \mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}_+^d,$$

where

$$\mathbf{a_0}(x) = x^{-d}\langle \log x\rangle^{-\gamma}, \ \forall x > 0,$$

where the function $\langle\cdot\rangle$ is defined in (1.16).

2.1.1 Reduction to one-dimensional weighted Hankel operators

The purpose of this section is to provide a technique of reduction to one-dimensional weighted Hankel operators. This reduction will prove to be very useful in the sequel for two reasons: first, it motivates the construction of some interesting examples of bounded, non-compact, discrete Hankel operators and second, it will be used, as we have already mentioned in the introduction, as a necessary tool to derive our main results.

2.1.1.1 Discrete case

By using induction in d, it is not difficult to check that

$$W_d(k) := |\{j \in \mathbb{N}_0^d : |j| = k\}| = \binom{k+d-1}{d-1}, \ \forall k \in \mathbb{N}_0. \tag{2.2}$$

Now we define the linear bounded operator $J : \ell^2(\mathbb{N}_0^d) \to \ell^2(\mathbb{N}_0)$, given by

$$\big(Jx\big)(i) := \big(W_d(i)\big)^{-\frac{1}{2}} \sum_{\{j \in \mathbb{N}_0^d : |j| = i\}} x(j), \ \forall i \in \mathbb{N}_0$$

and we can verify that, for an arbitrary Hankel operator $H_a : \ell^2(\mathbb{N}_0^d) \to \ell^2(\mathbb{N}_0^d)$, with kernel $a(i) = a_0(|i|)$, $\forall i \in \mathbb{N}_0^d$, where a_0 is a sequence defined on $\mathbb{N}_0$, the following relation holds true;

$$(H_a x, y) = (J^* \Gamma J x, y), \ \forall x, y \in \ell^2(\mathbb{N}_0^d),$$

where $\Gamma : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ is the weighted Hankel operator defined by

$$(\Gamma x)(i) = \sum_{j \geq 0} \sqrt{W_d(i)} a_0(i+j) \sqrt{W_d(j)} x(j), \ \forall i \in \mathbb{N}_0, \ \forall x \in \ell^2(\mathbb{N}_0). \tag{2.3}$$

Indeed, for sake of accuracy, observe that

$$\begin{aligned}(H_a x, y) &= \sum_{i,j \in \mathbb{N}_0^d} a(i+j) x(j) \overline{y(i)} \\ &= \sum_{i,j \in \mathbb{N}_0^d} a_0(|i| + |j|) x(j) \overline{y(i)} \\ &= \sum_{i,j \in \mathbb{N}_0} a_0(i+j) \sum_{\{k \in \mathbb{N}_0^d : |k| = j\}} x(k) \overline{\sum_{\{k \in \mathbb{N}_0^d : |k| = i\}} y(k)}.\end{aligned}$$

Besides, it is not difficult to check that the adjoint of J is given by

$$\big(J^* x\big)(i) = \big(W_d(|i|)\big)^{-\frac{1}{2}} x(|i|), \ \forall i \in \mathbb{N}_0^d, \ \forall x \in \ell^2(\mathbb{N}_0)$$

and in addition, J^* is a partial isometry. Indeed, for any $x \in \ell^2(\mathbb{N}_0)$,

$$\|J^* x\|_2^2 = \sum_{i \in \mathbb{N}_0^d} \big(W_d(|i|)\big)^{-1} \big|x(|i|)\big|^2 = \sum_{i \geq 0} |x(i)|^2 = \|x\|_2^2.$$

This indicates that also J is a partial isometry and therefore, instead of investigating the Schatten class inclusions of H_a, we investigate those of Γ. To achieve the latter, we define the sequence $u = \{u(j)\}_{j \in \mathbb{N}_0}$, with

$$u(j) = \left(\frac{W_d(j)}{(j+1)^{d-1}}\right)^{\frac{1}{2}}, \ \forall j \in \mathbb{N}_0,$$

and the diagonal matrix

$$\mathcal{U} = [u(j)]_{j \geq 0}.$$

Then we can see that

$$\Gamma = \mathcal{U} \Gamma_{a_0}^{\frac{d-1}{2}, \frac{d-1}{2}} \mathcal{U},$$

where $\Gamma_{a_0}^{\frac{d-1}{2}, \frac{d-1}{2}}$ has been defined in (1.3). Notice that $\mathcal{U}$ is an invertible bounded operator on $\ell^2(\mathbb{N}_0)$. The fact that $\mathcal{U}$ is bounded can be checked by noticing that $W_d(j) \sim \frac{j^{d-1}}{(d-1)!}$, when $j \to +\infty$. Thus, we can deduce results on the spectral behaviour of H_a by investigating $\Gamma_{a_0}^{\frac{d-1}{2}, \frac{d-1}{2}}$.

2.1.1.2 Continuous case

Notice that, for any Hankel operator $\mathbf{H_a} : L^2(\mathbb{R}^d_+) \to L^2(\mathbb{R}^d_+)$, with kernel $\mathbf{a}(x_1,\dots,x_d) = \mathbf{a_0}(x_1+\cdots+x_d)$, $\forall(x_1,\dots,x_d)\in\mathbb{R}^d_+$, and for every $f,g\in L^2(\mathbb{R}^d_+)$,

$$\begin{aligned}(\mathbf{H_a}f,g) &= \int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} \mathbf{a}(\mathbf{x}+\mathbf{y})f(\mathbf{y})\overline{g(\mathbf{x})}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\ &= \int_{\mathbb{R}_+}\cdots\int_{\mathbb{R}_+} \mathbf{a_0}\Big(\sum_{i=1}^d (x_i+y_i)\Big)f(y_1,\dots,y_d)\overline{g(x_1,\dots,x_d)}\,\mathrm{d}x_1\dots\mathrm{d}x_d\,\mathrm{d}y_1\dots\mathrm{d}y_d\end{aligned}$$

Let us make the change of variables $x=\sum_{i=1}^d x_i$, $y=\sum_{i=1}^d y_i$, and x_i, y_i remain unaltered for $i=1,\dots,d-1$. Then

$$\begin{aligned}(\mathbf{H_a}f,g) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathbf{a_0}(x+y) \underbrace{\int_0^y\int_0^{y-y_1}\cdots\int_0^{y-\sum_{i=1}^{d-2}y_i} f\Big(y_1,y_2,\dots,y_{d-1},y-\sum_{i=1}^{d-1}y_i\Big)\,\mathrm{d}y_1\dots\mathrm{d}y_{d-1}\times}\\ \times\int_0^x\int_0^{x-x_1}\cdots\int_0^{x-\sum_{i=1}^{d-2}x_i} g\Big(x_1,x_2,\dots,x_{d-1},x-\sum_{i=1}^{d-1}x_i\Big)\,\mathrm{d}x_1\dots\mathrm{d}x_{d-1}\,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned}$$

What precedes prompts us to define the bounded operator $J : L^2(\mathbb{R}^d_+)\to L^2(\mathbb{R}_+)$, given by the following formula:

$$\big(Jf\big)(x) := \sqrt{(d-1)!}\,x^{-\frac{d-1}{2}}\int_0^x\cdots\int_0^{x-\sum_{i=1}^{d-2}x_i} f\Big(x_1,\dots,x_{d-1},x-\sum_{i=1}^{d-1}x_i\Big)\,\mathrm{d}x_1\dots\mathrm{d}x_{d-1}\,,$$

for every $x\in\mathbb{R}_+$. So that we have

$$(\mathbf{H_a}f,g) = (J^*\mathbf{\Gamma_0}Jf,g),\ \forall f,g\in L^2(\mathbb{R}^d_+),$$

where $\mathbf{\Gamma_0} : L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)$ is the weighted Hankel operator defined by

$$\big(\mathbf{\Gamma_0}f\big)(x) = (d-1)!\int_0^{+\infty} x^{\frac{d-1}{2}}\mathbf{a_0}(x+y)y^{\frac{d-1}{2}}f(y)\,\mathrm{d}y\,,\ \forall x\in\mathbb{R}_+,\ \forall f\in L^2(\mathbb{R}_+). \tag{2.4}$$

Moreover, the adjoint of J, $J^* : L^2(\mathbb{R}_+)\to L^2(\mathbb{R}^d_+)$, is given by

$$\big(J^*f\big)(x_1,\dots,x_d) = \sqrt{(d-1)!}\Big(\sum_{i=1}^d x_i\Big)^{-\frac{d-1}{2}} f\Big(\sum_{i=1}^d x_i\Big),\ \forall(x_1,\dots,x_d)\in\mathbb{R}^d_+,\ \forall f\in L^2(\mathbb{R}_+),$$

and it can be checked that it is a partial isometry. Indeed,

$$\|J^*f\|_2^2 = (d-1)!\int_0^{+\infty}\int_0^{+\infty}\cdots\int_0^{+\infty}\left(\sum_{i=1}^d x_i\right)^{d-1}\left|f\left(\sum_{i=1}^d x_i\right)\right|^2\mathrm{d}x_d\dots\mathrm{d}x_2\,\mathrm{d}x_1\,.$$

By making the following change of variables

$$y_i = x_i,\ \forall i=1,2,\dots,d-1,\quad\text{and}\quad y_d = \sum_{i=1}^d y_i$$

we obtain

$$\begin{aligned}
\|J^* f\|_2^2 &= (d-1)! \int_0^{+\infty} \frac{|f(y)|^2}{y^{d-1}} \int_0^y \int_0^{y-y_1} \cdots \int_0^{y-\sum_{i=1}^{d-2}} \mathrm{d}y_{d-1} \dots \mathrm{d}y_2 \, \mathrm{d}y_1 \, \mathrm{d}y \\
&= (d-1)! \int_0^{+\infty} \frac{|f(y)|^2}{y^{d-1}} \int_0^y \int_0^{y-y_1} \cdots \int_0^{y-\sum_{i=1}^{d-3}} \left(y - \sum_{i=1}^{d-2} y_i \right) \mathrm{d}y_{d-2} \dots \mathrm{d}y_2 \, \mathrm{d}y_1 \, \mathrm{d}y \\
&\vdots \\
&= \frac{(d-1)!}{(d-2)!} \int_0^{+\infty} \frac{|f(y)|^2}{y^{d-1}} \int_0^y (y-y_1)^{d-2} \, \mathrm{d}y_1 \, \mathrm{d}y \\
&= \|f\|_2^2 .
\end{aligned}$$

Therefore, J is a partial isometry, too.

2.1.2 Examples of non-compact operators

In this section, we display some examples of explicitly diagonalisable Hankel operators. These examples show that the exponent $-d$ is a compactness threshold indeed, since we will construct non-compact Hankel operators H_a, with kernel $a(j) = a_0(|j|)$, where $a_0(n)$ is $O(n^{-d})$ but not $o(n^{-d})$, when $n \to +\infty$. To this end, we make use of a special family of weighted Hankel operators that are explicitly diagonalisable. This diagonalisation is fully described in [15]. We briefly quote the needed results from the cited paper.

Theorem 2.1. *Let α, β, γ be any positive real numbers with $\beta \le \gamma$ and define the weighted Hankel operator H acting on $\ell^2(\mathbb{N}_0)$ with entries $H_{i,j} = w(i)h(i+j)w(j)$, where*

$$w(j) = \sqrt{\frac{\Gamma(j+\beta)\Gamma(j+\gamma)}{\Gamma(j+\alpha)j!}} \quad \text{and} \quad h(j) = \frac{\Gamma(j+\alpha)}{\Gamma(j+\beta+\gamma)}, \ \forall j \in \mathbb{N}_0, \tag{2.5}$$

where Γ denotes the Gamma function. If $\alpha + \beta - \gamma \ge 0$, then H has solely simple continuous spectrum given by

$$\sigma_c(H) = [0, M(\alpha, \beta, \gamma)],$$

where

$$M(\alpha, \beta, \gamma) = \frac{1}{\Gamma(\beta+\gamma-\alpha)} \Gamma\left(\frac{\beta+\gamma-\alpha}{2}\right)^2 . \tag{2.6}$$

If $\alpha + \beta - \gamma < 0$, then H, besides the continuous spectrum, has also point spectrum, given by

$$\sigma_p(H) = \{\lambda_0, \lambda_1, \dots, \lambda_{N(\alpha,\beta,\gamma)}\},$$

where

$$N(\alpha, \beta, \gamma) = \left\lceil \frac{\gamma-\alpha-\beta}{2} \right\rceil - 1,$$

and

$$\lambda_k = \frac{\Gamma(\beta+k)\Gamma(\gamma-\alpha-k)}{\Gamma(\beta+\gamma-\alpha)}, \ \forall k = 0, 1, \dots, N(\alpha, \beta, \gamma).$$

Finally, it holds true that

$$\lambda_0 > \lambda_1 > \cdots > \lambda_{N(\alpha,\beta,\gamma)} > M(\alpha, \beta, \gamma).$$

Returning back to the d-dimensional case, we have seen that a Hankel operator H_a with kernel $a(j) = a_0(|j|)$, $\forall j \in \mathbb{N}_0^d$, is unitarily equivalent, modulo null-spaces, to the one dimensional weighted Hankel operator Γ_0, as it is described in (2.3). If we denote by $\Gamma_{i,j}$ the entries of Γ_0, we notice that

$$\Gamma_{i,j} = w(i)h(i+j)w(j),$$

where w as described in (2.5), for arbitrarily chosen $\alpha > 0$, $\beta = \alpha$ and $\gamma = d$ (or $\beta = d$ and $\gamma = \alpha$), and

$$h(j) = \frac{a_0(j)}{\Gamma(d)}, \ \forall j \in \mathbb{N}_0.$$

Therefore, for

$$a_0(j) = \frac{(d-1)!}{(j+\alpha)_d}, \ \forall j \in \mathbb{N}_0,$$

where

$$(x)_n := \prod_{j=0}^{n-1}(x+j)$$

is the Pochhammer symbol, we have the following:

Assume that $\alpha > 0$ and M be as defined in (2.6).

- *If $\alpha \in (0, \frac{d}{2})$, then $\sigma(H_a) = \sigma_c(H_a) \cup \sigma_p(H_a)$, where σ_p is given in Theorem 2.1*
- *If $\alpha \in [\frac{d}{2}, d]$, then $\sigma(H_a) = \sigma_c(H_a) = [0, M(\alpha, \alpha, d)]$.*
- *Finally, if $\alpha > d$, we have that $\sigma(H_a) = \sigma_c(H_a) = [0, M(\alpha, d, \alpha)] = [0, M(\alpha, \alpha, d)]$.*

Chapter 3

Main results

In this section we display our main results. Having already explained our interest in kernels (discrete or continuous) that depend on the sum of their arguments, we can extend our area of interest a bit more.

Regarding the discrete case, we can also assume that the sequence a_0 that generates the kernel a of the Hankel matrix H_a admits an error term g, so that

$$a_0(j) = j^{-d} (\log j)^{-\gamma} + g(j), \ \forall j \geq 2.$$

Moreover, notice that $a(j)$ was defined to be equal to $a_0(|j|)$, for all $j \in \mathbb{N}_0^d$. Notice that

$$|j| = j \cdot \mathbf{1}, \quad \text{where} \ \ \mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{R}_+^d,$$

and "·" denotes the usual inner product. This observation raises the question whether we can interpret a_0 as a real valued function on $\mathbb{R}_+$ and define kernels a of the form

$$a(j) = a_0(\kappa \cdot j), \ \forall j \in \mathbb{N}_0^d,$$

where $\kappa \in \mathbb{R}_+^d$. Then the case $a(j) = a_0(|j|)$ would be just a simple sub-case of this generalised version of kernels. Of course the same discussion can be developed for integral kernels, as well.

3.1 Discrete case

3.1.1 The special case of $\kappa = \mathbf{1}$

Lemma 3.1. *Let $\gamma > 0$ and $\{a_0(j)\}_{j \in \mathbb{N}_0}$ be a sequence of complex numbers such that it satisfies*

$$a_0^{(m)}(j) = O\big(j^{-d-m} (\log j)^{-\gamma}\big), \ j \to +\infty, \tag{3.1}$$

for every $m = 0, 1, \ldots, M(\gamma)$, where $M(\gamma)$ is defined in (1.14). Then the singular values of the corresponding Hankel operator H_a, where $a(j) = a_0(|j|)$, for all $j \in \mathbb{N}_0^d$, satisfy the following estimate

$$s_n(H_a) = O(n^{-\gamma}), \ n \to +\infty. \tag{3.2}$$

In addition, there exists a positive constant $C_\gamma = C(\gamma)$ such that

$$\|H_a\|_{\mathbf{S}_{p,\infty}} \leq C_\gamma \sup_{j \geq 0} (j+1)^{d+m} \big(\log(j+2)\big)^\gamma |a_0(j)|, \tag{3.3}$$

where $p = \frac{1}{\gamma}$.

Lemma 3.2. *Let $\{a_0(j)\}_{j\in\mathbb{N}_0}$ be a sequence of complex numbers such that*

$$a_0^{(m)}(j) = o\big(j^{-d-m}(\log j)^{-\gamma}\big),\ j\to+\infty,\ \forall m=0,1,\dots,M(\gamma), \tag{3.4}$$

where $\gamma>0$ and $M(\gamma)$ is defined in (1.14). Then the corresponding Hankel operator H_a, where $a(j)=a_0(|j|)$, for all $j\in\mathbb{N}_0^d$, is compact and its singular values satisfy the following estimate

$$s_n(H_a)=o(n^{-\gamma}),\ n\to+\infty.$$

Theorem 3.3. *Let $\gamma>0$ and a_0 be a real valued sequence of $\mathbb{N}_0$, such that*

$$a_0(j)=j^{-d}(\log j)^{-\gamma}+g(j),$$

where the error sequence $g(j)$ satisfies condition (3.4). If H_a is the Hankel operator, associated with the sequence $a(j)=a_0(|j|)$, $\forall j\in\mathbb{N}_0^d$, then it is compact and its eigenvalues satisfy the following asymptotic law

$$\lambda_n^\pm(H_a)=C_{d,\gamma}^\pm n^{-\gamma}+o(n^{-\gamma}),\ n\to+\infty, \tag{3.5}$$

where

$$C_{d,\gamma}^+=\frac{1}{2^d(d-1)!}\left(\int_{\mathbb{R}}\left(\mathcal{F}^{-1}k_d\right)^{\frac{1}{\gamma}}(x)\,\mathrm{d}x\right)^{\gamma}\quad\text{and}\quad C_{d,\gamma}^-=0, \tag{3.6}$$

where

$$k_d(x):=\frac{1}{\cosh^d\left(\frac{x}{2}\right)},\ \forall x\in\mathbb{R}.$$

3.1.2 The case of arbitrary $\kappa\in\mathbb{R}_+^d$

We proceed to the presentation of the general case, where κ is an arbitrary constant in $\mathbb{R}_+^d$. For any $\gamma>0$, define

$$M_0(\gamma)=\begin{cases}[\gamma]+1, & \gamma>1\\ 2, & \frac{1}{2}\le\gamma\le 1\\ 0, & \gamma\in\left(0,\frac{1}{2}\right)\end{cases}, \tag{3.7}$$

As it happens in the special case of $\kappa=\mathbf{1}$ with $M(\gamma)$ (see (1.14)), $M_0(\gamma)$ indicates the sufficient smoothness conditions that are required so that we obtain certain Schatten class inclusions. Observe that for the case of $\kappa=\mathbf{1}$, when $\gamma\in(\frac{1}{2},1)$, it is sufficient to require smoothness of order one, in contrast with the general case, where a higher order is required.

Lemma 3.4. *Let $\gamma>0$, $\kappa\in\mathbb{R}_+^d$, and a_0 be a real valued function such that*

$$a_0^{(m)}(t)=O\left(t^{-d-m}|\log t|^{-\gamma}\right),\ t\to+\infty, \tag{3.8}$$

for every $m=0,1,\dots,M_0$, where M_0 is defined in (3.7). Consider the Hankel operator $H_a:\ell^2(\mathbb{N}_0^d)\to\ell^2(\mathbb{N}_0^d)$, with kernel $a(j)=a_0(\kappa\cdot j)$, $\forall j\in\mathbb{N}_0^d$. Then H_a belongs to any Schatten class $\mathbf{S}_p$, for $p>\frac{1}{\gamma}$.

Theorem 3.5. *Let $\gamma > 0$, $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_d) \in \mathbb{R}_+^d$ and a_0 be a real valued function such that*

$$a_0(t) = t^{-d}(\log t)^{-\gamma} + g(t),$$

where the error term g satisfies

$$g^{(m)}(t) = O\left(t^{-d-m}|\log t|^{-\gamma-\varepsilon}\right), \ t \to +\infty, \tag{3.9}$$

for every $m = 0, 1, \dots, M_0$, where M_0 is defined in (3.7) and ε an arbitrary positive constant. Consider the Hankel operator $H_a : \ell^2(\mathbb{N}_0^d) \to \ell^2(\mathbb{N}_0^d)$, with kernel $a(j) = a_0(\kappa \cdot j)$, $\forall j \in \mathbb{N}_0^d$. Then H_a is a compact Hankel operator whose eigenvalue asymptotic behaviour is described by the following formula:

$$\lambda_n^{\pm}(H_a) = \frac{C_{d,\gamma}^{\pm}}{\kappa_1 \kappa_2 \dots \kappa_d} n^{-\gamma} + o(n^{-\gamma}), \ n \to +\infty,$$

where the constants $C_{d,\gamma}^{\pm}$ are defined in (3.6).

At this point it's good to have a few remarks. As it was highlighted in the introductory Chapter 0, the discrete case for general $\kappa \in \mathbb{R}_+^d$, in contrast with the special case of $\kappa = \mathbf{1}$ or the continuous case, hides some underlying difficulties. The way we try to overcome this kind of technicalities manifests itself from the very start with Lemma 3.4. This lemma is a substitute of Lemmas 3.1 and 3.2. The major difference is that the two aforementioned lemmas result inclusion in the $\mathbf{S}_{\frac{1}{\gamma},\infty}$ and $\mathbf{S}^0_{\frac{1}{\gamma},\infty}$ class, respectively. On the other hand, Lemma 3.4 gives only Schatten class inclusions for $p > \frac{1}{\gamma}$. Besides, observe that the smoothness condition for Lemma 3.4 is a bit stricter. For notice that for $\gamma \in [\frac{1}{2}, 1]$ we demand smoothness of degree 2, in contrast with Lemmas 3.1 and 3.2, where the respective degree was 1.

Finally, by a similar comparison between Theorems 3.3 and 3.5 we observe that, even if the eigenvalue asymptotics are actually alike, the preconditions differ. Particularly, notice that the error term in (3.9) is required to decay faster than the special case for $\kappa = \mathbf{1}$ (compare with condition (3.4)). More precisely, the condition (3.9) is a sub-case of the respective condition (3.4) for the special case of $\kappa = \mathbf{1}$.

3.2 Continuous case

In what follows, we remind that the function $\langle \cdot \rangle$ is defined in (1.16).

Lemma 3.6. *Let $\gamma > 0$ and $\mathbf{a_0}$ be a real valued function in $C^M(\mathbb{R}_+)$ such that*

$$\mathbf{a_0}^{(m)}(t) = O\left(t^{-d-m} \langle \log t \rangle^{-\gamma}\right), \ \text{when } t \to 0^+ \text{ and } t \to +\infty, \tag{3.10}$$

for $m = 0, 1, \cdots, M$, where $M = M(\gamma)$, as it is defined in (1.14). Then the Hankel operator $\mathbf{H_a} : L^2(\mathbb{R}_+^d) \to L^2(\mathbb{R}_+^d)$, where $\mathbf{a}(x_1, x_2, \dots, x_d) = \mathbf{a_0}(x_1 + x_2 + \dots + x_d)$, has singular values that obey the following asymptotic formula:

$$s_n(\mathbf{H_a}) = O(n^{-\gamma}), \ n \to +\infty.$$

In addition, there exists a positive constant $C_\gamma = C(\gamma)$ such that

$$\|\mathbf{H_a}\|_{\mathbf{S}_{p,\infty}} \le C_\gamma \sum_{m=0}^{M} \sup_{t>0} t^{d+m} \langle \log t \rangle^{\gamma} |\mathbf{a_0}^{(m)}(t)|, \tag{3.11}$$

where $p = \frac{1}{\gamma}$.

Lemma 3.7. *Let $\gamma > 0$ and $\mathbf{a_0}$ be a real valued function in $C^M(\mathbb{R}_+)$ such that*

$$|\mathbf{a_0}^{(m)}(t)| = o\left(t^{-d-m}\langle\log t\rangle^{-\gamma}\right), \text{ when } t \to 0 \text{ and } t \to +\infty, \tag{3.12}$$

for every $m = 0, 1, \dots, M$, where $M = M(\gamma)$, as it is defined in (1.14). Then

$$s_n(\mathbf{H_a}) = o(n^{-\gamma}), \; n \to +\infty. \tag{3.13}$$

Theorem 3.8. *Let b_0 and b_∞ be two non-negative numbers. Moreover, let $\gamma > 0$ and $\mathbf{a_0}$ be a real valued function in $L^1_{\text{loc}}(\mathbb{R}_+)$ which belongs to $C^M(\mathbb{R}_+)$, in case that $\gamma \geq \frac{1}{2}$; where $M = M(\gamma)$ is defined in (1.14). Assume that*

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left(\mathbf{a_0}(t) - b_0 t^{-d}|\log t|^{-\gamma}\right) = o\left(t^{-d-m}\langle\log t\rangle^{-\gamma}\right), \; t \to 0^+, \tag{3.14}$$

and

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left(\mathbf{a_0}(t) - b_\infty t^{-d}|\log t|^{-\gamma}\right) = o\left(t^{-d-m}\langle\log t\rangle^{-\gamma}\right), \; t \to +\infty, \tag{3.15}$$

for all $m = 0, 1, \dots, M$. Moreover, let $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{R}^d_+$. Then the Hankel operator $\mathbf{H_a}$, with kernel $\mathbf{a}(\mathbf{x}) = \mathbf{a_0}(\kappa \cdot \mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^d_+$, is compact and its eigenvalues obey the following asymptotic law:

$$\lambda_n^\pm(\mathbf{H_a}) = \frac{C^\pm_{d,\gamma}}{\kappa_1\kappa_2\dots\kappa_d} n^{-\gamma} + o(n^{-\gamma}), \; n \to +\infty, \tag{3.16}$$

where

$$C^+_{d,\gamma} = \frac{1}{2^d(d-1)!}\left(b_0^{\frac{1}{\gamma}} + b_\infty^{\frac{1}{\gamma}}\right)^\gamma \left(\int_{\mathbb{R}} \left(\mathcal{F}^{-1}k_d\right)^{\frac{1}{\gamma}}(x)\,\mathrm{d}x\right)^\gamma, \quad \text{and} \quad C^-_{d,\gamma} = 0, \tag{3.17}$$

and $k_d(x) = \left(\cosh\left(\frac{x}{2}\right)\right)^{-d}$, $\forall x \in \mathbb{R}$.

Chapter 4

Proofs

4.1 Outline

Our main goal is to find eigenvalue asymptotics for a given Hankel operator H_a (see Theorems 3.3, 3.5 and 3.8). Note that we have adopted the discrete case notation, but the same basic ideas can be applied for the continuous case, too. As it has been already illustrated in the introductory Chapter 0, the proving method relies on four basic ideas:

(i) The construction of a model operator,

(ii) reduction of the model operator to pseudo-differential operators,

(iii) reduction to one-dimensional weighted Hankel operators, and

(iv) Schatten class inclusions.

The model operator will be a Hankel operator, denoted by $\tilde{H}$, whose kernel behaves "similarly" to the kernel of H_a, a. $\tilde{H}$ will be responsible for the main contribution in the eigenvalue asymptotics. Its asymptotics will be obtained after a reduction to pseudo-differential operators. More precisely, $\tilde{H}$ will be expressed as a sum $\tilde{H} = S + E$, where S is unitarily equivalent to a pseudo-differential operator and E belongs to any Schatten class $\mathbf{S}_p$, for $p > 0$. The latter is proved in Lemmas 4.17 and 4.23, for the discrete and the continuous case, respectively. Notice that in Lemma 4.23, the operator E is just the identically zero operator.

Following the discussion above, the reduction to pseudo-differential operators is the technique that leads to the conclusion that S is unitarily equivalent (modulo null-spaces) to a pseudo-differential operator. This is a technique that was also applied by H. Widom in [33], as well as, by A. Pushnitski and D. Yafaev in [23]. In fact, we are going to prove that S can be expressed as a product of two operators, $S = L_w^* L_w$ and use the following lemma (cf. [4, §8.1, Theorem 4]):

Lemma 4.1. *Let T be a linear bounded operator, defined on a Hilbert space. Then the non-zero parts of T^*T and TT^* are unitarily equivalent.*

Then S will be unitarily equivalent (modulo null-spaces) to $L_w L_w^*$. The latter will be proved to be a pseudo-differential operator of the form $\mathcal{M}_\beta \alpha(D) \mathcal{M}_\beta$, whose eigenvalue asymptotics can be computed.

In order to order to obtain the eigenvalue asymptotics for H_a, we aim to express H_a as $H_a = \tilde{H} + (H_a - \tilde{H})$ and prove that the spectral contribution of the Hankel operator

$H_a - \tilde{H}$ is negligible, compared to that one of $\tilde{H}$. Then the eigenvalue asymptotics will be obtained by using K. Fan's lemma (see [12, Chapter II, §5, Theorem 2.3]):

Lemma 4.2 (K. Fan)**.** *Let S and T be two compact, self-adjoint operators on a Hilbert space. If*

$$\lambda_n^{\pm}(S) = K^{\pm} n^{-\gamma} + o(n^{-\gamma}), \quad \textit{and} \quad s_n(T) = o(n^{-\gamma}), \quad n \to +\infty,$$

then

$$\lambda_n^{\pm}(S+T) = K^{\pm} n^{-\gamma} + o(n^{-\gamma}),$$

for any positive constants $K^{\pm}$.

Finally, the Schatten class inclusions are used to prove that the spectral contribution of $H_a - \tilde{H}$ is indeed negligible. In fact, we prove that, under certain assumptions, the operator $H_a - \tilde{H}$ belongs to a specific compact operator ideal. These assumptions, for the discrete case, are examined by Lemmas 3.2, when $\kappa = \mathbf{1}$, and 3.4, for arbitrary $\kappa \in \mathbb{R}_+^d$. For the continuous case, the assumptions are given by Lemma 3.7.

4.2 Discrete case

4.2.1 The special case of $\kappa = \mathbf{1}$

In order to prove Lemma 3.1 we need to split the range of γ into two parts; $(0, \frac{1}{2})$ and $[\frac{1}{2}, +\infty)$. This choice is suggested by some interpolation methods which yield the asymptotic behaviour when $\gamma \in (0, \frac{1}{2})$. The remaining case is approached via weighted Hankel operators and interpolation, as well.

4.2.1.1 Proof of Lemmas 3.1 and 3.2

In order to deal with the case when $\gamma \in (0, \frac{1}{2})$, we need the following Lemma:

Lemma 4.3. *Let $\upsilon = \{\upsilon(j)\}_{j \in \mathbb{N}_0^d}$ be a positive valued sequence and suppose that there exist some positive constants M_2 and M_∞ such that*

$$\|H_a\|_{\mathbf{S}_2} \le M_2 \left\| \frac{a}{\upsilon} \right\|_{\ell^2_\upsilon} \tag{4.1}$$

and

$$\|H_a\| \le M_\infty \left\| \frac{a}{\upsilon} \right\|_{\ell^\infty}, \tag{4.2}$$

for every sequence a defined on $\mathbb{N}_0^d$, where H_a is the Hankel operator with kernel a. Then, for every $p \in (2, +\infty)$, there exists a positive constant M_p such that

$$\|H_a\|_{\mathbf{S}_{p,\infty}} \le M_p \left\| \frac{a}{\upsilon} \right\|_{\ell^{p,\infty}_\upsilon}.$$

Proof. Suppose that for a sequence a on $\mathbb{N}_0^d$ and $p \in (2, +\infty)$, $\frac{a}{\upsilon} \in \ell^{p,\infty}_\upsilon$. Then

$$\sum_{\{j \in \mathbb{N}_0^d : |\frac{a(j)}{\upsilon(j)}| > \lambda\}} \upsilon(j) \le \left(\frac{\|\frac{a}{\upsilon}\|_{\ell^{p,\infty}_\upsilon}}{\lambda} \right)^p, \ \forall \lambda > 0.$$

For an arbitrary $\lambda > 0$, let us define the sequences x_λ and y_λ as follows:

$$x_\lambda(j) = \begin{cases} \frac{a(j)}{\upsilon(j)}, & \text{if } \left|\frac{a(j)}{\upsilon(j)}\right| \le \frac{\lambda}{2M_\infty} \\ 0, & \text{otherwise} \end{cases}$$

and

$$y_\lambda(j) = \begin{cases} \frac{a(j)}{\upsilon(j)}, & \text{if } \left|\frac{a(j)}{\upsilon(j)}\right| > \frac{\lambda}{2M_\infty} \\ 0, & \text{otherwise} \end{cases},$$

for every $j \in \mathbb{N}_0^d$. Then

$$\pi_{H_a}(\lambda) \le \pi_{H_{\upsilon x_\lambda}}\left(\frac{\lambda}{2}\right) + \pi_{H_{\upsilon y_\lambda}}\left(\frac{\lambda}{2}\right).$$

The sequence of singular values $\{s_n(H_{\upsilon x_\lambda})\}_{n\in\mathbb{N}}$ is decreasing so that

$$s_n(H_{\upsilon x_\lambda}) \le s_1(H_{\upsilon x_\lambda}) = \|H_{\upsilon x_\lambda}\| \le M_\infty \frac{\lambda}{2M_\infty} = \frac{\lambda}{2}, \ \forall n \in \mathbb{N},$$

where the last inequality comes after (4.2). Consequently, $\pi_{H_{\upsilon x_\lambda}}(\frac{\lambda}{2}) = 0$ and

$$\pi_{H_a}(\lambda) \le \pi_{H_{\upsilon y_\lambda}}\left(\frac{\lambda}{2}\right). \tag{4.3}$$

Besides, $y_\lambda \in \ell^2_\upsilon$. In order to see this we make use of the following formula:

$$\|y_\lambda\|^2_{\ell^2_\upsilon} = 2\int_0^{+\infty} s \sum_{\{j\in\mathbb{N}_0^d : |y_\lambda(j)|>s\}} \upsilon(j)\,\mathrm{d}s\,;$$

see [13]. Thus, we obtain

$$\begin{aligned} \|y_\lambda\|^2_{\ell^2_\upsilon} &= 2\sum_{\left\{j\in\mathbb{N}_0^d : \left|\frac{a(j)}{\upsilon(j)}\right|>\frac{\lambda}{2M_\infty}\right\}} \upsilon(j)\int_0^{\frac{\lambda}{2M_\infty}} s\,\mathrm{d}s + 2\int_{\frac{\lambda}{2M_\infty}}^{+\infty} s \sum_{\left\{j\in\mathbb{N}_0^d : \left|\frac{a(j)}{\upsilon(j)}\right|>s\right\}} \upsilon(j)\,\mathrm{d}s \\ &\le 2\left\|\frac{a}{\upsilon}\right\|^p_{\ell^{p,\infty}_\upsilon} \lambda^{-p}(2M_\infty)^p \int_0^{\frac{\lambda}{2M_\infty}} s\,\mathrm{d}s + 2\left\|\frac{a}{\upsilon}\right\|^p_{\ell^{p,\infty}_\upsilon}\int_{\frac{\lambda}{2M_\infty}}^{+\infty} s^{1-p}\,\mathrm{d}s \\ &= \left\|\frac{a}{\upsilon}\right\|^p_{\ell^{p,\infty}_\upsilon} \lambda^{2-p}(2M_\infty)^{p-2} + \frac{2}{p-2}\left\|\frac{a}{\upsilon}\right\|^p_{\ell^{p,\infty}_\upsilon} \lambda^{2-p}(2M_\infty)^{p-2} \\ &= \frac{p}{p-2}\left\|\frac{a}{\upsilon}\right\|^p_{\ell^{p,\infty}_\upsilon} \lambda^{2-p}(2M_\infty)^{p-2} < +\infty, \end{aligned} \tag{4.4}$$

so $y_\lambda \in \ell^2_\upsilon$. Moreover, assumption (4.1) gives

$$\|H_{\upsilon y_\lambda}\|_{\mathbf{S}_{2,\infty}} \le \|H_{\upsilon y_\lambda}\|_{\mathbf{S}_2} \le M_2\|y_\lambda\|_{\ell^2_\upsilon}.$$

Therefore, a combination of (4.3) and (4.4) results

$$\begin{aligned} \pi_{H_a}(\lambda) &\le \pi_{H_{\upsilon y_\lambda}}\left(\frac{\lambda}{2}\right) \\ &\le 2^2\lambda^{-2}\|H_{\upsilon y_\lambda}\|^2_{\mathbf{S}_{2,\infty}} \\ &\le 2^2\lambda^{-2}M_2^2\|y_\lambda\|^2_{\ell^2_\upsilon} \\ &\le \lambda^{-p}\frac{2^p p M_2^2 M_\infty^{p-2}}{p-2}\left\|\frac{a}{\upsilon}\right\|^p_{\ell^{p,\infty}_\upsilon}. \end{aligned} \tag{4.5}$$

Now we set

$$M_p := \left(\frac{2^p p M_2^2 M_\infty^{p-2}}{p-2}\right)^{\frac{1}{p}}$$

and we notice that relation (4.5) does not depend on the choice of λ. Thus, after multiplying by λ^p both the two sides of (4.5) and taking supremum, we conclude that

$$\|H_a\|_{\mathbf{S}_{p,\infty}} \le M_p \left\|\frac{a}{v}\right\|_{\ell_v^{p,\infty}}.$$

□

Regarding the case when $\gamma \ge \frac{1}{2}$, we need to reduce the problem to the one dimensional case. This is achieved via a reduction to one-dimensional weighted Hankel operators and this procedure is fully described in §2.1.1. There we saw that H_a is unitarily equivalent (modulo null-spaces) to Γ (see (2.3)). We also observed that we can deduce Schatten class inclusions for Γ from those of $\Gamma_{a_0}^{\frac{d-1}{2},\frac{d-1}{2}}$ (see (1.3)). In order to obtain the aforementioned inclusions for $\Gamma_{a_0}^{\frac{d-1}{2},\frac{d-1}{2}}$, we will need the following lemma.

Lemma 4.4. *Define the measure space*

$$(\mathcal{M},\mu) := \bigoplus_{n\in\mathbb{N}_0} (\mathbb{T}, 2^n\mathbf{m}),$$

where $\mathbf{m}$ *is the Lebesgue measure on* $\mathbb{T}$*. Let* ϕ *be an analytic function in* $\mathbb{D}$*, described by*

$$\phi(z) = \sum_{n\in\mathbb{N}_0} a_0(n) z^n, \ \forall z \in \mathbb{D}.$$

Let $p \in (0,+\infty)$*, and* $q \in (0,+\infty]$*. If* $\bigoplus_{n\in\mathbb{N}_0} 2^{n(d-1)}\phi * V_n \in L^{p,q}(\mathcal{M},\mu)$*, where the polynomials* V_n *are defined in (1.6) and (1.7), then the weighted Hankel operator* $\Gamma_{a_0}^{\frac{d-1}{2},\frac{d-1}{2}}$ *belongs to the Schatten-Lorentz class* $\mathbf{S}_{p,q}$*.*

Proof. We consider the space $\mathcal{B}_{p,q}^{\frac{1}{p}+d-1}$ of all analytic functions f of $\mathbb{D}$ such that

$$\bigoplus_{n\in\mathbb{N}_0} 2^{n(d-1)} f * V_n \in L^{p,q}(\mathcal{M},\mu).$$

The claim is equivalent to the fact that the mapping $f \mapsto \Gamma_{\hat{f}}^{\frac{d-1}{2},\frac{d-1}{2}}$ is a bounded linear operator from $\mathcal{B}_{p,q}^{\frac{1}{p}+d-1}$ to $\mathbf{S}_{p,q}$. Notice that, according to Theorem 1.10, the mapping $f \mapsto \Gamma_{\hat{f}}^{\frac{d-1}{2},\frac{d-1}{2}}$ represents a bounded linear operator from $B_p^{\frac{1}{p}+d-1}$ to $\mathbf{S}_p \subset \mathbf{S}_\infty$, $\forall p \in (0,+\infty)$. Furthermore, by using the real interpolation method and the reiteration theorem (see Appendix A), it can be proved (cf. [16, (11)]) that, for every $p_0 \in (0,+\infty)$, $\theta \in (0,1)$ and $q \in (0,+\infty]$,

$$(\mathbf{S}_{p_0}, \mathbf{S}_\infty)_{\theta,q} = \mathbf{S}_{p,q}, \quad \text{where} \ \ p = \frac{p_0}{1-\theta}.$$

Thus, in order to prove the initial statement, it remains to prove that, for every $p_0, p_1 \in (0,+\infty)$, $\theta \in (0,1)$ and $q \in (0,+\infty]$,

$$\left(B_{p_0}^{\frac{1}{p_0}+d-1}, B_{p_1}^{\frac{1}{p_1}+d-1}\right)_{\theta,q} = \mathcal{B}_{p,q}^{\frac{1}{p}+d-1}, \quad \text{where} \ \ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

To this end, we make use of the retract argument (see Appendix A). First notice that for every $p_0, p_1 \in (0, +\infty)$, $\theta \in (0,1)$ and $q \in (0, +\infty]$,

$$\big(L^{p_0}(\mathcal{M},\mu), L^{p_1}(\mathcal{M},\mu)\big)_{\theta,q} = L^{p,q}(\mathcal{M},\mu), \quad \text{where } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1};$$

(cf. [3, Theorem 5.3.1]). Let $Hol(\mathbb{D})$ be the space of the holomorphic functions on $\mathbb{D}$ and define the linear operator

$$\mathcal{J}f = \bigoplus_{n\in\mathbb{N}_0} 2^{n(d-1)} f * V_n, \ \forall f \in Hol(\mathbb{D}),$$

where the polynomials V_n are defined in (1.6) and (1.7). Then, by the definition of the Besov space $B_p^{\frac{1}{p}+d-1}$, $\mathcal{J}$ is an isometry from $B_p^{\frac{1}{p}+d-1}$ to $L^p(\mathcal{M},\mu)$. In addition, we define the polynomials

$$\tilde{V}_0(z) = V_0(z) + V_1(z), \ \forall z \in \mathbb{T},$$

and, for every $n \in \mathbb{N}$,

$$\tilde{V}_n(z) = V_{n-1}(z) + V_n(z) + V_{n+1}(z), \ \forall z \in \mathbb{T}.$$

Notice that $V_n * \tilde{V}_n = V_n$, for every $n \in \mathbb{N}_0$. Now we define the linear operator

$$\mathcal{K} \bigoplus_{n\in\mathbb{N}_0} f_n = \sum_{n\in\mathbb{N}_0} 2^{-n(d-1)} f_n * \tilde{V}_n, \ \forall \bigoplus_{n\in\mathbb{N}_0} f_n \in L^p(\mathcal{M},\mu),$$

which is bounded from $L^p(\mathcal{M},\mu)$ to $B_p^{\frac{1}{p}+d-1}$. To see this, it is enough to check that

$$\sum_{n\in\mathbb{N}_0} 2^{n[1+p(d-1)]} \Big\| \sum_{m\in\mathbb{N}_0} 2^{-m(d-1)} f_m * \tilde{V}_m * V_n \Big\|_p^p < +\infty, \ \forall \bigoplus_{n\in\mathbb{N}_0} f_n \in L^p(\mathcal{M},\mu). \tag{4.6}$$

Now notice that

$$\sum_{m\in\mathbb{N}_0} f_m * \tilde{V}_m * V_0 = f_0 * V_0 \tag{4.7}$$

and, for every $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{m\in\mathbb{N}_0} f_m * \tilde{V}_m * V_n &= f_{n-1} * \tilde{V}_{n-1} * V_n + f_n * V_n * V_n + f_{n+1} * \tilde{V}_{n+1} * V_n \\ &= f_{n-1} * \tilde{V}_{n-1} * V_n + f_n * V_n + f_{n+1} * \tilde{V}_{n+1} * V_n.\end{aligned}$$

Thus, for every $n \in \mathbb{N}$,

$$\begin{aligned}\Big\| \sum_{m\in\mathbb{N}_0} 2^{-m(d-1)} f_m * \tilde{V}_m * V_n \Big\|_p^p \lesssim \|2^{-(n-1)(d-1)} f_{n-1} * \tilde{V}_{n-1} * V_n\|_p^p + \|2^{-n(d-1)} f_n * V_n\|_p^p + \\ + \|2^{-(n+1)(d-1)} f_{n+1} * \tilde{V}_{n+1} * V_n\|_p^p.\end{aligned} \tag{4.8}$$

Now we split the remaining of the proof into two cases: $p \in (1, +\infty)$, and $p \in (0, 1]$. Assume first that $p \in (1, +\infty)$. Moreover the function v (see §1.4.2.1) belongs to $C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz class on $\mathbb{R}$. Therefore, for any $m, n \in \mathbb{N}_0$,

$$\sup_{x\in\mathbb{R}} |x|^m |v^{(n)}(x)| < +\infty. \tag{4.9}$$

Thus, according to Theorem B.1, $v \in \mathscr{M}_p(\mathbb{R})$. Furthermore, according to Theorem B.2, $v_{n_{|\mathbb{Z}}} \in \mathscr{M}_p(\mathbb{T})$, with $\|v_{n_{|\mathbb{Z}}}\|_{\mathscr{M}_p(\mathbb{T})} \leq \|v\|_{\mathscr{M}_p(\mathbb{R})}$, for every $n \in \mathbb{N}$, where the functions v_n are defined in (1.5). Consequently, there exists a positive constant C_p, such that

$$\|f * V_n\|_p \leq C_p \|f\|_p, \ \forall f \in L^p(\mathbb{T}), \ \forall n \in \mathbb{N}. \tag{4.10}$$

For $n = 0$, notice that

$$\big(f * V_0\big)(z) = \hat{f}(-1)\overline{z} + \hat{f}(0) + \hat{f}(1)z, \ \forall z \in \mathbb{T},$$

and

$$\big|\big(f * V_0\big)(z)\big|^p \leq 3^p \left(\int_0^1 \big|f(e^{2\pi\theta})\big| \,\mathrm{d}\theta\right)^p, \ \forall z \in \mathbb{T}.$$

Thus, by applying Hölder's inequality and integrating over $\mathbb{T}$, we obtain

$$\|f * V_0\|_p \lesssim \|f\|_p \,, \ \forall f \in L^p(\mathbb{T}). \tag{4.11}$$

Thus, a combination of (4.10) and (4.11), yields that there exists a positive constant M_p such that

$$\|f * V_n\|_p \leq M_p \|f\|_p, \ \forall f \in L^p(\mathbb{T}), \ \forall n \geq 0. \tag{4.12}$$

Moreover, by (4.12) we get

$$\begin{aligned}\sum_{n\in\mathbb{N}} 2^{n[1+p(d-1)]} \|2^{-(n-1)(d-1)} f_{n-1} * \tilde{V}_{n-1} * V_n\|_p^p &= \sum_{n\in\mathbb{N}_0} 2^{(n+1)[1+p(d-1)]} \|2^{-n(d-1)} f_n * \tilde{V}_n * V_{n+1}\|_p^p \\ &\lesssim \sum_{n\in\mathbb{N}_0} 2^{n[1+p(d-1)]} \|2^{-n(d-1)} f_n\|_p^p,\end{aligned}$$

and similarly,

$$\sum_{n\in\mathbb{N}} 2^{n[1+p(d-1)]} \|2^{-(n+1)(d-1)} f_{n+1} * \tilde{V}_{n+1} * V_n\|_p^p \lesssim \sum_{n\in\mathbb{N}_0} 2^{n[1+p(d-1)]} \|2^{-n(d-1)} f_n\|_p^p.$$

Therefore, by applying (4.7), (4.8), and (4.12) we finally get that

$$\begin{aligned}\sum_{n\in\mathbb{N}_0} 2^{n[1+p(d-1)]} \Big\| \sum_{m\in\mathbb{N}_0} 2^{-m(d-1)} f_m * \tilde{V}_m * V_n \Big\|_p^p &= \|f_0 * V_0\|_p^p + \\ &+ \sum_{n\in\mathbb{N}} 2^{n[1+p(d-1)]} \Big\| \sum_{m\in\mathbb{N}_0} 2^{-m(d-1)} f_m * \tilde{V}_m * V_n \Big\|_p^p,\end{aligned}$$

where the RHS is bounded from above (modulo multiplicative constants) by

$$\begin{aligned}\|f_0\|_p^p + \sum_{n\in\mathbb{N}_0} 2^{n[1+p(d-1)]} \|2^{-n(d-1)} f_n\|_p^p &\lesssim \sum_{n\in\mathbb{N}_0} 2^{n[1+p(d-1)]} \|2^{-n(d-1)} f_n\|_p^p \\ &= \Big\| \bigoplus_{n\in\mathbb{N}_0} f_n \Big\|_p^p,\end{aligned}$$

which actually proves (4.6). Finally, we have that, for any analytic function f,

$$\begin{aligned}\mathcal{K}\mathcal{J} f &= \sum_{n\in\mathbb{N}_0} f_n * V_n * \tilde{V}_n \\ &= \sum_{n\in\mathbb{N}_0} f_n * V_n = f,\end{aligned}$$

so the proof for $p > 1$ is complete.

Now let $p \in (0, 1]$ and we again want to prove (4.6). To this end, we need to show that the sequence $\{V_n\}_{n\in\mathbb{N}_0}$ defines a uniformly bounded sequence of $\mathscr{M}_p(\mathbb{T})$ multipliers. Moreover, notice that, since we only deal with bounded operators, it makes sense to restrict our investigation on multipliers of the Hardy space $H^p(\mathbb{T})$. This happens because the kernel of a weighted Hankel operator could be assumed analytic (cf. [2, Theorem A]). Therefore, due to Theorem B.3, it is enough to prove that v defines a multiplier on $H^p(\mathbb{R})$. Observe that this will also prove that $V_0 \in \mathscr{M}_p(\mathbb{T})$, as it is explained below.
Let $\mathscr{R}$ and $\mathscr{L}$ be the right and the left shift, respectively, on $H^p(\mathbb{T})$. The operator $\mathscr{R}$ is actually multiplication by the function z and $\mathscr{L} = \mathbb{P}_+\overline{z}$; where $z \in \mathbb{T}$, and $\mathbb{P}_+$ denotes the analytic projection. Moreover, notice that $\mathscr{L}$ is bounded on the subspace $\{f = \sum_{n\in\mathbb{N}} a_n e^{in\cdot} \in H^p(\mathbb{T})\}$. The action of V_0, as a multiplier, on $H^p(\mathbb{T})$ could be described as

$$\big(V_0 f\big)(z) = a_0 + a_1 z, \forall z \in \mathbb{T}, \tag{4.13}$$

and every $f(z) = \sum_{n\geq 0} a_n z^n$, or equivalently,

$$\big(V_0 f\big)(z) = \big(\mathscr{L} v_{|\mathbb{Z}} \mathscr{R} f\big)(z) + \big(v_{|\mathbb{Z}} f\big)(z),\ \forall z \in \mathbb{T}, \tag{4.14}$$

where $v_{|\mathbb{Z}}$ denotes the action of the sequence $\{v(j)\}_{j\in\mathbb{N}}$ as a multiplier on $H^p(\mathbb{T})$. Therefore, (4.14) indicates that if $v_{|\mathbb{Z}}$ defines a multiplier on $H^p(\mathbb{T})$, V_0 defines a bounded operator on $H^p(\mathbb{T})$, whose action is described in (4.13).
Thus, as we mentioned, it only remains to prove that v defines a multiplier on $H^p(\mathbb{R})$. To this end, we only need to verify that v satisfies conditions (i) and (ii) of Theorem C.1. Notice that Theorem C.1 gives a sufficient condition in order a function defined on $(0, +\infty)$ to be a multiplier and therefore, the case of v_0 needs a slightly different manipulation, since $0 \in \operatorname{supp}(v_0)$. In order to prove that v satisfies indeed conditions (i) and (ii) of Theorem C.1, we choose some $k \in \mathbb{N}$ such that $k^{-1} < p$, and a positive number R. Ensuing, notice that $v \in C_c^\infty(\mathbb{R}_+)$, so it is bounded (by 1) and also, it satisfies (4.9). Now, for every $l = 1, \ldots, k$, define

$$C_{l,l} := \sup_{x\in\mathbb{R}_+} x^l |v^{(l)}(x)|$$

and notice that $C_{l,l} < +\infty$, for all $l = 1, \ldots, k$. Then it is easy to verify that

$$\int_R^{2R} |v^{(l)}(t)|^2 \,\mathrm{d}t \leq \frac{C_{l,l}^2}{2l-1}(1 - 2^{-2l+1})R^{-2l+1},\ \forall l = 1, \ldots, k.$$

Now define the following quantities

$$A_l := \frac{C_{l,l}^2}{2l-1}(1 - 2^{-2l+1}),\ \forall l = 1, \ldots, k, \quad \text{and} \quad A := \max_{1\leq l\leq k}\{1, A_l\},$$

so that

$$|v(t)| \leq A,\ \forall t \in \mathbb{R}_+, \quad \text{and} \quad \int_R^{2R} |v^{(l)}(t)|^2 \,\mathrm{d}t \leq AR^{-2l+1},\ \forall l = 1, \ldots, k,\ \forall R > 0,$$

and conditions (i) and (ii) of Theorem C.1 are satisfied, Q.E.D. □

Apart from Lemma 4.4, in order to handle the case when $\gamma \geq \frac{1}{2}$, we need the following lemma (cf. [24, Lemma 4.6]).

Lemma 4.5. *Assume that* $\gamma \geq \frac{1}{2}$ *and let* $M_\gamma := M(\gamma)$ *be as defined in (1.14). Moreover, let* $\{a_0(j)\}_{j\in\mathbb{N}_0}$ *be a real valued sequence which satisfies (3.1) and consider the function*

$$\phi(z) = \sum_{j\in\mathbb{N}_0} a_0(j) z^j, \ \forall z \in \mathbb{T}.$$

If V_n *are as defined in (1.7), then, for every* $q > \frac{1}{M_\gamma}$ *and every* $n \in \mathbb{N}$ *such that* $2^{n-1} \geq M_\gamma$,

$$\|\phi * V_n\|_\infty \leq \sum_{j=2^{n-1}}^{2^{n+1}} |a_0(j)|, \tag{4.15}$$

and

$$2^n \|\phi * V_n\|_q^q \leq C_q \left(\sum_{m=0}^{M_\gamma} \sum_{j=2^{n-1}-M_\gamma}^{2^{n+1}} (1+j)^m \left| a_0^{(m)}(j) \right| \right)^q, \tag{4.16}$$

for some positive constant C_q, *depending only on* q.

Now we are ready to proceed with the proof of Lemma 3.1.

Proof of Lemma 3.1. Recall (cf. §1.4.1) that the estimate (3.2) is equivalent to $H_a \in \mathbf{S}_{p,\infty}$, where $p = \frac{1}{\gamma}$; so, we aim to prove that $H_a \in \mathbf{S}_{p,\infty}$. First we deal with the case where $\gamma \in (0, \frac{1}{2})$. Observe that

$$\begin{aligned}
\|H_a\|_{\mathbf{S}_2}^2 &= \sum_{i,j\in\mathbb{N}_0^d} |a(i+j)|^2 \\
&= \sum_{i_1,j_1\geq 0} \sum_{i_2,j_2\geq 0} \cdots \sum_{i_d,j_d\geq 0} |a(i_1+j_1, i_2+j_2, \ldots, i_d+j_d)|^2 \\
&= \sum_{j_1,j_2,\ldots,j_d\geq 0} (j_1+1)(j_2+1)\ldots(j_d+1)|a(j_1,j_2,\ldots,j_d)|^2 \\
&\leq \sum_{j\in\mathbb{N}_0^d} (|j|+1)^d |a(j)|^2 \\
&= \sum_{j\in\mathbb{N}_0^d} (|j|+1)^{2d} |a(j)|^2 (|j|+1)^{-d}.
\end{aligned}$$

The latter suggests that $\|H_a\|_{\mathbf{S}_2} \leq \|\frac{a}{v}\|_{\ell^2_v}$, where v is given by

$$v(j) = \frac{1}{(|j|+1)^d}, \ \forall j \in \mathbb{N}_0^d.$$

For the same v, if $\frac{a}{v} \in \ell^\infty(\mathbb{N}_0^d)$, then

$$\begin{aligned}
|a(j)| &\leq \frac{\|\frac{a}{v}\|_{\ell^\infty}}{(|j|+1)^d} \\
&\leq \frac{\|\frac{a}{v}\|_{\ell^\infty}}{(j_1+1)(j_2+1)\ldots(j_d+1)}, \ \forall j \in \mathbb{N}_0^d.
\end{aligned}$$

Thus,

$$\begin{aligned}|(H_a x, y)| &\le \sum_{i,j\in\mathbb{N}_0^d} |a(i+j)|\,|x(j)|\,|y(i)| \\ &\le \left\|\frac{a}{\upsilon}\right\|_{\ell^\infty} \sum_{i_1,\dots,i_d,j_1,\dots,j_d\ge 0} \frac{|x(j)|\,|y(i)|}{(i_1+j_1+1)\dots(i_d+j_d+1)} \\ &\le \pi^d \left\|\frac{a}{\upsilon}\right\|_{\ell^\infty} \|x\|\,\|y\|\,,\ \forall x,y\in\ell^2(\mathbb{N}_0^d),\end{aligned}$$

where the last inequality comes by the boundedness of the tensor product of d Hilbert matrices and thus, $\|H_a\| \le \pi^d \|\frac{a}{\upsilon}\|_{\ell^\infty}$. In other words, we have shown that there are constants $M_2 = 1$ and $M_\infty = \pi^d$ such that

$$\|H_a\|_{\mathbf{S}_2} \le M_2 \left\|\frac{a}{\upsilon}\right\|_{\ell^2_\upsilon}$$

and

$$\|H_a\| \le M_\infty \left\|\frac{a}{\upsilon}\right\|_{\ell^\infty},$$

so that Lemma 4.3 is applicable and consequently, for every $p\in(2,+\infty)$, there exists a positive constant M_p such that

$$\|H_a\|_{\mathbf{S}_{p,\infty}} \le M_p \left\|\frac{a}{\upsilon}\right\|_{\ell^{p,\infty}_\upsilon}.$$

Now it remains to show that if a_0 satisfies (3.1), then $\frac{a}{\upsilon}\in\ell^{p,\infty}_\upsilon$, for every $p=\frac{1}{\gamma}\in(2,+\infty)$. For $\lambda>0$,

$$\begin{aligned}\left\{j\in\mathbb{N}_0^d : \frac{|a(j)|}{\upsilon(j)} > \lambda\right\} &= \left\{j\in\mathbb{N}_0^d : (|j|+1)^d|a_0(|j|)| > \lambda\right\} \\ &\subset \left\{j\in\mathbb{N}_0^d : \log(|j|+2) < \left(\frac{A_0}{\lambda}\right)^p\right\},\end{aligned}$$

where $A_0 := \sup_{j\ge 0}(j+1)^d\big(\log(j+2)\big)^\gamma|a_0(j)|$. Therefore,

$$\begin{aligned}\sum_{\left\{j\in\mathbb{N}_0^d:\ \frac{|a(j)|}{\upsilon(j)}>\lambda\right\}} \frac{1}{(|j|+1)^d} &\lesssim \sum_{\left\{j\in\mathbb{N}_0^d:\ \log(|j|+2)<\left(\frac{A_0}{\lambda}\right)^p\right\}} \frac{1}{(|j|+2)^d} \\ &= \sum_{\left\{j\in\mathbb{N}_0:\ \log(j+2)<\left(\frac{A_0}{\lambda}\right)^p\right\}} \frac{W_d(j)}{(j+2)^d} \\ &\lesssim \sum_{\left\{j\in\mathbb{N}_0:\ \log(j+2)<\left(\frac{A_0}{\lambda}\right)^p\right\}} \frac{(j+2)^{d-1}}{(j+2)^d} \\ &\lesssim \int_{\left\{\log(x+2)<\left(\frac{A_0}{\lambda}\right)^p\right\}} \frac{1}{x+2}\,\mathrm{d}x\,,\end{aligned}$$

where $W_d(j)$ was defined in (2.2). Thus, there exists some positive constant C such that

$$\lambda^p \sum_{\left\{j\in\mathbb{N}_0^d:\ \frac{|a(j)|}{\upsilon(j)}>\lambda\right\}} \frac{1}{(|j|+1)^d} \le (CA_0)^p,$$

which implies, by taking supremum over positive λs, that

$$\Big\|\frac{a}{v}\Big\|_{\ell_v^{p,\infty}} \le CA_0.$$

From the last relation and Lemma 4.3, we conclude that

$$\|H_a\|_{\mathbf{S}_{p,\infty}} \le M_p \Big\|\frac{a}{v}\Big\|_{\ell_v^{p,\infty}} \le M_p C A_0,$$

so that relation (3.3) comes true, by setting $C_\gamma = M_p C$.

Now assume that $\gamma \ge \frac{1}{2}$ and let ϕ be given by

$$\phi(z) = \sum_{j\in\mathbb{N}_0} a_0(j) z^j, \ \forall z \in \mathbb{T}.$$

According to the discussion that precedes Lemma 4.4 and the Lemma itself, it is enough to show that $\bigoplus_{n\ge0} 2^{n(d-1)}\phi * V_n \in L^{p,\infty}(\mathcal{M},\mu)$ or in other words, that

$$\sup_{s>0} s^p \sum_{n\in\mathbb{N}_0} 2^n \big|\{t\in[-\pi,\pi) : |2^{n(d-1)}(\phi * V_n)(e^{it})| > s\}\big| < +\infty. \tag{4.17}$$

For every non-negative integer n and any positive number s, set

$$E_n(s) := \{t\in[-\pi,\pi) : |2^{n(d-1)}(\phi * V_n)(e^{it})| > s\}.$$

Our goal is to find an estimate for $|E_n(s)|$ which proves the finiteness of (4.17). First of all, we notice that $E_n(s) = \varnothing$, for every $s \ge \|2^{n(d-1)}\phi * V_n\|_\infty$. An application of (4.15) gives that $E_n(s) = \varnothing$, for every $s \ge 2^{n(d-1)} \sum_{j=2^{n-1}}^{2^{n+1}} |a_0(j)|$. Let

$$A_m := \sup_{j\ge0} |a_0^{(m)}| (j+1)^{d+m} \big(\log(j+2)\big)^\gamma, \ \forall m \ge 0.$$

Therefore, condition (3.1) implies that $E_n(s) = \varnothing$ when

$$s \ge 2^{n(d-1)} A_0 \sum_{j=2^{n-1}}^{2^{n+1}} (j+1)^{-d} \big(\log(j+2)\big)^{-\gamma}.$$

Besides, for every $n \ge 3$ we have

$$\begin{aligned}
\sum_{j=2^{n-1}}^{2^{n+1}} (j+1)^{-d}\big(\log(j+2)\big)^{-\gamma} &\le \int_{2^{n-1}-1}^{2^{n+1}} (t+1)^{-d}\big(\log(t+2)\big)^{-\gamma}\,\mathrm{d}t \\
&\le \int_{2^{n-1}-1}^{2^{n+1}} t^{-d}(\log t)^{-\gamma}\,\mathrm{d}t \\
&\lesssim \int_{2^{n-1}}^{2^{n+1}} t^{-d}(\log t)^{-\gamma}\,\mathrm{d}t \\
&\lesssim \int_{n-1}^{n+1} 2^{-s(d-1)} s^{-\gamma}\,\mathrm{d}s \ \ (\text{change of variables } s = \log_2 t) \\
&\le 2^{1-(n-1)(d-1)}(n-1)^{-\gamma} \\
&\lesssim 2^{-n(d-1)} n^{-\gamma},
\end{aligned}$$

so that in general, we have that

$$\sum_{j=2^{n-1}}^{2^{n+1}} (j+1)^{-d}\big(\log(j+2)\big)^{-\gamma} \leq C2^{-n(d-1)} \langle n\rangle^{-\gamma}, \ \forall n \geq 0,$$

for some positive constant C. Now let q be an arbitrary number in $(\frac{1}{M_\gamma}, p)$, where $p = \gamma^{-1}$. Then, without loss of generality, we may assume that $C = C_q$, where C_q appears in (4.16). Therefore, $E_n(s) = \varnothing$, for every $n \geq 0$ such that $\langle n\rangle \geq N(s)$, where $N(s) := (\frac{C_q A_0}{s})^p$, $\forall s > 0$. Besides, by following exactly the same steps, it can be shown that

$$\sum_{j=2^{n-1}-M_\gamma}^{2^{n+1}} (j+1)^m |a_0^{(m)}(j)| \lesssim A_m 2^{-n(d-1)} \langle n\rangle^{-\gamma}, \ \forall m = 1, 2, \ldots, M_\gamma.$$

Thus, relation (4.16) gives that, for every $q \in (\frac{1}{M_\gamma}, p)$ and $n \in \mathbb{N}_0$ such that $M_\gamma \leq 2^{n-1}$,

$$2^n \|\phi * V_n\|_q^q \lesssim C_q A^q 2^{-n(d-1)q} \langle n\rangle^{-\gamma q},$$

where $A := \sum_{m=0}^{M_\gamma} A_m$. Now notice that, for any positive q,

$$s^q |E_n(s)| \leq 2^{n(d-1)q} \|\phi * V_n\|_q^q, \forall n \in \mathbb{N}_0.$$

Therefore, by putting all these together, we see that, for every $q \in (\frac{1}{M_\gamma}, p)$ and $s > 0$,

$$\begin{aligned}
s^p \sum_{n\in\mathbb{N}_0} 2^n |E_n(s)| &= s^{p-q}\left(s^q \sum_{\langle n\rangle \leq N(s)} 2^n |E_n(s)|\right)\\
&\leq s^{p-q} \sum_{\langle n\rangle \leq N(s)} 2^n 2^{n(d-1)q} \|\phi * V_n\|_q^q\\
&\lesssim s^{p-q} C_q A^q \sum_{\langle n\rangle \leq N(s)} 2^{n(d-1)q} 2^{-n(d-1)q} \langle n\rangle^{-\gamma q}\\
&\lesssim C_q A^q s^{p-q} N(s)^{1-\gamma q}, \ (\text{since } \gamma q = \frac{q}{p} \text{ and } q < p).
\end{aligned}$$

Finally, notice that $s^{p-q} N^{1-\gamma q}(s) = s^{p-q} (C_q A_0)^{p-q} s^{-(p-q)} = (C_q A_0)^{p-q}$, so there is a positive constant K, independent of s, such that

$$s^p \sum_{n\in\mathbb{N}_0} 2^n |E_n(s)| \leq K^p A^p, \ \forall s > 0.$$

The latter proves the validity of (4.17) (by taking supremum over all positive s). Besides, it is also equivalent to

$$\|\phi\|_{\mathcal{B}_{p,\infty}^{\frac{1}{p}+d-1}} \leq KA.$$

Finally, by Lemma 4.4, we know that there is a positive constant K_γ such that

$$\|\Gamma_{a_0}^{\frac{d-1}{2}, \frac{d-1}{2}}\|_{\mathbf{S}_{p,\infty}} \leq K_\gamma \|\phi\|_{\mathcal{B}_{p,\infty}^{\frac{1}{p}+d-1}}.$$

Therefore, by combining the last two relations,

$$\|\Gamma_{a_0}^{\frac{d-1}{2}, \frac{d-1}{2}}\|_{\mathbf{S}_{p,\infty}} \leq K_\gamma K A.$$

Finally, as it has been already mentioned, due to reduction of H_a to one-dimensional weighted Hankel operators,

$$\|H_a\|_{\mathbf{S}_{p,\infty}} = \|\Gamma\|_{\mathbf{S}_{p,\infty}} \lesssim \|\Gamma_{a_0}^{\frac{d-1}{2},\frac{d-1}{2}}\|_{\mathbf{S}_{p,\infty}},$$

where Γ is given by (2.3). Therefore, there exists a constant $C > 0$, such that

$$\|H_a\|_{\mathbf{S}_{p,\infty}} \leq CK_\gamma KA,$$

which gives us relation (3.3), with $C_\gamma = CK_\gamma K$. □

Proof of Lemma 3.2. By the definition of $\mathbf{S}^0_{p,\infty}$ (cf. §1.4.1), it is enough to show that $H_a \in \mathbf{S}^0_{p,\infty}$, for $p = \frac{1}{\gamma}$. The ideal $\mathbf{S}^0_{p,\infty}$ is the $\|\cdot\|_{\mathbf{S}_{p,\infty}}$-closure of finite rank operators. So, it is enough to approximate H_a by finite rank operators in the $\|\cdot\|_{\mathbf{S}_{p,\infty}}$ quasi-norm. For consider the cut-off function

$$\chi(t) = \begin{cases} 1, & t \in [0,1] \\ 0, & t \geq 2, \end{cases}$$

such that $\chi \in C^\infty(\mathbb{R}_+)$ and $\chi(\mathbb{R}_+) \subset [0,1]$. In addition, for every $N \in \mathbb{N}$, define the sequence $h_0^N(j) = a_0(j)\big(1 - \chi(\frac{j}{N})\big)$, $\forall j \in \mathbb{N}_0$, and let Γ_N be the Hankel operator with kernel $h_N(j) = h_0^N(|j|)$, $\forall j \in \mathbb{N}_0^d$. In other words, $\Gamma_N = H_a - H_N$, where H_N is a finite rank Hankel operator, with kernel equal to $a_0(|j|)\chi(\frac{|j|}{N})$, for every $j \in \mathbb{N}_0^d$. Then, by using the Leibniz rule,

$$\big(h_0^N\big)^{(m)}(j) = \sum_{n=0}^{m} \binom{m}{n} a_0^{(m-n)}(j+n)\big(1-\chi\big)^{(n)}\big(\frac{j}{N}\big),\ \forall j \in \mathbb{N}_0.$$

Therefore, for every $j \geq 2$,

$$\left|\big(h_0^N\big)^{(m)}(j) j^{d+m}(\log j)^\gamma\right| \leq \sum_{n=0}^{m} \binom{m}{n} \left|a_0^{(m-n)}(j+n)\right| j^{d+m-n}(\log j)^\gamma j^n \big(1-\chi\big)^{(n)}\big(\frac{j}{N}\big)$$

Then the RHS is less than or equal to

$$\sum_{n=0}^{m} \binom{m}{n} \left|a_0^{(m-n)}(j+n)\right| (j+n)^{d+m-n}(\log(j+n))^\gamma j^n \big(1-\chi\big)^{(n)}\big(\frac{j}{N}\big).$$

So that, for every $j \geq 2$,

$$\left|\big(h_0^N\big)^{(m)}(j) j^{d+m}(\log j)^\gamma\right| \leq \sum_{n=0}^{m} \binom{m}{n} \left|a_0^{(m-n)}(j+n)\right| \times \\ \times (j+n)^{d+m-n}(\log(j+n))^\gamma j^n \left|\big(1-\chi\big)^{(n)}\big(\frac{j}{N}\big)\right|. \quad (4.18)$$

Moreover, observe that, for any $n \in \mathbb{N}$,

$$t^n \big(1-\chi\big)^{(n)}\big(\frac{t}{N}\big) = (-1)^n \big(\frac{t}{N}\big)^n \chi^{(n)}\big(\frac{t}{N}\big),\ \forall t \in (N, 2N),$$

and

$$t^n \big(1-\chi\big)^{(n)}\big(\frac{t}{N}\big) = 0,\ \forall t \in (0,N) \cup (2N, +\infty).$$

As a result, for every $n \in \mathbb{N}_0$,

$$\sup_{t>0} \left| t^n \big(1-\chi\big)^{(n)}\Big(\frac{t}{N}\Big) \right| \le 2^n \max_{t\in[1,2]} \left|\chi^{(n)}(t)\right|, \ \forall N \in \mathbb{N}.$$

By considering $K := \max_{0\le n\le M(\gamma)}\{2^n \max_{t\in[1,2]} \left|\chi^{(n)}(t)\right|\}$, (4.18) yields, that for every $j \ge 2$,

$$\left|\big(h_0^N\big)^{(m)}(j)\right| j^{d+m}(\log j)^\gamma \le K \sum_{n=0}^{m} \binom{m}{n} \left|a_0^{(m-n)}(j+n)\right| \times \\ \times (j+n)^{d+m-n}(\log(j+n))^\gamma. \quad (4.19)$$

Under the assumption (3.4) for a_0, we see that, for any $N \in \mathbb{N}$, h_0^N satisfies assumption (3.1) of Lemma 3.1. Consequently, Γ_N satisfies relation (3.3). Thus, there exists a constant C_γ such that

$$\|H_a - H_N\|_{\mathbf{S}_{p,\infty}} \le C_\gamma \sup_{j\in\mathbb{N}_0} \left|\big(h_0^N\big)^m(j)\right| (j+1)^{d+m}\big(\log(j+2)\big)^\gamma, \ \forall m = 0,1,\dots,M(\gamma).$$

Then (4.19) implies that

$$\|H_a - H_N\|_{\mathbf{S}_{p,\infty}} \lesssim \sum_{m=0}^{M(\gamma)} \sum_{n=0}^{m} \binom{m}{n} \sup_{j>N} \left|a_0^{(m-n)}(j+n)\right| (j+n)^{d+m-n}(\log(j+n))^\gamma,$$

where the sum in the RHS converges to zero, as $N \to +\infty$, due to assumption (3.4). □

4.2.1.2 The model operator and the proof of Theorem 3.3

In order to prove Theorem 3.3, as we have already mentioned, we need to introduce a *model Hankel operator* $\tilde{H}$ which will play a dominant role in the investigation of the spectral properties of H_a. For consider a positive valued function $\chi_0 \in C^\infty(\mathbb{R}_+)$ such that

$$\chi_0(t) = \begin{cases} 1, & 0 < t \le \frac{1}{2} \\ 0, & t \ge \frac{3}{4} \end{cases}, \quad (4.20)$$

and $\chi_0(\mathbb{R}_+) = [0,1]$. Determine the function

$$w(t) = \frac{1}{(d-1)!} t^{d-1} |\log t|^{-\gamma} \chi_0(t), \ \forall t > 0. \quad (4.21)$$

Moreover, consider the sequence

$$\tilde{a}_0(j) = \big(\mathcal{L}w\big)(j), \ \forall j \in \mathbb{N}_0, \quad (4.22)$$

where $\mathcal{L}$ denotes the Laplace transform; namely,

$$\big(\mathcal{L}w\big)(t) = \int_0^{+\infty} w(\lambda) e^{-\lambda t}\, \mathrm{d}\lambda, \ \forall t \ge 0.$$

Ensuing, we correspond to $\tilde{a}_0$ the Hankel operator $\tilde{H} := H_{\tilde{a}}$. Our goal is to find an asymptotic formula for the sequence $\tilde{a}_0$ and the eigenvalues of $\tilde{H}$, but first, we need to give a necessary lemma, which will play an important role to obtain the asymptotic behaviour of $\tilde{a}_0$ (cf. [23, Lemma 3.3 and 3.4]).

Lemma 4.6.

(i) Let

$$I_n(t) = \int_0^{\lambda_0} |\log\lambda|^{-\gamma}\lambda^n e^{-\lambda t}\,\mathrm{d}\lambda\,,$$

where $\gamma > 0$, $n \in \mathbb{N}_0$ and $\lambda_0 \in (0,1)$. Then

$$I_n(t) = n!\; t^{-1-n}|\log t|^{-\gamma}\left(1 + O(|\log t|^{-1})\right),\; t \to +\infty.$$

(ii) Let

$$I_n(t) = \int_{\lambda_\infty}^{\infty} |\log\lambda|^{-\gamma}\lambda^n e^{-\lambda t}\,\mathrm{d}\lambda\,,$$

where $\gamma > 0$, $n \in \mathbb{N}_0$ and $\lambda_\infty > 1$. Then

$$I_n(t) = n!\; t^{-1-n}|\log t|^{-\gamma}\left(1 + O(|\log t|^{-1})\right),\; t \to 0^+.$$

Lemma 4.7. *Let w be the function that was described in (4.21) and*

$$\tilde{a}_0(t) = \big(\mathcal{L}w\big)(t),\;\; \forall t \geq 0.$$

Then $\tilde{a}_0$ satisfies the following formula:

$$\tilde{a}_0(t) = t^{-d}|\log t|^{-\gamma} + \tilde{g}(t),\;\; \forall t > 0,$$

where the error kernel $\tilde{g}$ is a $C^\infty(\mathbb{R}_+)$ function which presents the following asymptotic behaviour:

$$\tilde{g}^{(m)}(t) = O\left(t^{-d-m}(\log t)^{-\gamma-1}\right),\; t \to +\infty, \tag{4.23}$$

for all $m \in \mathbb{N}_0$.

Proof. We can express $\tilde{g}$ as

$$\tilde{g}(t) = \frac{1}{(d-1)!}\int_0^{+\infty} \lambda^{d-1}|\log\lambda|^{-\gamma}\chi_0(\lambda)e^{-\lambda t}\,\mathrm{d}\lambda - t^{-d}|\log t|^{-\gamma},\;\; \forall t > 0,$$

and notice that $\tilde{g} \in C^\infty(\mathbb{R}_+)$. More precisely, for every $m \in \mathbb{N}$ and any $t > 1$,

$$\tilde{g}^{(m)}(t) = \frac{(-1)^m}{(d-1)!}\int_0^{+\infty} \lambda^{d+m-1}|\log\lambda|^{-\gamma}\chi_0(\lambda)e^{-\lambda t}\,\mathrm{d}\lambda - \\ - \sum_{n=0}^{m}\binom{m}{n}\left(\frac{\mathrm{d}^n t^{-d}}{\mathrm{d}t^n}\right)\left(\frac{\mathrm{d}^{m-n}(\log t)^{-\gamma}}{\mathrm{d}t^{m-n}}\right). \tag{4.24}$$

Moreover, for every $m \in \mathbb{N}_0$ and any $t > 0$,

$$\int_0^{+\infty} \lambda^{d+m-1}|\log\lambda|^{-\gamma}\chi_0(\lambda)e^{-\lambda t}\,\mathrm{d}\lambda = \int_0^{\frac{1}{2}} \lambda^{d+m-1}|\log\lambda|^{-\gamma}e^{-\lambda t}\,\mathrm{d}\lambda +$$

$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \lambda^{d+m-1} |\log \lambda|^{-\gamma} e^{-\lambda t} \chi_0(\lambda)\, \mathrm{d}\lambda\,.$$

Notice that the second integral converges to zero exponentially fast when $t \to +\infty$. Thus Lemma 4.6 yields

$$\int_0^{+\infty} \lambda^{d+m-1} |\log \lambda|^{-\gamma} \chi_0(\lambda) e^{-\lambda t}\, \mathrm{d}\lambda = (d+m-1)! t^{-d-m} (\log t)^{-\gamma} \left(1 + O\left((\log t)^{-1}\right)\right), \quad (4.25)$$

when $t \to +\infty$. Besides, notice that, for every $k \in \mathbb{N}$,

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} (\log t)^{-\gamma} = O\left(t^{-k} (\log t)^{-\gamma-1}\right), \ t \to +\infty.$$

Thus, it is easily verified that,

$$\sum_{n=0}^{m} \binom{m}{n} \left(\frac{\mathrm{d}^n t^{-d}}{\mathrm{d}t^n}\right) \left(\frac{\mathrm{d}^{m-n} (\log t)^{-\gamma}}{\mathrm{d}t^{m-n}}\right) = \frac{(-1)^m}{(d-1)!} (d+m-1)! t^{-d-m} (\log t)^{-\gamma} + \\ + O\left(t^{-d-m} (\log t)^{-\gamma-1}\right), \text{ when } t \to +\infty. \quad (4.26)$$

Then by putting (4.25) and (4.26) back to (4.24), we obtain (4.23). □

Lemma 4.8. *Let w be as it has been defined in (4.21) and consider the sequence $\{\tilde{a}_0(j)\}_{j \in \mathbb{N}_0}$, where $\tilde{a}_0(j)$ is defined in (4.22). Then*

$$\tilde{a}_0(j) = j^{-d} |\log j|^{-\gamma} + \tilde{g}(j), \ \forall j \in \mathbb{N}_0, \quad (4.27)$$

where the error sequence $\{\tilde{g}(j)\}_{j \in \mathbb{N}_0}$ satisfies the smoothness condition presented in (3.4).

Proof. Firstly, (4.27) holds clearly true by Lemma 4.7. Regarding the smoothness condition for error term, it is enough to notice (by using induction) that

$$\tilde{g}^{(m)}(j) = \int_0^1 \int_0^1 \cdots \int_0^1 g^{(m)}(j + t_1 + t_2 + \cdots + t_m)\, \mathrm{d}t_m \ldots \mathrm{d}t_2\, \mathrm{d}t_1\,.$$

Then the desired smoothness condition is obtained immediately by Lemma 4.7. □

Lemma 4.9. *Let $\tilde{a}_0$ be the sequence described in (4.22) and $\tilde{H} = H_{\tilde{a}}$ be the corresponding Hankel operator. Then $\tilde{H}$ is compact and we have the following asymptotic formula for its eigenvalues*

$$\lambda_n^{\pm}(\tilde{H}) = C^{\pm} n^{-\gamma} + o(n^{-\gamma}), \ n \to +\infty,$$

where the constants $C^{\pm}$ are described in (3.6).

Proof. The result will arise immediately as a sub-case of Lemma 4.17. □

Proof of Theorem 3.3. Let $\tilde{a}_0$ be the sequence which generates the model operator $\tilde{H}$, and notice that $H_a = \tilde{H} + (H_a - \tilde{H})$. Then the eigenvalue asymptotics of $\tilde{H}$ are given by Lemma 4.9 and they are of type (3.5). Thus, in order to prove that H_a has the same asymptotics, according to Lemma 4.2, it is enough to prove that

$$s_n(H_a - \tilde{H}) = o(n^{-\gamma}), \text{ for } n \to +\infty. \quad (4.28)$$

In order to prove this, we need to apply Lemma 3.2. For notice that $H_a - \tilde{H}$ is a Hankel operator with kernel $\big(a-\tilde{a}\big)(j) = \big(a_0 - \tilde{a}_0\big)(|j|)$, for all $j \in \mathbb{N}_0^d$. Thus, by combining the smoothness conditions for the error terms g and $\tilde{g}$ (see Lemma 4.8), we observe that

$$\big(a_0 - \tilde{a}_0\big)(j) = o\left(j^{-d-m}(\log j)^{-\gamma}\right), \ j \to +\infty.$$

Then Lemma 3.2 implies indeed (4.28). □

4.2.2 The general case of an arbitrary $\kappa \in \mathbb{R}_+^d$

4.2.2.1 Proof of Lemma 3.4

Like in the "simple" case before, in order to prove Lemma 3.4 we need to split again the range of γ; this time into three cases: $\gamma \in (0, \frac{1}{2})$, $\gamma \in [\frac{1}{2}, 1]$ and $\gamma > 1$. Thus, there will be need for a set of three lemmas which will help us with the proof.

Lemma 4.10. *Let $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{R}_+^d$. For any function $f : \mathbb{R}_+ \to \mathbb{R}$, define*

$$S_f(x) := \sup\{t^d|f(t)|, \ \kappa_- x \le t \le \kappa_+ x\}, \ \forall x > 0, \tag{4.29}$$

where

$$\kappa_- := \min\{\kappa_1, \dots, \kappa_d\}, \quad \kappa_+ := \max\{\kappa_1, \dots, \kappa_d\}. \tag{4.30}$$

In addition, for any $p \in [1, \infty]$, define the spaces

$$X_p = \left\{ f \in L^\infty(\mathbb{R}) : \ \{S_f(n)\}_{n\in\mathbb{N}} \in \ell^p_{\frac{1}{n}}(\mathbb{N}) \right\},$$

endowed with the norm

$$\|f\|_p := \|S_f\|_{\ell^p_{\frac{1}{n}}}, \ \forall f \in X_p.$$

Then

$$(X_2, X_\infty)_{[\theta]} = X_{p_\theta}, \quad \text{where} \quad \frac{1}{p_\theta} = \frac{1-\theta}{2}, \ \theta \in (0,1).$$

Proof. We begin our proof with a remark. Notice that X_p are understood as normed spaces under the convention of identifying the functions that are equal on the intervals $I_n := [\kappa_- n, \kappa_+ n]$, for all $n \in \mathbb{N}$.

The proof is based on the complex interpolation method, which is briefly described in the Appendix A. Observe that here, instead of the spaces X_0, X_1, which is the notation that was introduced in the Appendix, we interpolate between X_2 and X_∞. First observe that X_2 and X_∞ are complete. Indeed, if $\{f^j\}_{j\in\mathbb{N}} \subset X_2$ is a $\|\cdot\|_2$-Cauchy sequence, then it can be checked that $\{f^j\}_{j\in\mathbb{N}}$ is Cauchy, with respect to the supremum norm, in every of I_ns. Let $f_n := \lim_{j\to+\infty} f^j|_{I_n}$, for all $n \in \mathbb{N}$. Then it can be checked that $\{f^j\}_{j\in\mathbb{N}}$ converges in the $\|\cdot\|_2$ sense to a function f that is identical to f_n on every interval I_n. Same reasoning could be applied on X_∞, too.

Our goal is to prove that $\|\cdot\|^{(\theta)} = \|\cdot\|_{p_\theta}$. We first prove that $\|\cdot\|^{(\theta)} \le \|\cdot\|_{p_\theta}$. For let $f \in X_{p_\theta}$. Assume, without loss of generality, that $\|f\|_{p_\theta} = 1$ (otherwise work with $\frac{f}{\|f\|_{p_\theta}}$) and define the function $F : S \to X$ (where $X = X_2 + X_\infty$ and S a closed strip defined in the Appendix), with

$$F(z) = (\cdot)^{-d} \left|(\cdot)^d f(\cdot)\right|^{\frac{p_\theta}{2}(1-z)} e^{i\mathrm{Arg} f(\cdot)}, \ \forall z \in S,$$

where Arg stands for the principal argument of a complex number. Then notice that, for any $y \in \mathbb{R}$,

$$t^{2d} \left|\big(F(iy)\big)(t)\right|^2 = \big(t^d \left|f(t)\right|\big)^{p_\theta}, \ \forall t > 0.$$

Consequently, $\big(S_{F(iy)}(n)\big)^2 = (S_f(n))^2$, for every $n \in \mathbb{N}$, and therefore, $\|F(iy)\|_2 = \|f\|_{p_\theta} = 1$. Similarly, notice that, for any $y \in \mathbb{R}$,

$$t^d \left|\big(F(1+iy)\big)(t)\right| = 1, \ \forall t > 0,$$

so that $\|F(1+iy)\|_\infty = \|F(iy)\|_2 = 1$, for all $y \in \mathbb{R}$. Thus, $|||F||| = 1 = \|f\|_{p_\theta}$. Since $F(\theta) = f$,

$$\|f\|_{(\theta)} = \|F(\theta)\|_{(\theta)} = \inf_{\{g \in \mathscr{F}(X_2, X_\infty):\ g(\theta)=f\}} |||g||| \leq |||F||| = \|f\|_{p_\theta}.$$

Conversely, let $f \in X_{p_\theta}$ and we prove that $\|f\|_{p_\theta} \leq \|f\|_{(\theta)}$. Observe that

$$\|f\|_{(\theta)} = \inf_{\{F \in \mathscr{F}(X_2, X_\infty):\ F(\theta)=f\}} |||F|||$$

thus, it is enough to prove that

$$\|F(\theta)\|_{p_\theta} \leq |||F|||, \ \forall F \in \mathscr{F}(X_2, X_\infty):\ F(\theta) = f. \tag{4.31}$$

The idea is to prove that $F(\theta)$ acts as a linear functional on the space X_{q_θ}, where $p_\theta^{-1} + q_\theta^{-1} = 1$. For let $F \in \mathscr{F}(X_2, X_\infty)$ such that $F(\theta) = f$, $g \in X_{q_\theta}$ and define the function

$$G(z) = (\cdot)^{-d} \left|(\cdot)^d g(\cdot)\right|^{q_\theta\left(\frac{1-z}{2}+z\right)} e^{i\mathrm{Arg}g(\cdot)}, \ \forall z \in S.$$

For every $n \in \mathbb{N}$ choose an element $t_n \in I_n$ and consider the complex valued function $\Lambda : S \to \mathbb{C}$, described by

$$\Lambda(z) = \sum_{n \in \mathbb{N}} \frac{1}{n} t_n^{2d} \left[\big(G(z)\big)(t_n)\right] \overline{\left[\big(F(z)\big)(t_n)\right]}, \ \forall z \in S.$$

Then, since $F : S \to X$ is analytic on S° and bounded (as an X-valued function), the Hadamard's three lines theorem (together with the maximum principle) implies that:

$$|\Lambda(\theta)| \leq \sup_{y \in \mathbb{R}} \left\{|\Lambda(iy)|, |\Lambda(1+iy)|\right\}. \tag{4.32}$$

Moreover, notice that

$$|\Lambda(iy)| \leq \sum_{n \in \mathbb{N}} \frac{1}{n} S_{G(iy)}(n) S_{F(iy)}(n) \leq \|G(iy)\|_2 \|F(iy)\|_2, \quad \forall y \in \mathbb{R}.$$

Substituting iy with $1 + iy$ gives

$$|\Lambda(1+iy)| \leq \|G(1+iy)\|_1 \|F(1+iy)\|_\infty, \ \forall y \in \mathbb{R}.$$

Thus, going back to (4.32), we get

$$\begin{aligned}
|\Lambda(\theta)| &\leq \sup_{y \in \mathbb{R}} \left\{\|G(iy)\|_2 \|F(iy)\|_2, \|G(1+iy)\|_1 \|F(1+iy)\|_\infty\right\} \\
&\leq \left(\sup_{y \in \mathbb{R}} \left\{\|G(iy)\|_2, \|G(1+iy)\|_1\right\}\right) \left(\sup_{y \in \mathbb{R}} \left\{\|F(iy)\|_2, \|F(1+iy)\|_\infty\right\}\right) \\
&= |||G||| |||F||| \\
&\leq \|g\|_{q_\theta} |||F|||;
\end{aligned}$$

or equivalently,

$$\left|\sum_{n\in\mathbb{N}} \frac{1}{n} t_n^{2d} g(t_n)\overline{f(t_n)}\right| \le \|g\|_{q_\theta} \, |||F|||, \ \forall g \in X_{q_\theta} \tag{4.33}$$

and this relation holds true for any choice of $t_n \in I_n$. Besides, for every $n \in \mathbb{N}$, there exists a sequence $\{t_j^n\}_{j\in\mathbb{N}} \subset I_n$, such that $\lim_{j\to+\infty} f(t_j^n) = \sup_{t\in I_n} |f(t)|$. In addition, we can define the function

$$g(t) = f(t)\left(t^d|f(t)|\right)^{p_\theta - 2}, \ \forall t > 0,$$

which can be checked that belongs to X_{q_θ}. Then (4.33) yields that

$$\|f\|_{p_\theta} \le |||F|||,$$

which proves (4.31). □

Lemma 4.11. *Let $p \in [2, +\infty)$, $\kappa \in \mathbb{R}_+^d$ and $H_h : \ell^2(\mathbb{N}_0^d) \to \ell^2(\mathbb{N}_0^d)$ be a Hankel operator with kernel $h(j) = h_0(\kappa\cdot j)$, for some real valued function h_0 defined on $\mathbb{R}_+$. If the sequence $\{S_{h_0}(n)\}_{n\in\mathbb{N}}$ belongs to $\ell^p_{\frac{1}{n}}(\mathbb{N})$, then the operator H_h belongs to $\mathbf{S}_p$.*

Proof. It is easy to check that

$$\begin{aligned}
\|H_h\|_2^2 &= \sum_{i,j\in\mathbb{N}_0^d} |h(i+j)|^2 \\
&\le \sum_{i_1,\dots,i_d\in\mathbb{N}_0} (i_1+\dots+i_d+1)^d |h(i_1,\dots,i_d)|^2 \\
&= \sum_{i_1,\dots,i_d\in\mathbb{N}_0} (i_1+\dots+i_d+1)^d |h_0((i_1,\dots,i_d)\cdot\kappa)|^2.
\end{aligned}$$

Notice that

$$\kappa_- \mathbf{1}\cdot t \le \kappa\cdot t \le \kappa_+ \mathbf{1}\cdot t, \ \forall t \in \mathbb{R}_+^d.$$

Let us assume that $h_0(0) = 0$. This convention makes sense because alternations of the kernel of H_h at the point $(0, 0, \dots, 0)$ are just perturbations by rank 1 operators and as a result, the spectral asymptotic analysis remains unaffected. Then

$$\begin{aligned}
\|H_h\|_2^2 &\le \sum_{i_1,\dots,i_d\in\mathbb{N}_0} (i_1+\dots+i_d+1)^d |h_0((i_1,\dots,i_d)\cdot\kappa)|^2 \\
&\le \sum_{n\in\mathbb{N}_0} W_d(n)(n+1)^d \sup_{t\in[\kappa_- n,\kappa_+ n]} |h_0(t)|^2 \\
&\lesssim \sum_{n\in\mathbb{N}} n^{2d-1} \sup_{t\in[\kappa_- n,\kappa_+ n]} |h_0(t)|^2,
\end{aligned} \tag{4.34}$$

where W_d is defined in (2.2). Now notice that, for every $t \in [\kappa_- n, \kappa_+ n]$,

$$n^{2d-1}|h_0(t)|^2 \le \frac{t^{2d-1}}{\kappa_-^{2d-1}}|h_0(t)|^2 = \frac{1}{\kappa_-^{2d-1}} t^{-1}\left(t^d|h_0(t)|\right)^2.$$

So, we see that

$$n^{2d-1} \sup_{t\in[\kappa_- n,\kappa_+ n]} |h_0(t)|^2 \le \frac{1}{\kappa_-^{2d}} \frac{S_{h_0}(n)^2}{n}, \ \forall n \in \mathbb{N},$$

and eventually,

$$\|H_h\|_2^2 \lesssim \sum_{n\in\mathbb{N}} \frac{S_{h_0}(n)^2}{n}.$$

In other words, this means that if the sequence $\{S_{h_0}(n)\}_{n\in\mathbb{N}}$ belongs to $\ell^2_{\frac{1}{n}}(\mathbb{N})$, then $H_h \in \mathbf{S}_2$. In addition, we see that, for any x and y in $\ell^2(\mathbb{N}_0^d)$,

$$|\,(H_h x, y)\,| \leq \sum_{i,j\in\mathbb{N}_0^d} |h_0\,(\kappa \cdot (i+j))|\,|x(j)|\,|y(i)|\,.$$

Let $i, j \in \mathbb{N}_0^d$, with $i = (i_1, \dots, i_d)$ and $j = (j_1, \dots, j_d)$, and notice that, for every $t \in [\kappa_- \mathbf{1} \cdot (i+j), \kappa_+ \mathbf{1} \cdot (i+j)]$,

$$\begin{aligned}|h_0(t)| &= \frac{|h_0(t)|}{(i_1+j_1+1)\dots(i_d+j_d+1)} \prod_{k=1}^d (i_k+j_k+1)\\ &\leq (\mathbf{1}\cdot(i+j)+1)^d \, \frac{|h_0(t)|}{(i_1+j_1+1)\dots(i_d+j_d+1)}.\end{aligned}$$

By assuming again that $h_0(0) = 0$, we see that

$$\begin{aligned}|h_0(t)| &\lesssim (\mathbf{1}\cdot(i+j))^d \, \frac{|h_0(t)|}{(i_1+j_1+1)\dots(i_d+j_d+1)}\\ &\leq \frac{t^d}{\kappa_-^d} \frac{|h_0(t)|}{(i_1+j_1+1)\dots(i_d+j_d+1)}\\ &\leq \frac{1}{\kappa_-^d} \frac{S_{h_0}(\mathbf{1}\cdot(i+j))}{(i_1+j_1+1)\dots(i_d+j_d+1)}\\ &\leq \frac{1}{\kappa_-^d} \sup_{n\in\mathbb{N}} S_{h_0}(n) \frac{1}{(i_1+j_1+1)\dots(i_d+j_d+1)}, \quad \forall t \in [\kappa_- \mathbf{1}\cdot(i+j), \kappa_+ \mathbf{1}\cdot(i+j)].\end{aligned}$$

Therefore,

$$|(H_h x, y)| \lesssim \sup_{n\in\mathbb{N}} S_{h_0}(n) (\mathcal{H}|x|, |y|),$$

where $\mathcal{H}$ is the tensor product of d Hilbert matrices on $\ell^2(\mathbb{N}_0^d)$, $|x|(i) := |x(i)|$, and $|y|(i) := |y(i)|$, $\forall i \in \mathbb{N}_0^d$. Thus, the boundedness of Hilbert operator implies that if the sequence $\{S_{h_0}(n)\}_{n\in\mathbb{N}}$ belongs to $\ell^\infty_{\frac{1}{n}}(\mathbb{N})$, then $H_h \in \mathbf{B}$, where $\mathbf{B}$ is the set of bounded operators. In other words, we have shown that the linear operator $T : h \mapsto H_h$ is bounded from X_2 to $\mathbf{S}_2$ and from X_∞ to $\mathbf{B}$, where the spaces X_p, for $p \in [2, \infty]$, have been defined in the previous Lemma. Then, since (see [25, "Appendix to IX.4 - Abstract interpolation", Proposition 8])

$$(\mathbf{S}_2, \mathbf{B})_{[\theta]} = \mathbf{S}_{p_\theta}, \quad \text{where} \quad \frac{1}{p_\theta} = \frac{1-\theta}{2}, \; \theta \in (0,1),$$

the retract argument and the previous Lemma give that T is bounded from X_p to $\mathbf{S}_p$, for any $p > 2$, too. Therefore, if $\{S_{h_0}(n)\}_{n\in\mathbb{N}} \in \ell^p_{\frac{1}{n}}(\mathbb{N})$, then H_h belongs to $\mathbf{S}_p$, indeed. □

Lemma 4.12. *Let $p \in [1,2]$ and $H_h : \ell^2(\mathbb{N}_0^d) \to \ell^2(\mathbb{N}_0^d)$ be a Hankel operator with kernel $h(j) = h_0(\kappa \cdot j)$, for some real valued function h_0 defined on $\mathbb{R}_+$, and $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{R}_+^d$. Then H_h belongs to the Schatten class $\mathbf{S}_p$, whenever $\mathcal{F}^{-1}h_0$ belongs to the Besov class $B^d_{1,p}(\mathbb{R})$.*

Proof. Consider the sequence of intervals $\{[2^{n-1}, 2^{n+1}]\}_{n\in\mathbb{N}_0}$. For every $n \in \mathbb{N}$, define

$$m_0(n) := \min\{m \in \mathbb{N}_0 : \; n\kappa_- \le 2^{m+1}\},$$

and

$$m_1(n) := \min\{m \in \mathbb{N}_0 : \; n\kappa_+ \le 2^{m+1}\},$$

where κ_- and κ_+ are defined in (4.30). If we assume that $n > \frac{2}{\kappa_-}$, we can say that

$$m_0(n) = \lceil \log_2(n\kappa_-) \rceil - 1, \quad m_1(n) = \lceil \log_2(n\kappa_+) \rceil - 1,$$

where $\lceil\cdot\rceil$ denotes the *ceiling function*. Therefore,

$$[\kappa_- n, \kappa_+ n] \subset [2^{m_0(n)-1}, 2^{m_1(n)+1}] = \bigcup_{m=m_0(n)}^{m_1(n)} [2^{m-1}, 2^{m+1}], \; \forall n \in \mathbb{N}. \tag{4.35}$$

Then, by repeating some of the arguments that were presented in Lemma 4.11 and assuming that $h_0(t)$ is zero for small enough values of t, relation (4.34) gives

$$\begin{aligned}\|H_h\|_2^2 &\lesssim \sum_{n\in\mathbb{N}} n^{2d-1} \sup_{t\in[\kappa_- n, \kappa_+ n]} |h_0(t)|^2 \\ &\le \sum_{n\in\mathbb{N}} n^{2d-1} \left(\sum_{m=m_0(n)}^{m_1(n)} \sup_{t\in[2^{m-1},2^{m+1}]} |h_0(t)| \right)^2 .\end{aligned}$$

Observe that the assumption of h_0 being zero for small enough values of t does not cause any change in the spectral asymptotic analysis, since this is translated into perturbations of H_h by finite rank operators. Next, notice that

$$\begin{aligned} m_1(n) - m_0(n) &= \lceil \log_2(n\kappa_+) \rceil - \lceil \log_2(n\kappa_-) \rceil \\ &\le \lceil \log_2 \frac{\kappa_+}{\kappa_-} \rceil \\ &\le \log_2 \frac{2\kappa_+}{\kappa_-}. \end{aligned}$$

Thus, relation (4.35) yields

$$\begin{aligned}\|H_h\|_2^2 &\le \log_2 \frac{4\kappa_+}{\kappa_-} \sum_{n\in\mathbb{N}} \sum_{m=m_0(n)}^{m_1(n)} n^{2d-1} \sup_{t\in[2^{m-1},2^{m+1}]} |h_0(t)|^2 \\ &\lesssim \sum_{n\in\mathbb{N}} \sum_{m=m_0(n)}^{m_1(n)} 2^{(2d-1)m_1(n)} \sup_{t\in[2^{m-1},2^{m+1}]} |h_0(t)|^2. \end{aligned} \tag{4.36}$$

Now let $m \in \mathbb{N}_0$. If the interval $[2^{m-1}, 2^{m+1}]$ is appeared in the previous relation, then there exists a large enough natural number n such that $m \in \{m_0(n), \ldots, m_1(n)\}$. By the definition of $m_0(n)$, we get that $\kappa_- n \le 2^{m+1}$. This suggests that every interval $[2^{m-1}, 2^{m+1}]$ in (4.36) is used at most $\max\{1, 2^{m+1}\lceil \kappa_-^{-1} \rceil\}$ times. Moreover, (4.35) helps us to verify that, for every large enough natural number n,

$$2^{(2d-1)m_1(n)} \le \left(\frac{2\kappa_+}{\kappa_-}\right)^{2d-1} 2^{(2d-1)m}, \; \forall m = m_0(n), \ldots, m_1(n).$$

Thus, going back to (4.36),

$$\begin{aligned}\|H_h\|_2^2 &\lesssim \sum_{m\in\mathbb{N}_0}\left(\frac{2\kappa_+}{\kappa_-}\right)^{2d-1} 2^{(2d-1)m}2^{m+1}\max\{1,\lceil\kappa_-^{-1}\rceil\}\sup_{t\in[2^{m-1},2^{m+1}]}|h_0(t)|^2\\ &\lesssim \sum_{m\in\mathbb{N}_0} 2^{2dm}\sup_{t\in[2^{m-1},2^{m+1}]}|h_0(t)|^2 \\ &= \sum_{m\in\mathbb{Z}} 2^{2dm}\sup_{t\in[2^{m-1},2^{m+1}]}|h_0(t)|^2,\end{aligned} \tag{4.37}$$

where the last equality holds since h_0 has been assumed to vanish for t small enough. Now let $\{v_n\}_{n\in\mathbb{Z}}$ be the sequence of smooth functions described in (1.8). Then $h_0(t) = \sum_{n\in\mathbb{Z}} h_{0n}(t)$, where $h_{0n}(t) = h_0(t)v_n(t)$. More precisely,

$$h_0(t) = \sum_{j=m-1}^{m+1} h_{0_j}(t) = \sum_{j=m-1}^{m+1} \mathcal{F}\left[\left(\mathcal{F}^{-1}h_0\right)*V_j\right](t),\ \forall t\in[2^{m-1},2^{m+1}].$$

Notice that, for any $j\in\mathbb{Z}$,

$$\left|\mathcal{F}\left[\left(\mathcal{F}^{-1}h_0\right)*V_j\right](t)\right| \le \left\|\left(\mathcal{F}^{-1}h_0\right)*V_j\right\|_1,\ \forall t>0,$$

so that

$$\sup_{t\in[2^{m-1},2^{m+1}]}|h_0(t)|^2 \le 3\sum_{j=m-1}^{m+1}\left\|\left(\mathcal{F}^{-1}h_0\right)*V_j\right\|_1^2,\ \forall m\in\mathbb{Z}.$$

Thus, going back to (4.37) results

$$\begin{aligned}\|H_h\|_2^2 &\lesssim \sum_{m\in\mathbb{Z}} 2^{2dm}\sup_{t\in[2^{m-1},2^{m+1}]}|h_0(t)|^2\\ &\le 3\sum_{m\in\mathbb{Z}} 2^{2dm}\sum_{j=m-1}^{m+1}\left\|\left(\mathcal{F}^{-1}h_0\right)*V_j\right\|_1^2\\ &\lesssim \sum_{m\in\mathbb{Z}} 2^{2dm}\left\|\left(\mathcal{F}^{-1}h_0\right)*V_j\right\|_1^2 = \left\|\mathcal{F}^{-1}h_0\right\|_{B^d_{1,2}}^2.\end{aligned}$$

The latter suggests that if $\mathcal{F}^{-1}h_0$ belongs to the Besov class $B^d_{1,2}(\mathbb{R}_+)$, then $H_h\in\mathbf{S}_2$. This observation prompts us to aim for interpolation between the Besov classes $B^d_{1,2}(\mathbb{R}_+)$ and $B^d_{1,1}(\mathbb{R}_+)$. So we want to prove that if $\mathcal{F}^{-1}h_0\in B^d_{1,1}$, then $H_h\in\mathbf{S}_1$.

Since $h_0(t) = \sum_{n\in\mathbb{N}_0} h_{0n}(t)$, we can write $H_h = \sum_{n\in\mathbb{N}_0} H_n$, where H_n is the Hankel operator corresponding to the kernel $h_n(j) = h_{0n}(\kappa\cdot j)$, $\forall j\in\mathbb{N}_0$. In addition, since $\mathrm{supp} h_{0n}\subset[2^{n-1},2^{n+1}]$, we can interpret H_n as a finite matrix acting on $\{0,1,\dots,N_n\}^d$, where $N_n := \lceil 2^{n+1}\kappa_-^{-1}\rceil$.

We now proceed to estimating the trace norm of H_n. Since

$$h_{0n}(\kappa\cdot j) = \int_{\mathbb{R}}\left(\mathcal{F}^{-1}h_{0_n}\right)(x)e^{-2\pi i x(\kappa\cdot j)}\,\mathrm{d}x\,,\ \forall j\in\mathbb{N}_0^d,$$

the matrix $[H_n]_{j,k}$ could be represented as

$$\int_{\mathbb{R}}\left(\mathcal{F}^{-1}h_{0_n}\right)(x)e^{-2\pi i x(\kappa\cdot j)}e^{-2\pi i x(\kappa\cdot k)}\,\mathrm{d}x\,,\ j,k\in\{0,1,\dots,N_n\}^d.$$

It is enough to estimate the trace norm of the rank one matrices $[e^{-2\pi ix(\kappa\cdot j)}e^{-2\pi ix(\kappa\cdot k)}]_{j,k}$. Notice that since these matrices are rank one operators, their trace norm equals the operator norm which is bounded, according to Schur test, by $\sum_{j\in\{0,\dots,N_n\}^2}1=N_n^d$. Therefore, $\|H_n\|_1\le N_n^d\|\mathcal{F}^{-1}h_{0_n}\|_1$ and consequently,

$$\begin{aligned}\|H_h\|_1&\le\sum_{n\in\mathbb{N}_0}\|H_n\|_1\\&\le\sum_{n\in\mathbb{N}_0}N_n^d\left\|\mathcal{F}^{-1}h_{0_n}\right\|_1\\&\le\frac{2^d}{\kappa_-}\sum_{n\in\mathbb{N}_0}2^{dn}\left\|\mathcal{F}^{-1}h_{0_n}\right\|_1=\frac{2^d}{\kappa_-}\left\|\mathcal{F}^{-1}h_0\right\|_{B^d_{1,1}}.\end{aligned}$$

Now notice that

$$(\mathbf{S}_1,\mathbf{S}_2)_{\theta,p}=\mathbf{S}_p\quad\text{and}\quad\left(B^d_{1,1},B^d_{1,2}\right)_{\theta,p}=B^d_{1,p},\quad\text{where}\quad\frac{1}{p}=1-\theta+\frac{\theta}{2};$$

see [3, §7.3 and Theorem 6.4.5], respectively. Therefore, by interpolation, we get that $H_h\in\mathbf{S}_p$, whenever $\mathcal{F}^{-1}h_0\in B^d_{1,p}$, $\forall p\in[1,2]$. In order to verify this, it is enough to note that the linear mapping $\eta_0\mapsto\mathcal{F}\eta_0=:h_0\mapsto H_h$ is bounded from $B^d_{1,1}$ to $\mathbf{S}_1$ and from $B^d_{1,2}$ to $\mathbf{S}_2$. □

Lemma 4.13. *Let $p\in(0,1)$ and $H_h:\ell^2(\mathbb{N}_0^d)\to\ell^2(\mathbb{N}_0^d)$ be a Hankel operator with kernel $h(j)=h_0(\kappa\cdot j)$, for some real valued function h_0 defined in $L^\infty(\mathbb{R}_+)$, and $\kappa=(\kappa_1,\dots,\kappa_d)\in\mathbb{R}^d_+$. Then H_h belongs to the Schatten class $\mathbf{S}_p$, whenever $\mathcal{F}^{-1}h_0$ belongs to the Besov class $B^{d-1+\frac{1}{p}}_{p,p}(\mathbb{R})$.*

Proof. Let us consider again the decomposition $\{[2^{n-1},2^{n+1}]\}_{n\in\mathbb{N}_0}$. Then as we saw before, $H_h=\sum_{n\in\mathbb{N}_0}H_n$, where H_n is the Hankel operator corresponding to the kernel $h_n(j)=h_{0n}(\kappa\cdot j)$, $\forall j\in\mathbb{N}_0$. Since $p\in(0,1)$, instead of the usual triangle inequality we have that $\|H_h\|_p^p\le\sum_{n\in\mathbb{N}_0}\|H_n\|_p^p$, so it is enough to study the p-norm of H_n. We can still interpret H_n as a finite matrix acting on $\{0,1,\dots,N_n\}^d$, where $N_n:=\lceil 2^{n+1}\kappa_-^{-1}\rceil$ and κ_- is defined in (4.30). Unlike Lemma 4.12, an integral representation does not help here. Instead, notice that $h_{0n}\in L^2([0,N_n])$ and consider the orthonormal basis of $L^2([0,N_n])$, $\{\sqrt{N_n^{-1}}e^{2\pi i\frac{m}{N_n}\cdot}\}_{m\in\mathbb{Z}}$. Then

$$\begin{aligned}h_{0n}(x)&=\frac{1}{N_n}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}h_{0n}(y)e^{-2\pi i\frac{m}{N_n}y}\,dy\,e^{2\pi i\frac{m}{N_n}x}\\&=\frac{1}{N_n}\sum_{m\in\mathbb{Z}}\hat{h}_{0n}(mN_n^{-1})e^{2\pi i\frac{m}{N_n}x},\ \forall x\in[0,N_n],\end{aligned}$$

and

$$h_{0n}(\kappa\cdot(j+k))=\frac{1}{N_n}\sum_{m\in\mathbb{Z}}\hat{h}_{0n}(mN_n^{-1})e^{2\pi i\frac{m}{N_n}\kappa\cdot j}e^{2\pi i\frac{m}{N_n}\kappa\cdot k},\ \forall j,k\in\{0,1,\dots,N_n\}^d.$$

Therefore, by applying again the modified triangle inequality $\|A+B\|_p^p\le\|A\|_p^p+\|B\|_p^p$, we see that it is enough to bound the p-norm of the rank one operator represented by the matrix

$$\left[e^{2\pi i\frac{m}{N_n}\kappa\cdot j}e^{2\pi i\frac{m}{N_n}\kappa\cdot k}\right]_{j,k\in\{0,1,\dots,N_n\}^d}.$$

Since the matrix above is of rank one, its p-norm equals the operator norm, which is bounded (Schur test) by N_n^d. Thus,

$$\|H_n\|_p^p \leq N_n^{-p} \sum_{m\in\mathbb{Z}} |\hat{h}_{0n}(mN_n^{-1})|^p N_n^{dp}, \ \forall n \in \mathbb{N}_0. \tag{4.38}$$

Finally, let us define the function $h_{0n_{N_n}}(x) := h_{0n}(xN_n)$, $\forall x \in [0,1]$. Then $\operatorname{supp}(h_{0n_{N_n}}) \subset [-1,1]$ and $h_{0n_{N_n}} \in L^2(\mathbb{R})$, so the Paley-Wiener Theorem implies that $\mathcal{F}h_{0n_{N_n}}$ is of exponential type 1. Thus, the Polya-Plancherel inequality gives that

$$\sum_{m\in\mathbb{Z}} \left|\left(\mathcal{F}h_{0n_{N_n}}\right)(m)\right|^p \lesssim \left\|\mathcal{F}h_{0n_{N_n}}\right\|_{L^p(\mathbb{R})}^p, \ \forall n \in \mathbb{N}_0; \tag{4.39}$$

for the respective theory, see Appendix B. Now notice that $\left(\mathcal{F}h_{0n_{N_n}}\right)(x) = N_n^{-1}\hat{h}_{0n}(N_n^{-1}x)$, $\forall x \in \mathbb{R}$ and thus, substituting in (4.39), yields

$$\sum_{m\in\mathbb{Z}} |\hat{h}_{0n}(mN_n^{-1})|^p \lesssim \|\hat{h}_{0n}(\cdot N_n^{-1})\|_p^p = N_n\|\hat{h}_{0n}\|_p^p.$$

Moreover, notice that, for any function $f : \mathbb{R} \to \mathbb{R}$,

$$\hat{f}(-x) = \left(\mathcal{F}^{-1}f\right)(x), \ \forall x \in \mathbb{R} \ \text{ and } \ \|f(\cdot)\|_{L^p(\mathbb{R})} = \|f(-\cdot)\|_{L^p(\mathbb{R})}, \ \forall p > 0;$$

provided that the expressions above make sense. Therefore, (4.38) eventually gives that for every index $n \geq 0$,

$$\begin{aligned}\|H_n\|_p^p &\lesssim N_n^{(d-1)p} N_n \|\hat{h}_{0n}\|_p^p \\ &\simeq 2^{np(d-1+\frac{1}{p})}\|\hat{h}_{0n}\|_p^p \ \ \text{(by substituting } N_n := \lceil 2^{n+1}\kappa_-^{-1}\rceil) \\ &= 2^{np(d-1+\frac{1}{p})}\left\|\mathcal{F}^{-1}h_{0n}\right\|_p^p.\end{aligned}$$

The latter implies that

$$\|H_h\|_p^p \lesssim \sum_{n\in\mathbb{N}_0} 2^{np(d-1+\frac{1}{p})}\left\|\mathcal{F}^{-1}h_{0n}\right\|_p^p = \sum_{n\in\mathbb{N}_0} 2^{np(d-1+\frac{1}{p})}\left\|\left(\mathcal{F}^{-1}h_0\right) * V_n\right\|_p^p.$$

If we assume that $h_0(t)$ is zero for small enough values of t (same justification as in Lemma 4.12), the right hand side equals $\|\mathcal{F}^{-1}h_0\|^p_{B_{pp}^{d-1+\frac{1}{p}}}$, so we get the desired result. □

Lemma 4.14 ([24], Lemma 3.5)**.** *Let $\gamma \geq \frac{1}{2}$ and M as defined in (3.7). In addition, let $f : \mathbb{R}_+ \to \mathbb{R}$ be a $C^2(\mathbb{R}_+)$ function such that*

$$f^{(m)}(t) = O\left(t^{-d-m}(\log t)^{-\gamma}\right), \ \textit{when } t \to +\infty, \ \ \textit{for } m = 0, 1, \ldots, M.$$

For every $n \in \mathbb{Z}$, define $f_n := fv_n$, where v_n are defined in (1.8). Then, for any $q > \frac{1}{M}$,

$$2^n\left\|\mathcal{F}^{-1}f_n\right\|_q^q \lesssim \left(\sum_{m=0}^{M}\int_{2^{n-1}}^{2^{n+1}} x^m|f^{(m)}(x)|\,\mathrm{d}x\right)^q, \ \forall n \in \mathbb{Z}.$$

Proof of Lemma 3.4. We divide the proof into three steps according to the range of γ, indicated in (3.7).

Let $\gamma \in (0, 1/2)$. According to Lemma 4.11, it is enough to prove that $\{S_{a_0}(n)\}_{n\in\mathbb{N}}$ belongs to $\ell^p_{n^{-1}}(\mathbb{N})$, for all $p > \frac{1}{\gamma}$, where S_{a_0} (for $a_0 = f$) is defined in (4.29). Notice that for every function a_0 described by (3.8), we have that $S_0(n) \lesssim (\log n)^{-\gamma}$, $\forall n \in \mathbb{N}$. Moreover, the series $\sum_{n\in\mathbb{N}} S^p_{a_0}(n) n^{-1}$ converges for every $p > 1/\gamma$. Therefore, $\{S_{a_0}(n)\}_{n\in\mathbb{N}}$ belongs, indeed, to $\ell^p_{n^{-1}}(\mathbb{N})$, for all $p > \frac{1}{\gamma}$.

Let $\gamma \in [\frac{1}{2}, 1]$ and a_0 be as described in (3.8). Then, for $M = 2$, Lemma 4.14 gives

$$\begin{aligned} 2^n \left\|\mathcal{F}^{-1} a_{0_n}\right\|_1 &\lesssim \sum_{m=0}^{2} \int_{2^{n-1}}^{2^{n+1}} x^m |a_0^{(m)}(x)| \, dx \\ &\lesssim \sum_{m=0}^{2} \int_{2^{n-1}}^{2^{n+1}} x^{-d} |\log x|^{-\gamma} \, dx \\ &\lesssim \sum_{m=0}^{2} 2^{-(d-1)n} n^{-\gamma}, \ \forall n \in \mathbb{N}_0. \end{aligned}$$

Therefore,

$$\left\|\mathcal{F}^{-1} a_0\right\|^p_{B^d_{1,p}} \lesssim \sum_{n\in\mathbb{N}_0} 2^{2npd} 2^{-2npd} n^{-p\gamma},$$

and the series in the right hand side converges if and only if $p > 1/\gamma$. Consequently, Lemma 4.12 results that $H_a \in \mathbf{S}_p$, $\forall p > 1/\gamma$.

Let $\gamma > 1$ and a_0 be as described in (3.8). By repeating the arguments that were presented right before, it is not difficult to see that, for $M = [\gamma] + 1$,

$$2^n \left\|\mathcal{F}^{-1} a_{0_n}\right\|^p_p \lesssim 2^{-(d-1)np} n^{-\gamma p},$$

for every index n sufficiently large. Thus,

$$\left\|\mathcal{F}^{-1} a_0\right\|^p_{B^{d-1+\frac{1}{p}}_{pp}} \lesssim \sum_{n\in\mathbb{N}_0} n^{-\gamma p},$$

where the latter series converges whenever $p > \frac{1}{\gamma}$. Thus, Lemma 4.13 gives that $H_a \in \mathbf{S}_p$, for all $p > \frac{1}{\gamma}$. $\square$

4.2.2.2 The model operator and the proof of Theorem 3.5

As we did in the particular case of $\kappa = \mathbf{1}$, the proof of Theorem 3.5 will be obtained with the help of a suitable *model operator*. To construct this operator, we consider again the function χ_0 as it has been defined in (4.20) and also,

$$w(t) = \frac{1}{(d-1)!} t^{d-1} |\log t|^{-\gamma} \chi_0(t), \ \forall t > 0, \tag{4.40}$$

where γ is the same real number that was defined in Theorem 3.5. Now consider the real valued function

$$\tilde{a}_0(t) = (\mathcal{L}w)(t), \ \forall t > 0,$$

where $\mathcal{L}$ denotes the Laplace transform. Ensuing, for $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{R}_+^d$ we define the Hankel operator $\tilde{H} : \ell^2(\mathbb{N}_0^d) \to \ell^2(\mathbb{N}_0^d)$ such that

$$\big(\tilde{H}x\big)(n) = \sum_{j\in\mathbb{N}_0^d} \tilde{a}_0\big(\kappa\cdot(n+j)\big)x(j), \ \forall n \in \mathbb{N}_0^d, \ \forall x \in \ell^2(\mathbb{N}_0^d), \tag{4.41}$$

where "$\cdot$" denotes the usual inner product. For sake of accuracy, notice that $\tilde{a}_0$ is in fact defined only on $\mathbb{R}_+$, while formula (4.41) suggests that it attains a value at 0. This a convention that is made, since alternations of the kernel of $\tilde{H}$ at the point $(0,0,\dots,0)$ are just perturbations by rank 1 operators and as a result, the spectral asymptotic analysis remains unaffected.

Lemma 4.15. *Let $d \in \mathbb{N}$, $\{\kappa_i\}_{i=1}^d \subset \mathbb{R}_+$ and define the function*

$$v(\lambda) = \prod_{i=1}^d \frac{1}{1-e^{-\lambda\kappa_i}}, \ \forall\lambda > 0.$$

Then

$$v(\lambda) = \frac{1}{\kappa_1\dots\kappa_d}\lambda^{-d} + v_1(\lambda), \ \forall\lambda > 0, \tag{4.42}$$

where v_1 is a meromorphic function. Moreover 0 is its only pole in $\mathbb{R}$, of order $d-1$.

Proof. Notice that, for any $\alpha > 0$,

$$1 - e^{-\alpha\lambda} = \lambda g_\alpha(\lambda),$$

where $g_\alpha(\lambda)$ is an entire function, such that $g_\alpha(0) = \alpha$. Then, by setting $g_i := g_{\kappa_i}$, for $i = 1, \dots, d$,

$$\begin{aligned}\prod_{i=1}^d \big(1-e^{-\lambda\kappa_i}\big) &= \prod_{i=1}^d \big(\lambda g_i(\lambda)\big)\\ &= \lambda^d \prod_{i=1}^d \big(\kappa_i + O(\lambda)\big), \ \lambda \to 0^+.\end{aligned}$$

Then,

$$v(\lambda) = \lambda^{-d}\prod_{i=1}^d \big(\kappa_i + O(\lambda)\big)^{-1}, \ \lambda \to 0^+,$$

and, by combining with (4.42), we get

$$\begin{aligned}v_1(\lambda) &= \frac{1}{\lambda^d\prod_{i=1}^d \kappa_i}\left(\frac{1}{\prod_{i=1}^d\big(1+\kappa_i^{-1}O(\lambda)\big)} - 1\right)\\ &= \frac{1}{\lambda^d\prod_{i=1}^d \kappa_i}\left(\frac{1}{1+O(\lambda)} - 1\right)\\ &= \frac{1}{\lambda^{d-1}\prod_{i=1}^d \kappa_i}O(1), \quad \lambda \to 0^+.\end{aligned}$$

Finally, notice that all the involved error terms in the above calculations represent functions that are analytic on $\mathbb{R}_+$. Thus, v_1 has indeed a pole of order $d-1$ at 0. ∎

Lemma 4.16. *Let w be a bounded, compactly supported function on $\mathbb{R}_+$ and set $b_- = \min\{x \in \operatorname{supp}(w)\}$ and $b_+ = \max\{x \in \operatorname{supp}(w)\}$. In addition, let ϕ be a complex valued function, which is defined analytically on a neighbourhood V of $[2b_-, 2b_+]$. Then the integral Hankel operator $\Gamma : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, with*

$$\big(\Gamma f\big)(x) = \int_0^{+\infty} w(x)\phi(x+y)w(y)f(y)\,\mathrm{d}y\,,\ \forall x \in \mathbb{R}_+,\ \forall f \in L^2(\mathbb{R}_+),$$

belongs to the Schatten class $\mathbf{S}_p$, for all $p > 0$.

Proof. By the analyticity of ϕ on V, for every $x \in V \cap \mathbb{R}$, there exists a positive number r_x, such that $(x - r_x, x + r_x) \subset V$ and

$$\phi(z) = \sum_{n \in \mathbb{N}_0} \frac{\phi^{(n)}(x)}{n!}(z-x)^n,\ \forall z \in D(x; r_x).$$

Then, due to the compactness of $[2b_-, 2b_+]$, there exists $m \in \mathbb{N}$, such that $\{x_i\}_{i=1}^m \subset [2b_-, 2b_+]$ and

$$[2b_-, 2b_+] \subset \bigcup_{i=1}^m (x_i - r_i, x_i + r_i), \quad \text{where } r_i := r_{x_i}.$$

In the sequel, we choose a partition $Q = \{y_0 = b_-, y_1, \dots, y_k = b_+\}$ of $[b_-, b_+]$ with $|y_{i+1} - y_i| = \delta$, for $i = 0, 1, \dots, k-1$, where δ is chosen such that for every $i, j \in \{0, 1, \dots, k-1\}$, there exists $l = l(i,j) \in \{1, 2, \dots, m\}$ such that

$$[z_i + z_j - \delta, z_i + z_j + \delta] \subset \left(x_l - \frac{r_l}{2}, x_l + \frac{r_l}{2}\right), \tag{4.43}$$

where

$$z_i := \frac{y_{i+1} + y_i}{2},\ \forall i = 0, 1, \dots, k-1. \tag{4.44}$$

Then, for every $l = l(i,j)$, as it is defined by (4.43), and every $N \in \mathbb{N}$, define the polynomials

$$p_{l_N}(z) := \sum_{n=0}^N \frac{\phi^{(n)}(x_l)}{n!}(z - x_l)^n,\ \forall z \in \mathbb{C}.$$

Then, the Cauchy integral formula for the Taylor coefficients gives that, for every $R_l \in (0, r_l)$,

$$\left|\frac{\phi^{(n)}(x_i)}{n!}\right| \le R_l^{-n} \max_{|z-x_l|=R_l} |\phi(z)|,\ \forall n \in \mathbb{N}_0.$$

Thus, for any $0 < \rho_l < R_l < r_l$,

$$\begin{aligned}\big|\phi(z) - p_{l_{N-1}}(z)\big| &\le \max_{|z-x_l|=R_l} |\phi(z)| \sum_{n \ge N} \left(\frac{\rho_l}{R_l}\right)^n \\ &= \max_{|z-x_l|=R_l} |\phi(z)| \left(\frac{\rho_l}{R_l}\right)^N \frac{R_l}{R_l - \rho_l},\ \forall z \in D(x_l; \rho_l).\end{aligned}$$

So, if $\rho_l = \frac{r_l}{2}$ and $R_l = \frac{3\rho_l}{2}$, we get

$$\big|\phi(z) - p_{l_{N-1}}(z)\big| \le 3 \max_{|z-x_l|=\frac{3r_l}{4}} |\phi(z)| \left(\frac{2}{3}\right)^N,\ \forall z \in D\left(x_l; \frac{r_l}{2}\right). \tag{4.45}$$

In addition observe that for

$$w_i := w\chi_{[y_i,y_{i+1}]}, \ \forall i = 0, 1, \ldots, k-1,$$

and

$$\big(\Gamma_{ij} f\big)(x) := \int_0^{+\infty} w_i(x)\phi(x+y)w_j(y)f(y)\,\mathrm{d}y\,, \ \forall x > 0, \ \forall f \in L^2(\mathbb{R}_+),$$

where $i, j \in \{0, 1, \ldots, k-1\}$, we have that

$$\Gamma = \sum_{i,j}^{k-1} \Gamma_{ij}.$$

Besides, the N-th singular value of Γ denotes its distance from operators of rank at most $N-1$. Therefore, if $l = l(i,j)$, as it is defined by (4.43), and

$$\big(P_{l_{N-1}} f\big)(x) := \int_0^{+\infty} w_i(x)p_{l_{N-1}}(x+y)w_j(y)f(y)\,\mathrm{d}y\,, \ \forall x > 0, \ \forall f \in L^2(\mathbb{R}_+),$$

then

$$s_N(\Gamma) \le \left\| \sum_{i,j=0}^{k-1} \big(\Gamma_{ij} - P_{l_{N-1}}\big) \right\| \le \left\| \sum_{i,j=0}^{k-1} \big(\Gamma_{ij} - P_{l_{N-1}}\big) \right\|_2 \le \sum_{i,j=0}^{k-1} \left\|\Gamma_{ij} - P_{l_{N-1}}\right\|_2,$$

where $\|\cdot\|_2$ is the Hilbert-Schmidt norm. Thus, for any $p > 0$,

$$s_N^p(\Gamma) \lesssim \sum_{i,j=0}^{k-1} \left\|\Gamma_{ij} - P_{l_{N-1}}\right\|_2^p, \ \forall N \in \mathbb{N}. \tag{4.46}$$

If

$$M_w := \max_{x \in \operatorname{supp}(w)} |w(x)|,$$

then, for any $i, j \in \{0, 1, \ldots, k-1\}$,

$$\left\|\Gamma_{ij} - P_{l_{N-1}}\right\|_2^2 \le M_w^4 \iint\limits_{[y_i,y_{i+1}]\times[y_j,y_{j+1}]} \left|\phi(x+y) - p_{l_{N-1}}(x+y)\right|^2 \mathrm{d}y\,\mathrm{d}x\,.$$

Besides, for any $(x,y) \in [y_i, y_{i+1}] \times [y_j, y_{j+1}]$, $x + y \in [z_i + z_j - \delta, z_i + z_j + \delta]$, where z_i is defined in (4.44) and $\delta = y_{i+1} - y_i$, for all $i = 0, 1, \ldots, k-1$. But then, (4.43) implies that $x + y \in (x_l - \frac{r_l}{2}, x_l + \frac{r_l}{2}) \subset D(x_l; \frac{x_l}{2})$. Thus, if

$$M_\phi := \max_{1 \le l \le m} \left\{ \max_{|z - x_l| = \frac{3r_l}{4}} |\phi(z)|, \right\}$$

relation (4.45) yields that

$$\left\|\Gamma_{ij} - P_{l_{N-1}}\right\|_2 \le 3\delta M_w^2 M_\phi \left(\frac{2}{3}\right)^N, \ \forall N \in \mathbb{N}.$$

Therefore,

$$\sum_{N\in\mathbb{N}} \left\|\Gamma_{ij} - P_{l_{N-1}}\right\|_2^p \lesssim \sum_{N\in\mathbb{N}} \left(\frac{2}{3}\right)^{pN} < +\infty, \ \forall p > 0.$$

Finally, by applying this to (4.46) we get that $\Gamma \in \mathbf{S}_p$, for all $p > 0$. □

Lemma 4.17. *The model operator $\tilde{H}$ is unitarily equivalent to the sum $\Psi + E$, where $E \in \mathbf{S}_p$, for any $p > 0$, and Ψ is a pseudo-differential operator of the form $\mathcal{M}_\beta \alpha(D) \mathcal{M}_\beta$, where*

$$\alpha(x) = \begin{cases} 0, & x \to +\infty \\ \\ \frac{1}{2^d (d-1)! \prod_{i=1}^d \kappa_i} |x|^{-\gamma} \big(1 + o(1)\big), & x \to -\infty \end{cases}, \tag{4.47}$$

and

$$\beta = \sqrt{\mathcal{F}^{-1} \left(\cosh\left(\frac{\cdot}{2}\right)\right)^{-d}}. \tag{4.48}$$

Proof. First of all, notice that β is not ill-defined, since $\mathcal{F}^{-1} \left(\cosh\left(\frac{\cdot}{2}\right)\right)^{-d}$ is positive on $\mathbb{R}$. Indeed, it is enough to observe that $\mathcal{F}^{-1} \left(\cosh\left(\frac{\cdot}{2}\right)\right)^{-1} = 2\pi \left(\cosh(2\pi^2 \cdot)\right)^{-1}$, which is a positive function. Then the result is obtained by noticing that the convolution of positive functions is positive.

The result arises by showing that $\tilde{H}$ can be written as a product of two operators $L_w^w L_w$. $L_w^* L_w$ will be unitarily equivalent to $L_w L_w^*$ and the latter to Ψ. To see this, let $x, y \in \ell^2(\mathbb{N}_0^d)$. Then

$$\begin{aligned} (\tilde{H}x, y) &= \sum_{i,j \in \mathbb{N}_0^d} \tilde{a}(i+j) x(j) \overline{y(i)} \\ &= \sum_{i,j \in \mathbb{N}_0^d} \tilde{a}_0\big(\kappa \cdot (i+j)\big) x(j) \overline{y(i)} \\ &= \sum_{i,j \in \mathbb{N}_0^d} \int_0^{+\infty} e^{-t\left(\kappa \cdot (i+j)\right)} w(t) \,\mathrm{d}t \, x(j) \overline{y(i)} \\ &= (L_w x, L_w y). \end{aligned}$$

The latter implies that $\tilde{H} = L_w^* L_w$, where $L_w : \ell^2(\mathbb{N}_0^d) \to L^2(\mathbb{R}_+)$ with

$$\big(L_w x\big)(t) = \sqrt{w(t)} \sum_{j \in \mathbb{N}_0^d} e^{-t(\kappa \cdot j)} x(j), \;\; \forall t > 0, \;\; \forall x \in \ell^2(\mathbb{N}_0^d).$$

Moreover, for any $x \in \ell^2(\mathbb{N}_0^d)$ and any $f \in L^2(\mathbb{R}_+)$, we have

$$\begin{aligned} (L_w x, f) &= \int_0^{+\infty} \overline{f(t)} \sqrt{w(t)} \sum_{j \in \mathbb{N}_0^d} e^{-t(\kappa \cdot j)} x(j) \,\mathrm{d}t \\ &= \sum_{j \in \mathbb{N}_0^d} x(j) \int_0^{+\infty} \overline{f(t)} \sqrt{w(t)} e^{-t(\kappa \cdot j)} \,\mathrm{d}t \end{aligned}$$

and consequently, $L_w^* : L^2(\mathbb{R}_+) \to \ell^2(\mathbb{N}_0^d)$ with

$$\big(L_w^* f\big)(j) = \big(\mathcal{L} \sqrt{w} f\big)(\kappa \cdot j), \;\; \forall j \in \mathbb{N}_0^d, \;\; \forall f \in L^2(\mathbb{R}_+).$$

Then, according to Lemma 4.1, the non-zero parts of $\tilde{H}$ and $S_w := L_w L_w^*$ are unitarily equivalent. We also need to find a precise formula for S_w. To this end, let $f \in L^2(\mathbb{R}_+)$.

Then

$$\begin{aligned}(S_w f)(t) &= \sqrt{w(t)} \sum_{j\in\mathbb{N}_0^d} e^{-t(\kappa\cdot j)} (L_w^* f)(j) \\ &= \sqrt{w(t)} \sum_{j\in\mathbb{N}_0^d} \int_0^{+\infty} f(s)\sqrt{w(s)} e^{-(t+s)\kappa\cdot j} \,\mathrm{d}s \\ &= \int_0^{+\infty} \sqrt{w(t)} \frac{f(s)}{\prod_{i=1}^d (1-e^{-(s+t)\kappa_i})} \sqrt{w(s)} \,\mathrm{d}s\,, \ \forall t \in \mathbb{R}_+.\end{aligned}$$

Observe that, according to Lemma 4.15,

$$\frac{1}{\prod_{i=1}^d (1-e^{-(s+t)\kappa_i})} = \left(\prod_{i=1}^d \frac{1}{\kappa_i}\right) \frac{1}{(s+t)^d} + v_1(s+t), \ \forall s,t \in \mathbb{R}_+,$$

where v_1 is a meromorphic function with pole of order $d-1$ at 0. Thus, $S_w = S + E$, where

$$(Sf)(t) = \left(\prod_{i=1}^d \frac{1}{\kappa_i}\right) \int_0^{+\infty} \sqrt{w(t)} \frac{f(s)}{(t+s)^d} \sqrt{w(s)} \,\mathrm{d}s\,, \ \forall t > 0, \ \forall f \in L^2(\mathbb{R}_+),$$

and

$$(Ef)(t) = \int_0^{+\infty} \sqrt{w(t)} v_1(t+s) \sqrt{w(s)} f(s) \,\mathrm{d}s\,, \ \forall t > 0, \ \forall f \in L^2(\mathbb{R}_+).$$

Now, with the help of the unitary transformation $U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$, where

$$(Uf)(x) = e^{\frac{x}{2}} f(e^x), \tag{4.49}$$

we can see that $\left(\prod_{i=1}^d \kappa_i\right) S = U^* \mathcal{M}_{\alpha_0}^{\frac{1}{2}} T \mathcal{M}_{\alpha_0}^{\frac{1}{2}} U$, where

$$\alpha_0(x) = 2^{-d} e^{-(d-1)x} w(e^x), \ \forall x \in \mathbb{R}, \tag{4.50}$$

and $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, with

$$(Tf)(x) = \int_{\mathbb{R}} \frac{1}{\cosh^d\left(\frac{x-y}{2}\right)} f(y) \,\mathrm{d}y\,, \ \forall f \in L^2(\mathbb{R}), \ \forall x \in \mathbb{R}.$$

Notice that $Tf = \left(\cosh\left(\frac{\cdot}{2}\right)\right)^{-d} * f$, so that $T = \mathcal{F}\mathcal{M}_\beta^2\mathcal{F}^*$, where β is given by (4.48). As a result we have the following chain of unitary equivalences:

$$\begin{aligned}\left(\prod_{i=1}^d \kappa_i\right) S &= U^* \mathcal{M}_{\alpha_0}^{\frac{1}{2}} T \mathcal{M}_{\alpha_0}^{\frac{1}{2}} U \\ &\sim \mathcal{M}_{\alpha_0}^{\frac{1}{2}} T \mathcal{M}_{\alpha_0}^{\frac{1}{2}} \\ &= \mathcal{M}_{\alpha_0}^{\frac{1}{2}} \mathcal{F}\mathcal{M}_\beta^2\mathcal{F}^* \mathcal{M}_{\alpha_0}^{\frac{1}{2}} \\ &\sim \mathcal{F}^*\mathcal{M}_{\alpha_0}^{\frac{1}{2}} \mathcal{F}\mathcal{M}_\beta^2\mathcal{F}^* \mathcal{M}_{\alpha_0}^{\frac{1}{2}}\mathcal{F} \\ &= \alpha_0(D)^{\frac{1}{2}} \mathcal{M}_\beta^2 \alpha_0(D)^{\frac{1}{2}} \\ &\sim \mathcal{M}_\beta \alpha_0(D) \mathcal{M}_\beta,\end{aligned}$$

where the last equivalence is obtained by Lemma 4.1 and the rest of them by the unitary character of U and $\mathcal{F}$. Therefore, by defining $\alpha := \left(\prod_{i=1}^{d} \frac{1}{\kappa_i}\right) \alpha_0$, where α_0 was definied in (4.50), we see that S is unitarily equivalent (modulo null-spaces) to $\Psi = \mathcal{M}_\beta \alpha(D) \mathcal{M}_\beta$. Then, it remains to notice that α is indeed described by (4.47).

Finally, it remains to prove that $E \in \mathbf{S}_p$, for any $p > 0$. To this end, notice that Lemma 4.15 suggests that v_1 is analytic on $\mathbb{R} \setminus \{0\}$, with a pole of order $d-1$ at 0. Thus,

$$v_1(t) = \sum_{j=1}^{d-1} \frac{c_{-j}}{t^j} + v_{1,\mathrm{an}}(t), \ \forall t \neq 0,$$

where $v_{1,\mathrm{an}}$ is analytic, and c_{-j} are real constants. According to Lemma 4.16, the integral operator with kernel $\sqrt{w(\lambda)} v_{1,\mathrm{an}}(\lambda+\mu) \sqrt{w(\mu)}$ belongs to every Schatten class. Indeed, this is an immediate consequence of the continuity of w and the compactness of its support ($\mathrm{supp}(w) = [0, \frac{3}{4}]$), where w is defined in (4.40), and the analyticity of $v_{1,\mathrm{an}}$. Finally, it is not difficult to see that, for $j = 1, 2, \ldots, d-1$, the integral operator with kernel $\sqrt{w(\lambda)}(\lambda+\mu)^{-j}\sqrt{w(\mu)}$ is unitarily equivalent to $U^* \mathcal{M}_{\alpha_j}^{1/2} T_j \mathcal{M}_{\alpha_j}^{1/2} U$, where U is defined in (4.49), T_j is an integral operator acting on $L^2(\mathbb{R})$ with kernel $\cosh^{-j}\left(\frac{\cdot}{2}\right)$, and

$$\alpha_j(x) = 2^{-j} e^{-(j-1)x} w(e^x), \ \forall x \in \mathbb{R}.$$

Since multiplication by $t|\log t|^{\gamma'-\gamma}\chi_0(t)$, for any $\gamma' > 0$, does not affect the Schatten class inclusions, we can assume that $w(t) = \frac{1}{j!} t^{j-1} |\log t|^{-\gamma'}$ near 0. Then notice that

$$\alpha_j(x) = \begin{cases} 0, & x \to +\infty \\ \frac{1}{2^j j!}|x|^{-\gamma'}, & x \to -\infty \end{cases}.$$

Therefore, by applying Lemma D.1, we see that

$$\lambda_n^{\pm}(\mathcal{M}_{\alpha_j}^{1/2} T_1 \mathcal{M}_{\alpha_j}^{1/2}) = O(n^{-\gamma'}), \ n \to +\infty.$$

Since γ' is arbitrary, this indicates that the operator $\mathcal{M}_{\alpha_j}^{1/2} T_1 \mathcal{M}_{\alpha_j}^{1/2}$ belongs to any Schatten class. Finally, notice that $E \sim \sum_{j=1}^{d-1} c_{-j} \mathcal{M}_{\alpha_j}^{1/2} T_1 \mathcal{M}_{\alpha_j}^{1/2}$ and thus, E belongs to all $\mathbf{S}_p$, indeed. □

Lemma 4.18. *The model operator $\tilde{H}$ is compact and for its eigenvalues we have the following asymptotic formula*

$$\lambda_n^{\pm}(\tilde{H}) = \frac{C^{\pm}}{\kappa_1 \ldots \kappa_d} n^{-\gamma} + o\left(n^{-\gamma}\right), \ n \to +\infty, \tag{4.51}$$

where the constants $C^{\pm}$ are described in (3.6).

Proof. In Lemma 4.17 was proved that $\tilde{H} \sim \Psi + E$, where Ψ is a compact pseudo-differential operator of the form $\Psi = \mathcal{M}_\beta \alpha(D) \mathcal{M}_\beta$, with α and β being given by (4.47) and (4.48), respectively, and $E \in \mathbf{S}_p$, for any $p > 0$. Therefore, $\tilde{H}$ is a sum of two compact operators so that $\tilde{H} \in \mathbf{S}_\infty$. Finally, Ψ satisfies the conditions of Lemma D.1. Indeed, α is already given by (4.47) in the desired form and regarding β, by differentiating, we can see that $\left(\cosh\left(\frac{\cdot}{2}\right)\right)^{-d} \in \mathcal{S}(\mathbb{R})$ and consequently, $\beta^2 \in \mathcal{S}(\mathbb{R})$. As a result, condition (D.6) is satisfied, too. Therefore, Lemma D.1 results that the eigenvalue asymptotics for Ψ are described by (4.51). To see that this is the case for $\tilde{H}$, too, it is enough to apply Lemma 4.2. For it only needs to notice that, since $E \in \mathbf{S}_p$, for any $p > 0$, $s_n(E) = o(n^{-\gamma})$, when $n \to +\infty$. □

Proof of Theorem 3.5. We aim to apply Lemma 4.2. For consider the model operator $\tilde{H}$ that is described in (4.41) and express H_a as the sum $H_a = \tilde{H} + (H_a - \tilde{H})$. Notice that $H_a - \tilde{H}$ is a Hankel operator acting on $\ell^2(\mathbb{N}_0^d)$ with kernel $\big(g - \tilde{g}\big)(\kappa \cdot j)$, for all $j \in \mathbb{N}_0^d$. In addition, Lemma 4.7 and (3.9) give that condition (3.9) holds true for $g - \tilde{g}$, too. Therefore, according to Proposition 3.4, $g - \tilde{g}$ produces a Hankel operator in any Schatten class $\mathbf{S}_q$, for $q > \frac{1}{\gamma+\varepsilon}$, and consequently, it belongs to $\mathbf{S}_{\frac{1}{\gamma}}$, too. As a result, its eigenvalues decay faster than $n^{-\gamma}$ as $n \to +\infty$. Thus, Lemma 4.2 implies the desired formulae. □

4.3 Continuous case

4.3.1 Proofs of Lemmas 3.6 and 3.7

In order to prove Lemma 3.6 we need to split the range of γ into two parts; $(0, \frac{1}{2})$ and $[\frac{1}{2}, +\infty)$. This choice is suggested by some interpolation methods which yield the asymptotic behaviour when $\gamma \in (0, \frac{1}{2})$. The remaining case is approached via weighted Hankel operators and interpolation, as well.

Lemma 4.19. *Consider a function $v : \mathbb{R}_+^d \to \mathbb{R}_+$ and suppose that there exist some positive constants M_2 and M_∞ such that*

$$\|\mathbf{H}_\mathbf{a}\|_{\mathbf{S}_2} \leq M_2 \left\| \frac{\mathbf{a}}{v} \right\|_{L_v^2}$$

and

$$\|\mathbf{H}_\mathbf{a}\| \leq M_\infty \left\| \frac{\mathbf{a}}{v} \right\|_{L^\infty}, \tag{4.52}$$

for every function $\mathbf{a}$ *defined on* $\mathbb{R}_+^d$*, where* $\mathbf{H}_\mathbf{a}$ *is the Hankel operator with kernel* $\mathbf{a}$*. Then, for every* $p \in (2, +\infty)$*, there exists a positive constant* M_p *such that*

$$\|\mathbf{H}_\mathbf{a}\|_{\mathbf{S}_{p,\infty}} \leq M_p \left\| \frac{\mathbf{a}}{v} \right\|_{L_v^{p,\infty}} .$$

Proof. Suppose that for a function $\mathbf{a}$ on $\mathbb{R}_+^d$ and $p \in (2, +\infty)$, $\frac{\mathbf{a}}{v} \in L_v^{p,\infty}(\mathbb{R}_+^d)$. Then

$$\int\limits_{\{\mathbf{x} \in \mathbb{R}_+^d : \ |f(\mathbf{x})| > \lambda\}} v(\mathbf{x})\, \mathrm{d}\mathbf{x} \leq \left(\frac{\left\| \frac{\mathbf{a}}{v} \right\|_{L_v^{p,\infty}}}{\lambda} \right)^p, \ \forall \lambda > 0.$$

For an arbitrary $\lambda > 0$, let us define the functions f_λ and g_λ as follows:

$$f_\lambda(\mathbf{x}) = \begin{cases} \frac{\mathbf{a}(\mathbf{x})}{v(\mathbf{x})}, & \text{if } \left| \frac{\mathbf{a}(\mathbf{x})}{v(\mathbf{x})} \right| \leq \frac{\lambda}{2M_\infty} \\ 0, & \text{otherwise} \end{cases}$$

and

$$g_\lambda(\mathbf{x}) = \begin{cases} \frac{\mathbf{a}(\mathbf{x})}{v(\mathbf{x})}, & \text{if } \left| \frac{\mathbf{a}(\mathbf{x})}{v(\mathbf{x})} \right| > \frac{\lambda}{2M_\infty} \\ 0, & \text{otherwise} \end{cases},$$

for every $\mathbf{x} \in \mathbb{R}^d_+$. Then

$$\pi_{\mathbf{H_a}}(\lambda) \leq \pi_{\mathbf{H}_{\upsilon f_\lambda}}\left(\frac{\lambda}{2}\right) + \pi_{\mathbf{H}_{\upsilon g_\lambda}}\left(\frac{\lambda}{2}\right).$$

The sequence of singular values $\{s_n(\mathbf{H}_{\upsilon f_\lambda})\}_{n\in\mathbb{N}}$ is decreasing so that

$$s_n(\mathbf{H}_{\upsilon f_\lambda}) \leq s_1(\mathbf{H}_{\upsilon f_\lambda}) = \|\mathbf{H}_{\upsilon f_\lambda}\| \leq M_\infty \frac{\lambda}{2M_\infty} = \frac{\lambda}{2}, \ \forall n \in \mathbb{N},$$

where the last inequality comes after (4.52). Consequently, $\pi_{\mathbf{H}_{\upsilon f_\lambda}}(\frac{\lambda}{2}) = 0$ and

$$\pi_{\mathbf{H_a}}(\lambda) \leq \pi_{\mathbf{H}_{\upsilon g_\lambda}}\left(\frac{\lambda}{2}\right). \tag{4.53}$$

Besides, $g_\lambda \in L^2_\upsilon(\mathbb{R}^d_+)$. In order to see this we make use of the following formula:

$$\|g_\lambda\|^2_{L^2_\upsilon} = 2\int_0^{+\infty} s \int\limits_{\{\mathbf{x}\in\mathbb{R}^d_+ :\ |g_\lambda(\mathbf{x})|>s\}} \upsilon(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\,;$$

see [13]. Thus, we obtain

$$\begin{aligned}
\|g_\lambda\|^2_{L^2_\upsilon} &= 2 \int\limits_{\left\{\mathbf{x}\in\mathbb{R}^d_+ :\ \left|\frac{\mathbf{a}(\mathbf{x})}{\upsilon(\mathbf{x})}\right| > \frac{\lambda}{2M_\infty}\right\}} \upsilon(\mathbf{x})\,\mathrm{d}\mathbf{x} \int_0^{\frac{\lambda}{2M_\infty}} s\,\mathrm{d}s + 2\int_{\frac{\lambda}{2M_\infty}}^{+\infty} s \int\limits_{\left\{\mathbf{x}\in\mathbb{R}^d_+ :\ \left|\frac{\mathbf{a}(\mathbf{x})}{\upsilon(\mathbf{x})}\right| > s\right\}} \upsilon(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&\leq 2\left\|\frac{\mathbf{a}}{\upsilon}\right\|^p_{L^{p,\infty}_\upsilon} \lambda^{-p}(2M_\infty)^p \int_0^{\frac{\lambda}{2M_\infty}} s\,\mathrm{d}s + 2\left\|\frac{\mathbf{a}}{\upsilon}\right\|^p_{L^{p,\infty}_\upsilon} \int_{\frac{\lambda}{2M_\infty}}^{+\infty} s^{1-p}\,\mathrm{d}s \\
&= \left\|\frac{\mathbf{a}}{\upsilon}\right\|^p_{L^{p,\infty}_\upsilon} \lambda^{2-p}(2M_\infty)^{p-2} + \frac{2}{p-2}\left\|\frac{\mathbf{a}}{\upsilon}\right\|^p_{L^{p,\infty}_\upsilon} \lambda^{2-p}(2M_\infty)^{p-2} \\
&= \frac{p}{p-2}\left\|\frac{\mathbf{a}}{\upsilon}\right\|^p_{L^{p,\infty}_\upsilon} \lambda^{2-p}(2M_\infty)^{p-2} < +\infty,
\end{aligned} \tag{4.54}$$

so $g_\lambda \in L^2_\upsilon(\mathbb{R}^d_+)$. Moreover, assumption (4.52) gives

$$\|\mathbf{H}_{\upsilon g_\lambda}\|_{\mathbf{S}_{2,\infty}} \leq \|\mathbf{H}_{\upsilon g_\lambda}\|_{\mathbf{S}_2} \leq M_2\|g_\lambda\|_{L^2_\upsilon}.$$

Therefore, a combination of (4.53) and (4.54) results

$$\begin{aligned}
\pi_{\mathbf{H_a}}(\lambda) &\leq \pi_{\mathbf{H}_{\upsilon g_\lambda}}\left(\frac{\lambda}{2}\right) \\
&\leq 2^2\lambda^{-2}\|\mathbf{H}_{\upsilon g_\lambda}\|^2_{\mathbf{S}_{2,\infty}} \\
&\leq 2^2\lambda^{-2}M_2^2\|g_\lambda\|^2_{L^2_\upsilon} \\
&\leq \lambda^{-p}\frac{2^p p M_2^2 M_\infty^{p-2}}{p-2}\left\|\frac{\mathbf{a}}{\upsilon}\right\|^p_{L^{p,\infty}_\upsilon}.
\end{aligned} \tag{4.55}$$

Now we set

$$M_p := \left(\frac{2^p p M_2^2 M_\infty^{p-2}}{p-2}\right)^{\frac{1}{p}}$$

and we notice that relation (4.55) does not depend on the choice of λ. Thus, after multiplying by λ^p both the two sides of (4.55) and taking supremum, we conclude that

$$\|\mathbf{H_a}\|_{\mathbf{S}_{p,\infty}} \leq M_p \left\| \frac{\mathbf{a}}{\upsilon} \right\|_{L^{p,\infty}_{\upsilon}} .$$

□

In order to prove the case for $\gamma \geq \frac{1}{2}$, we work in a way similar to that one of the discrete case. This means that we are going to make use of the reduction to one-dimensional weighted Hankel operators. In §2.1.1 we saw that $\mathbf{H_a}$ is unitarily equivalent (modulo null-spaces) to $\mathbf{\Gamma_0}$ (see (2.4)), which allows us to deduce the spectral behaviour of $\mathbf{H_a}$ from that of $\mathbf{\Gamma_0}$. To achieve the latter, we will need the following lemma.

Lemma 4.20. *Define the measure space*

$$(\mathcal{M}, \mu) := \bigoplus_{n\in\mathbb{Z}} (\mathbb{R}, 2^n \mathbf{m}) , \tag{4.56}$$

where $\mathbf{m}$ *is the Lebesgue measure on* $\mathbb{R}$*. Let* ϕ *be an analytic function on* $\mathbb{R}$*,* $p \in (0, +\infty)$*, and* $q \in (0, +\infty]$*. If* $\bigoplus_{n\in\mathbb{Z}} 2^{n(d-1)}\phi * V_n \in L^{p,q}(\mathcal{M}, \mu)$*, then the weighted Hankel operator* $\mathbf{\Gamma}_{\hat{\phi}}^{\frac{\mathbf{d-1}}{\mathbf{2}},\frac{\mathbf{d-1}}{\mathbf{2}}}$ *belongs to* $\mathbf{S}_{p,q}$*, where* $\hat{\phi} = \mathcal{F}\phi$*.*

Proof. Like in the discrete case, we define the space

$$\mathcal{B}^{\frac{1}{p}+d-1}_{p,q}(\mathbb{R}_+) := \left\{ f \in Hol(\mathbb{C}_+) : \bigoplus_{n\in\mathbb{Z}} 2^{n(d-1)} f * V_n \in L^{p,q}(\mathcal{M}, \mu) \right\}$$

and we aim to prove that the mapping $f \mapsto \mathbf{\Gamma}_{\hat{f}}^{\frac{d-1}{2},\frac{d-1}{2}}$ is a bounded linear operator from $\mathcal{B}^{\frac{1}{p}+d-1}_{p,q}(\mathbb{R}_+)$ to $\mathbf{S}_{p,q}$. According to Theorem 1.10, the mapping $f \mapsto \mathbf{\Gamma}_{\hat{f}}^{\frac{d-1}{2},\frac{d-1}{2}}$ represents a bounded linear operator from $B^{\frac{1}{p}+d-1}_{p}$ to $\mathbf{S}_p$, $\forall p \in (0, +\infty)$. Furthermore, by using the real interpolation method and the reiteration theorem it can be proved that, for every $p_0, p_1 \in (0, +\infty)$, $\theta \in (0, 1)$ and $q \in (0, +\infty]$,

$$(\mathbf{S}_{p_0}, \mathbf{S}_{p_1})_{\theta,q} = \mathbf{S}_{p,q}, \text{ where } p = (1-\theta)p_0 + \theta p_1.$$

Thus, in order to prove the initial statement, it remains to prove that, for every $p_0, p_1 \in (0, +\infty)$, $\theta \in (0, 1)$ and $q \in (0, +\infty]$,

$$\left(B^{\frac{1}{p_0}+d-1}_{p_0}, B^{\frac{1}{p_1}+d-1}_{p_1} \right)_{\theta,q} = \mathcal{B}^{\frac{1}{p}+d-1}_{p,q}$$

where $p = (1-\theta)p_0 + \theta p_1$. To this end, we make use again of the retract argument. For let

$$\mathcal{J} f = \bigoplus_{n\in\mathbb{Z}} 2^{n(d-1)} f * V_n, \ \forall f \in B^{\frac{1}{p}+d-1}_{p}.$$

Then, by the definition of the Besov space $B^{\frac{1}{p}+d-1}_{p}$ (see 1.10), $\mathcal{J}$ is an isometry from $B^{\frac{1}{p}+d-1}_{p}$ to $L^p(\mathcal{M}, \mu)$, where $(\mathcal{M}, \mu)$ is defined in (4.56). In addition, for every $n \in \mathbb{Z}$, we define the polynomials

$$\tilde{V}_n(x) = V_{n-1}(x) + V_n(x) + V_{n+1}(x), \ \forall x \in \mathbb{R}_+,$$

where V_n are described in (1.9). Notice that $V_n * \tilde{V}_n = V_n$, for every $n \in \mathbb{Z}$. Now we define the linear operator

$$\mathcal{K} \bigoplus_{n\in\mathbb{Z}} f_n = \sum_{n\in\mathbb{N}_0} 2^{-n(d-1)} f_n * \tilde{V}_n, \ \forall \bigoplus_{n\in\mathbb{Z}} f_n \in L^p(\mathcal{M},\mu),$$

which is bounded from $L^p(\mathcal{M},\mu)$ to $B_p^{\frac{1}{p}+d-1}$. To see this, it is enough to check that

$$\sum_{n\in\mathbb{Z}} 2^{n[1+p(d-1)]} \| \sum_{m\in\mathbb{Z}} 2^{-m(d-1)} f_m * \tilde{V}_m * V_n \|_p^p < +\infty, \ \forall \bigoplus_{n\in\mathbb{Z}} f_n \in L^p(\mathcal{M},\mu). \tag{4.57}$$

Observe that for every $n \in \mathbb{Z}$,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} f_m * \tilde{V}_m * V_n &= f_{n-1} * \tilde{V}_{n-1} * V_n + f_n * V_n * V_n + f_{n+1} * \tilde{V}_{n+1} * V_n \\ &= f_{n-1} * \tilde{V}_{n-1} * V_n + f_n * V_n + f_{n+1} * \tilde{V}_{n+1} * V_n.\end{aligned}$$

Thus, for every $n \in \mathbb{Z}$,

$$\begin{aligned}\| \sum_{m\in\mathbb{Z}} 2^{-m(d-1)} f_m * \tilde{V}_m * V_n \|_p^p \lesssim \|2^{-(n-1)(d-1)} f_{n-1} * \tilde{V}_{n-1} * V_n\|_p^p + \|2^{-n(d-1)} f_n * V_n\|_p^p + \\ + \|2^{-(n+1)(d-1)} f_{n+1} * \tilde{V}_{n+1} * V_n\|_p^p. \end{aligned} \tag{4.58}$$

Now we split the remaining of the proof into two cases: $p \in (1,+\infty)$, and $p \in (0,1]$. Assume first that $p \in (1,+\infty)$. Moreover the function v (see (1.8)) belongs to $C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz class on $\mathbb{R}$. Therefore, for any $m, n \in \mathbb{N}_0$,

$$\sup_{x\in\mathbb{R}} |x|^m |v^{(n)}(x)| < +\infty.$$

Thus, according to Theorem B.1, $v \in \mathscr{M}_p(\mathbb{R})$. Furthermore, Theorem B.2 yields that $v_n \in \mathscr{M}_p(\mathbb{R})$, with $\|v_n\|_{\mathscr{M}_p(\mathbb{R})} = \|v\|_{\mathscr{M}_p(\mathbb{R})}$, $\forall n \in \mathbb{Z}$. Consequently,

$$\|f * V_n\|_p \leq \|v\|_{\mathscr{M}_p(\mathbb{R})} \|f\|_p, \ \forall f \in L^p(\mathbb{R}), \ \forall n \in \mathbb{Z}. \tag{4.59}$$

Moreover, by (4.59) we get

$$\begin{aligned}\sum_{n\in\mathbb{Z}} 2^{n[1+p(d-1)]} \|2^{-(n-1)(d-1)} f_{n-1} * \tilde{V}_{n-1} * V_n\|_p^p &= \sum_{n\in\mathbb{Z}} 2^{(n+1)[1+p(d-1)]} \|2^{-n(d-1)} f_n * \tilde{V}_n * V_{n+1}\|_p^p \\ &\lesssim \sum_{n\in\mathbb{Z}} 2^{n[1+p(d-1)]} \|2^{-n(d-1)} f_n\|_p^p,\end{aligned}$$

and similarly,

$$\sum_{n\in\mathbb{Z}} 2^{n[1+p(d-1)]} \|2^{-(n+1)(d-1)} f_{n+1} * \tilde{V}_{n+1} * V_n\|_p^p \lesssim \sum_{n\in\mathbb{Z}} 2^{n[1+p(d-1)]} \|2^{-n(d-1)} f_n\|_p^p.$$

Therefore, by applying (4.58), and (4.59) we finally get

$$\begin{aligned}\sum_{n\in\mathbb{Z}} 2^{n[1+p(d-1)]} \| \sum_{m\in\mathbb{Z}} 2^{-m(d-1)} f_m * \tilde{V}_m * V_n \|_p^p &\lesssim \sum_{n\in\mathbb{Z}} 2^{n[1+p(d-1)]} \|2^{-n(d-1)} f_n\|_p^p \\ &= \| \bigoplus_{n\in\mathbb{Z}} f_n \|_p^p,\end{aligned}$$

which actually proves (4.57). Finally, we have that

$$\begin{aligned}\mathcal{K}\mathcal{J}f &= \sum_{n\in\mathbb{Z}} f_n * V_n * \tilde{V}_n \\ &= \sum_{n\in\mathbb{Z}} f_n * V_n = f, \ \forall f \in B_p^{\frac{1}{p}+d-1},\end{aligned}$$

so, according to the retract argument, the proof for $p > 1$ is complete.

Now let $p \in (0,1]$ and we again want to prove (4.57). To this end, we need to show that the sequence $\{V_n\}_{n\in\mathbb{Z}}$ defines a uniformly bounded sequence of $\mathscr{M}_p(\mathbb{R})$ multipliers. Moreover, notice that, since we only deal with bounded operators, it makes sense to restrict again our investigation on multipliers of the Hardy space $H^p(\mathbb{R})$. Therefore, due to Theorem B.2, it is enough to prove that v defines a multiplier on $H^p(\mathbb{R})$. To this end, we only need to verify that v satisfies conditions (i) and (ii) of Theorem C.1, which is already done in the proof of Lemma 4.4. □

Lemma 4.21. *Let $\gamma \geq \frac{1}{2}$ and $\mathbf{a_0}$ be a complex valued function in $C^M(\mathbb{R}_+)$ which satisfies relation (3.10). Then*

i) $\|2^{n(d-1)}(\mathcal{F}^{-1}\mathbf{a_0}) * V_n\|_\infty \leq \int_{2^{n-1}}^{2^{n+1}} |2^{n(d-1)}\mathbf{a_0}(x)| \,\mathrm{d}x$ *and,*

ii) $2^n\|2^{n(d-1)}(\mathcal{F}^{-1}\mathbf{a_0}) * V_n\|_q^q \leq C_q \left(\sum_{m=0}^{M} \int_{2^{n-1}}^{2^{n+1}} x^m |2^{n(d-1)}\mathbf{a_0}^{(m)}(x)| \,\mathrm{d}x\right)^q, \ \forall q > \frac{1}{M},$

for every $n \in \mathbb{Z}$, where C_q is a positive constant that depends only on q.

Proof. The proof of this lemma is obtained by repeating the proof of [24, Lemma 3.5] with $\check{\mathbf{h}}_n = 2^{n(d-1)}(\mathcal{F}^{-1}\mathbf{a_0}) * V_n$. □

Proof of Lemma 3.6. Firstly, we prove the case for $\gamma \in (0, \frac{1}{2})$. If we set $\gamma = \frac{1}{p}$, essentially, we want to prove that $\mathbf{H_a} \in \mathbf{S}_{p,\infty}$. This will be done by proving that there are some positive constants M_2 and M_∞ such that

$$\|\mathbf{H_a}\|_{\mathbf{S}_2} \leq M_2 \left\|\frac{\mathbf{a}}{v}\right\|_{L^2_v}$$

and

$$\|\mathbf{H_a}\| \leq M_\infty \left\|\frac{\mathbf{a}}{v}\right\|_{L^\infty},$$

where

$$v(\mathbf{x}) = \left(\sum_{j=1}^{d} x_j\right)^{-d}, \ \forall \mathbf{x} \in \mathbb{R}^d_+.$$

Then Lemma 4.19 implies the desired result. We know that

$$\|\mathbf{H_a}\|^2_{\mathbf{S}_2} = \iint\limits_{\mathbb{R}^d_+} |\mathbf{a}(\mathbf{x}+\mathbf{y})|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,.$$

As a result,

$$\begin{aligned}\|\mathbf{H_a}\|^2_{\mathbf{S}_2} &= \int_0^{+\infty} \cdots \int_0^{+\infty} |\mathbf{a}(x_1+y_1,\ldots,x_d+y_d)|^2 \,\mathrm{d}x_1\ldots\mathrm{d}x_d\,\mathrm{d}y_1\ldots\mathrm{d}y_d \\ &= \int_0^{+\infty} \cdots \int_0^{+\infty} \int_0^{z_1} \cdots \int_0^{z_d} |\mathbf{a}(z_1,\ldots,z_d)|^2 \,\mathrm{d}z_1\ldots\mathrm{d}z_d\,\mathrm{d}x'_1\ldots\mathrm{d}x'_d\end{aligned},$$

where the last line comes by making the change of variables $x'_j = x_j$ and $z_j = x_j + y_j$, $\forall j = 1, \dots, d$. Thus,

$$\begin{aligned}\|\mathbf{H}_{\mathbf{a}}\|^2_{\mathbf{S}_2} &= \int_0^{+\infty} \cdots \int_0^{+\infty} \prod_{j=1}^d x_j |\mathbf{a}(x_1, \dots, x_d)|^2 \, \mathrm{d}x_1 \dots \mathrm{d}x_d \\ &\le \int_0^{+\infty} \cdots \int_0^{+\infty} \Big(\sum_{j=1}^d x_j\Big)^d |\mathbf{a}(x_1, \dots, x_d)|^2 \, \mathrm{d}x_1 \dots \mathrm{d}x_d = \left\|\frac{\mathbf{a}}{v}\right\|^2_{L^2_v}\end{aligned}$$

so that $M_2 = 1$. Moreover, if $\frac{\mathbf{a}}{v} \in L^\infty(\mathbb{R}^d_+)$, then

$$\begin{aligned}|\mathbf{a}(\mathbf{x})| &\le \frac{\left\|\frac{\mathbf{a}}{v}\right\|_\infty}{(x_1 + \dots + x_d)^d} \\ &\le \frac{\left\|\frac{\mathbf{a}}{v}\right\|_\infty}{x_1 \dots x_d}, \ \forall \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d_+,\end{aligned}$$

and

$$\begin{aligned}|(\mathbf{H}_{\mathbf{a}} f, g)| &\le \iint_{\mathbb{R}^d_+} |\mathbf{a}(\mathbf{x} + \mathbf{y})| \, |f(\mathbf{y})| \, |g(\mathbf{x})| \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x} \\ &\le \left\|\frac{\mathbf{a}}{v}\right\|_\infty \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{|f(y_1, \dots, y_d)| \, |g(x_1, \dots, x_d)|}{(x_1 + y_1) \dots (x_d + y_d)} \, \mathrm{d}x_1 \dots \mathrm{d}x_d \, \mathrm{d}y_1 \dots \mathrm{d}y_d \\ &\le \pi^d \left\|\frac{\mathbf{a}}{v}\right\|_\infty \|f\|_2 \, \|g\|_2 \,, \ \forall f, g \in L^2(\mathbb{R}^d_+),\end{aligned}$$

where the last inequality follows by the boundedness of Carleman operator. More precisely, the last integral is the inner product $\big(\bigotimes_{j=1}^d \mathscr{C} f, g\big)$, where $\mathscr{C}$ denotes the Carleman operator. Moreover, the last computation shows that $M_\infty = \pi^d$. Finally, in order to use Lemma 4.19 and get the desired result, it remains to prove that if $\mathbf{a_0}$ satisfies relation (3.10), then $\frac{\mathbf{a}}{v} \in L^{p,\infty}_v$. For $\lambda > 0$,

$$\begin{aligned}\left\{\mathbf{x} \in \mathbb{R}^d_+ : \ \frac{|\mathbf{a}(x)|}{v(x)} > \lambda\right\} &= \left\{\mathbf{x} \in \mathbb{R}^d_+ : \ \Big(\sum_{j=1}^d x_j\Big)^d |\mathbf{a_0}(x_1 + x_2)| > \lambda\right\} \\ &\subset \left\{\mathbf{x} \in \mathbb{R}^d_+ : \ A_0 \left\langle \log \Big(\sum_{j=1}^d x_j\Big) \right\rangle^{-\frac{1}{p}} > \lambda\right\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^d_+ : \ \left\langle \log \Big(\sum_{j=1}^d x_j\Big) \right\rangle < \left(\frac{A_0}{\lambda}\right)^p\right\},\end{aligned}$$

where $A_0 := \sup_{t>0} t^2 \langle \log t \rangle^\gamma |\mathbf{a_0}(t)|$. Therefore,

$$\int \cdots \int_{\left\{\mathbf{x} \in \mathbb{R}^d_+ : \ \frac{|\mathbf{a}(x)|}{v(x)} > \lambda\right\}} \frac{\mathrm{d}x_1 \dots \mathrm{d}x_d}{(x_1 + \dots + x_d)^d} \le \int \cdots \int_{\left\{\mathbf{x} \in \mathbb{R}^d_+ : \ \langle \log(x_1 + \dots + x_d) \rangle < \left(\frac{A_0}{\lambda}\right)^p\right\}} \frac{\mathrm{d}x_1 \dots \mathrm{d}x_d}{(x_1 + \dots + x_d)^d}. \tag{4.60}$$

Set $x_1' = x_1$, $x_2' = x_2, \dots, x_{d-1}' = x_{d-1}$, and $x = x_1 + \cdots + x_d$. Then (4.60) gives

$$\begin{aligned}\int\cdots\int_{\left\{\mathbf{x}\in\mathbb{R}_+^d:\ \frac{|\mathbf{a}(x)|}{v(x)}>\lambda\right\}} \frac{\mathrm{d}x_1\ldots\mathrm{d}x_d}{(x_1+\cdots+x_d)^d} &\le \int\cdots\int_{\left\{\mathbf{x}\in\mathbb{R}_+^d:\ \langle\log(x_1+\cdots+x_d)\rangle<\left(\frac{A_0}{\lambda}\right)^p\right\}} \frac{\mathrm{d}x_1\ldots\mathrm{d}x_d}{(x_1+\cdots+x_d)^d}\\ &\le \int_{A_\lambda}\int_0^x \mathrm{d}x_1\int_0^x \mathrm{d}x_2\cdots\int_0^x \mathrm{d}x_{d-1}\,\frac{\mathrm{d}x}{x^d}\\ &= \int_{A_\lambda}\frac{\mathrm{d}x}{x},\end{aligned}$$

where

$$A_\lambda := \left\{x\in\mathbb{R}_+ :\ \langle\log x\rangle < \left(\frac{A_0}{\lambda}\right)^p\right\}.$$

Besides, if we set $A_\lambda^- := A_\lambda\cap(0,1)$, $A_\lambda^+ := A_\lambda\cap[1,+\infty)$, and $x_0(\lambda) := \lambda^{-p}\sqrt{A_0^{2p}-\lambda^{2p}}$ (provided that $\lambda\in(0,A_0)$), then it can be verified that $A_\lambda^- = (e^{-x_0(\lambda)},1)$ and $A_\lambda^+ = [1,e^{x_0(\lambda)})$, so that $A_\lambda = (e^{-x_0(\lambda)},e^{x_0(\lambda)})$. As a result,

$$\int\cdots\int_{\left\{\mathbf{x}\in\mathbb{R}_+^d:\ \frac{|\mathbf{a}(x)|}{v(x)}>\lambda\right\}} \frac{\mathrm{d}x_1\ldots\mathrm{d}x_d}{(x_1+\cdots+x_d)^d} \le 2x_0(\lambda)\le 2\lambda^{-p}A_0^p.$$

Thus,

$$\lambda^p \iint_{\left\{\mathbf{x}\in\mathbb{R}_+^2:\ \frac{|\mathbf{a}(x)|}{v(x)}>\lambda\right\}} \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{(x_1+x_2)^2}\le 2A_0^p,$$

and taking supremum over $\lambda\in(0,A_0)$ implies that

$$\left\|\frac{\mathbf{a}}{v}\right\|_{L_v^{p,\infty}} \le 2^{\frac{1}{p}}A_0.$$

From the last relation and by using Lemma 4.19, we conclude that there exists a positive constant M_p, such that

$$\|\mathbf{H_a}\|_{\mathbf{S}_{p,\infty}} \le M_p\left\|\frac{\mathbf{a}}{v}\right\|_{L_v^{p,\infty}} \le 2^\gamma M_pA_0,$$

so that relation (3.11) comes true, by setting $C_\gamma = 2^\gamma M_p$.

Now assume that $\gamma\ge\frac12$ and let $p=\frac1\gamma$. According to the discussion that precedes Lemma 4.20 and the Lemma itself, in order to prove that $\mathbf{H_a}\in\mathbf{S}_{p,\infty}$, it is enough to show that $\bigoplus_{n\in\mathbb{Z}}2^{n(d-1)}(\mathcal{F}^{-1}\mathbf{a_0})*V_n\in L^{p,\infty}(\mathcal{M},\mu)$, where the measure space $(\mathcal{M},\mu)$ is described in Lemma 4.20. Equivalently, we need to show that

$$\sup_{s>0}s^p\sum_{n\in\mathbb{Z}}2^n\left|\{x\in\mathbb{R} : |2^{n(d-1)}\big((\mathcal{F}^{-1}\mathbf{a_0})*V_n\big)(x)|>s\}\right| < +\infty. \tag{4.61}$$

For every $n\in\mathbb{Z}$ and $s>0$, set

$$E_n(s) := \left\{x\in\mathbb{R} : |2^{n(d-1)}\big((\mathcal{F}^{-1}\mathbf{a_0})*V_n\big)(x)|>s\right\}.$$

Then, for any $q>0$,

$$s^q|E_n(s)|\le\|2^{n(d-1)}\big((\mathcal{F}^{-1}\mathbf{a_0})*V_n\big)\|_q^q,\ \ \forall s>0.$$

Thus, for any $q > 0$,

$$s^p \sum_{n\in\mathbb{Z}} 2^n |E_n(s)| = s^{p-q} \sum_{n\in\mathbb{Z}} 2^n s^q |E_n(s)| \\ \le s^{p-q} \sum_{n\in\mathbb{Z}} 2^n \|2^{n(d-1)}\big((\mathcal{F}^{-1}\mathbf{a_0}) * V_n\big)\|_q^q, \ \forall s > 0.$$

Moreover, Lemma 4.21 gives that, for any $q \in (M^{-1}, p)$, where M is defined in (1.14),

$$s^p \sum_{n\in\mathbb{Z}} 2^n |E_n(s)| \le C_q s^{p-q} \sum_{n\in\mathbb{Z}} \left(\sum_{m=0}^{M} \int_{2^{n-1}}^{2^{n+1}} x^m |2^{n(d-1)} \mathbf{a_0}^{(m)}(x)| \, \mathrm{d}x \right)^q, \ \forall s > 0. \quad (4.62)$$

Moreover, for every $m = 0, 1, \ldots, M$, let A_m be such that

$$|\mathbf{a_0}^{(m)}(t)| \le A_m t^{-d-m} \langle \log t \rangle^{-\gamma}, \ \forall t > 0.$$

Then, for all $n \in \mathbb{Z}$ and $m = 0, 1, \ldots, M$,

$$\int_{2^{n-1}}^{2^{n+1}} x^m |2^{n(d-1)} \mathbf{a_0}^{(m)}(x)| \, \mathrm{d}x \le 2^{n(d-1)} A_m \int_{2^{n-1}}^{2^{n+1}} x^{-d} \langle \log x \rangle^{-\gamma} \, \mathrm{d}x \\ = 2^{n(d-1)} A_m \log 2 \int_{n-1}^{n+1} 2^{-(d-1)t} \langle t \log 2 \rangle^{-\gamma} \, \mathrm{d}t \,, \ \text{for } t = \log_2 x \\ \lesssim 2^{n(d-1)} A_m \int_{n-1}^{n+1} 2^{-(d-1)t} \langle t \rangle^{-\gamma} \, \mathrm{d}t \\ \lesssim A_m \langle n \rangle^{-\gamma}.$$

So there exists a positive constant C such that

$$\int_{2^{n-1}}^{2^{n+1}} x^m |2^{n(d-1)} \mathbf{a_0}^{(m)}(x)| \, \mathrm{d}x \le C A_m \langle n \rangle^{-\gamma}, \ \forall n \in \mathbb{Z}, \ \forall m = 0, 1, \ldots, M,$$

and, for $\mathbf{A} := \sum_{m=0}^{M} A_m$, (4.62) gives that, for any $q \in (M^{-1}, p)$,

$$s^p \sum_{n\in\mathbb{Z}} 2^n |E_n(s)| \le \mathbf{A}^q C'_q s^{p-q} \sum_{n\in\mathbb{Z}} \langle n \rangle^{-\gamma q}, \ \forall s > 0. \quad (4.63)$$

In addition, notice that, for any $s \ge \|2^{n(d-1)}\big((\mathcal{F}^{-1}\mathbf{a_0}) * V_n\big)\|_\infty$, $E_n(s) = \varnothing$. Consequently, by using the results of Lemma 4.21, we infer that if $E_n(s) \ne \varnothing$, then

$$s < \int_{2^{n-1}}^{2^{n+1}} |2^{n(d-1)} \mathbf{a_0}(x)| \, \mathrm{d}x \\ \le 2^{n(d-1)} A_0 \int_{2^{n-1}}^{2^{n+1}} x^{-d} \langle \log x \rangle^{-\gamma} \, \mathrm{d}x \\ \lesssim A_0 \langle n \rangle^{-\gamma}, \ n \in \mathbb{Z}.$$

So, if $E_n(s) \ne \varnothing$, then

$$\langle n \rangle \le \left(\frac{C A_0}{s} \right)^p =: N(s),$$

for some positive constant C. Thus, returning back to (4.63) gives

$$\begin{aligned} s^p \sum_{n\in\mathbb{Z}} 2^n |E_n(s)| &\le \mathbf{A}^q C_q' s^{p-q} \sum_{n\in\mathbb{Z}} \langle n\rangle^{-\gamma q} \\ &= \mathbf{A}^q C_q' s^{p-q} \sum_{\langle n\rangle \le N(s)} \langle n\rangle^{-\gamma q} \\ &\lesssim \mathbf{A}^q s^{p-q} \frac{1}{1-\gamma q} N(s)^{1-\gamma q} \end{aligned}$$

and, by substituting $N(s)$, we finally see that there exists a positive constant C', independent of s, such that

$$s^p \sum_{n\in\mathbb{Z}} 2^n |E_n(s)| \le C' \mathbf{A}^q, \ \forall s > 0,$$

which indicates that (4.61) is finite. More precisely, we see that

$$\sup_{s>0} s^p \sum_{n\in\mathbb{Z}} 2^n \big|\{x \in \mathbb{R} : |2^{n(d-1)}\big((\mathcal{F}^{-1}\mathbf{a_0}) * V_n\big)(x)| > s\}\big| \le C'\mathbf{A}^q, \ \forall q \in (M^{-1}, p);$$

thus taking the limit $q \to p$ gives

$$\Big\| \bigoplus_{n\in\mathbb{Z}} 2^{n(d-1)} \big((\mathcal{F}^{-1}\mathbf{a_0}) * V_n\big) \Big\|_{L^{p,\infty}(\mathcal{M},\mu)} \le C'^{\frac{1}{p}} \mathbf{A}. \tag{4.64}$$

Finally, an application of Lemma 4.20 combined with (4.64) gives immediately (3.11). $\square$

Proof of Lemma 3.7. Essentially, we want to prove that $\mathbf{H_a} \in \mathbf{S}^0_{p,\infty}$, where $p = \gamma^{-1}$. To this end, it is enough to approximate $\mathbf{H_a}$ in $\mathbf{S}_{p,\infty}$ by a sequence of $\mathbf{S}^0_{p,\infty}$ operators which satisfy (3.13), since $\mathbf{S}^0_{p,\infty}$ is a closed subspace of $\mathbf{S}_{p,\infty}$. Notice that, for any $\gamma > 0$, there exists $\gamma' > \gamma$ such that $M(\gamma') = M(\gamma)$. In addition, if we assume that $\mathbf{a_0}(t)$ equals zero for all small and large values of t, then Lemma 3.6 implies that $s_n(\mathbf{H_a}) = O(n^{-\gamma'})$ and thus, $s_n(\mathbf{H_a}) = o(n^{-\gamma})$, when $n \to +\infty$.

Now let χ_0 and χ_∞ be two smooth cut-off functions, defined as follows:

$$\chi_0(t) = \begin{cases} 1, & t \in (0, \frac{1}{4}] \\ 0, & t \ge \frac{1}{2} \end{cases}, \ \text{ and } \ \chi_\infty(t) = \begin{cases} 0, & t \in (0, 2] \\ 1, & t \ge 4 \end{cases}.$$

We also define the sequence of functions $\{\zeta_N\}_{N\in\mathbb{N}}$, with $\zeta_N(t) = \chi_0(\frac{t}{N})\chi_\infty(Nt)$, for all $t > 0$. Then notice that

$$\zeta_N(t) = \begin{cases} 0, & t \in (0, \frac{2}{N}] \cup [\frac{N}{2}, +\infty) \\ 1, & t \in [\frac{4}{N}, \frac{N}{4}] \end{cases}.$$

In the sequel, for every $N \in \mathbb{N}$, we define the Hankel operators $\mathbf{H}_N$ with kernel

$$\big(\mathbf{a_0}\zeta_N\big)(x_1 + x_2 + \cdots + x_d), \ \forall (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d_+.$$

Observe that $\mathbf{a_0}\zeta_N$ equals zero for all small and large values of t, so, by our initial discussion, $\mathbf{H}_N \in \mathbf{S}^0_{p,\infty}$. Now it remains to prove that

$$\lim_{N\to+\infty} \|\mathbf{H_a} - \mathbf{H}_N\|_{\mathbf{S}_{p,\infty}} = 0.$$

According to Lemma 3.6, it is enough to prove that

$$\lim_{N\to+\infty}\sup_{t>0} t^{d+m}\langle\log t\rangle^{\gamma}\left|\left[\mathbf{a_0}\left(1-\zeta_N\right)\right]^{(m)}(t)\right| = 0,\ \forall m=0,1,\dots,M(\gamma). \tag{4.65}$$

Notice that, for any $m = 0, 1, \dots, M(\gamma)$,

$$\left[\mathbf{a_0}(t)\big(1-\zeta_N(t)\big)\right]^{(m)} = \sum_{k=0}^{m}\binom{m}{k}\mathbf{a_0}^{(k)}(t)\big(1-\zeta_N(t)\big)^{(m-k)},\ \forall t>0,$$

and

$$t^{d+m}\langle\log t\rangle^{\gamma}\left|\left[\mathbf{a_0}\left(1-\zeta_N\right)\right]^{(m)}(t)\right| \le \sum_{k=0}^{m}\binom{m}{k}\left|\mathbf{a_0}^{(k)}(t)\right| t^{d+k}\langle\log t\rangle^{\gamma}\times$$
$$\times\, t^{m-k}\left|\big(1-\zeta_N(t)\big)^{(m-k)}\right|,\ \forall t>0. \tag{4.66}$$

Moreover, we choose $N \ge 4$. Then for any $t \in (0, \frac{2}{N}] \cup [\frac{4}{N}, \frac{N}{4}] \cup [\frac{N}{2}, +\infty)$, it can be easily seen that

$$t^{k}\left|\big(1-\zeta_N(t)\big)^{(k)}\right| < +\infty,\ \forall k=1,2,\dots,m. \tag{4.67}$$

In addition, for any $k = 1, 2, \dots, m$, and any $t \in (\frac{2}{N}, \frac{4}{N}) \cup (\frac{N}{4}, \frac{N}{2})$,

$$t^{k}\frac{\mathrm{d}}{\mathrm{d}t^{k}}\left(1-\zeta_N\right)(t) = (-1)^{k}\sum_{j=0}^{k}\binom{k}{j}\Big(\frac{t}{N}\Big)^{j}\chi_0^{(j)}\Big(\frac{t}{N}\Big)\,(Nt)^{k-j}\,\chi_\infty^{(k-j)}(Nt). \tag{4.68}$$

When $k = 0$, $1-\zeta_N$ is always bounded by 1. Moreover, notice that, since we have assumed that $N \ge 4$,

$$\chi_0\Big(\frac{t}{N}\Big) = 1,\ \forall t\in\Big(\frac{2}{N},\frac{4}{N}\Big),\quad\text{and}\quad \chi_\infty(Nt)=1,\ \forall t\in\Big(\frac{N}{4},\frac{N}{2}\Big). \tag{4.69}$$

Therefore, (4.68) becomes

$$t^{k}\frac{\mathrm{d}}{\mathrm{d}t^{k}}\left(1-\zeta_N\right)(t) = (-1)^{k}\left[\chi_0\Big(\frac{t}{n}\Big)(Nt)^{k}\chi_\infty^{(k)}(Nt) + \Big(\frac{t}{N}\Big)^{k}\chi_0^{(k)}\Big(\frac{t}{N}\Big)\chi_\infty(Nt)\right],$$

for any $t \in (\frac{2}{N}, \frac{4}{N}) \cup (\frac{N}{4}, \frac{N}{2})$. Thus, (4.69) implies that, for any $k = 1, 2, \dots, m$,

$$t^{k}\left|\frac{\mathrm{d}}{\mathrm{d}t^{k}}\left(1-\zeta_N\right)(t)\right| \le 4^{k}\max_{t\in[2,4]}\left|\chi_\infty^{(k)}(t)\right|,\ \forall t\in\Big(\frac{2}{N},\frac{4}{N}\Big), \tag{4.70}$$

and

$$t^{k}\left|\frac{\mathrm{d}}{\mathrm{d}t^{k}}\left(1-\zeta_N\right)(t)\right| \le 2^{-k}\max_{t\in[\frac{1}{4},\frac{1}{2}]}\left|\chi_0^{(k)}(t)\right|,\ \forall t\in\Big(\frac{N}{4},\frac{N}{2}\Big). \tag{4.71}$$

As a result, relations (4.67), (4.70) and (4.71) yield that, for every $m = 0, 1, \dots, M(\gamma)$, and every $N \ge 4$,

$$\sup_{t>0} t^{m-k}\left|\big(1-\zeta_N(t)\big)^{(m-k)}\right| < +\infty,\ \text{for every } k=0,1,\dots,m.$$

Finally, observe that hypothesis (3.12) implies that, for any $k = 0, 1, \dots, m$,

$$\left|\mathbf{a_0}^{(k)}(t)\right| t^{d+k}\langle\log t\rangle^{\gamma}\to 0,\quad\text{when } t\to 0^{+}\text{ or } t\to+\infty.$$

Therefore, the last two relations combined with (4.66), easily prove (4.65). □

4.3.2 The model operator and the proof of Theorem 3.8

Like we did in the discrete case, in order to prove Theorem 3.8, we need to introduce a model Hankel operator $\tilde{\mathbf{H}}$, which eventually, will give the spectral asymptotics of the Hankel operator that is defined in Theorem 3.8. To this end, let b_0, b_∞ be two non-negative constants, $\gamma > 0$, and χ_0, χ_∞ be two real valued functions in $C^\infty(\mathbb{R}_+)$ defined as follows:

$$\chi_0(t) = \begin{cases} 1, & t \in (0, \frac{1}{4}] \\ 0, & t \geq \frac{1}{2} \end{cases}, \quad \text{and} \quad \chi_\infty(t) = \begin{cases} 0, & t \in (0, 2] \\ 1, & t \geq 4 \end{cases}. \tag{4.72}$$

We now define the function σ as

$$\sigma(t) = \frac{b_\infty}{(d-1)!} t^{d-1} |\log t|^{-\gamma} \chi_0(t) + \frac{b_0}{(d-1)!} t^{d-1} |\log t|^{-\gamma} \chi_\infty(t), \ \forall t > 0. \tag{4.73}$$

Lemma 4.22. *Let $\gamma > 0$ and σ as described in (4.73) and denote by $\tilde{\mathbf{a}}_\mathbf{0}$ the Laplace transform of σ, $\mathcal{L}\sigma$. Then*

$$\tilde{\mathbf{a}}_\mathbf{0}(t) = t^{-d} |\log t|^{-\gamma} \left(b_0 \chi_0(t) + b_\infty \chi_\infty(t) \right) + \tilde{\mathbf{g}}(t), \ \forall t > 0,$$

where the error kernel $\tilde{\mathbf{g}} \in C^\infty(\mathbb{R}_+)$ satisfies

$$|\tilde{\mathbf{g}}^{(m)}(t)| \leq C_m t^{-d-m} \langle \log t \rangle^{-\gamma-1}, \ \forall t > 0, \ \forall m \in \mathbb{N}_0. \tag{4.74}$$

Proof. First assume that $b_0 = 0$ and $b_\infty = 1$. Then

$$\tilde{\mathbf{g}}(t) = \frac{1}{(d-1)!} \int_0^{+\infty} \lambda^{d-1} |\log \lambda|^{-\gamma} \chi_0(\lambda) e^{-\lambda t} \, \mathrm{d}\lambda - t^{-d} |\log t|^{-\gamma} \chi_\infty(t), \ \forall t > 0.$$

First of all, it can be easily seen that $\tilde{\mathbf{g}} \in C^\infty(\mathbb{R}_+)$. Therefore, relation (4.74) holds true in any compact interval of $\mathbb{R}_+$. Moreover, by the way χ_0 and χ_∞ have been defined, we can easily see that $\tilde{\mathbf{g}}^{(m)}(t) = O(1)$, when $t \to 0^+$, for any $m \in \mathbb{N}_0$, so that (4.74) holds again. It now remains to prove that (4.74) also stays true when $t \to +\infty$, since then a combination of these three cases will give the validity of (4.74) for any positive t. Indeed,

$$\tilde{\mathbf{g}}(t) = \frac{1}{(d-1)!} \int_0^{\frac{1}{4}} \lambda^{d-1} |\log \lambda|^{-\gamma} e^{-\lambda t} \, \mathrm{d}\lambda + \\ + \frac{1}{(d-1)!} \int_{\frac{1}{4}}^{\frac{1}{2}} \lambda^{d-1} |\log \lambda|^{-\gamma} e^{-\lambda t} \chi_0(\lambda) \, \mathrm{d}\lambda - t^{-d} |\log t|^{-\gamma}, \ \forall t \geq 4.$$

It is easy to see that both the three terms, in the expression above, are $C^\infty(\mathbb{R}_+)$ functions of t and all the derivatives of the second integral decay exponentially fast as $t \to +\infty$. Therefore, the derivatives of this integral satisfy (4.74). Finally, by using Lemma 4.6, notice that

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} \left(\int_0^{\frac{1}{4}} \lambda^{d-1} |\log \lambda|^{-\gamma} e^{-\lambda t} \, \mathrm{d}\lambda - t^{-d} |\log t|^{-\gamma} \right) =$$

$$= (-1)^m \left(\int_0^{\frac{1}{4}} \lambda^{m+d-1} |\log\lambda|^{-\gamma} e^{-\lambda t}\, \mathrm{d}\lambda - (m+d-1)! t^{-d-m} |\log t|^{-\gamma} \right) +$$

$$+ \sum_{k=1}^{m} \binom{m}{k} \left(t^{-d}\right)^{(m-k)} \left[(\log t)^{-\gamma}\right]^{(k)},$$

and the RHS is equal to

$$t^{-d-m} O(|\log t|^{-\gamma-1}) + \sum_{k=1}^{m} t^{-d-m} O(|\log t|^{-\gamma-1}) = t^{-d-m} O(|\log t|^{-\gamma-1}),\ t \to +\infty.$$

Thus, (4.74) is satisfied. A similar approach proves that the (4.74) holds valid when $b_\infty = 0$ and $b_0 = 1$. A combination of these two cases gives eventually that (4.74) holds true for any $b_0, b_\infty \geq 0$. □

Lemma 4.23. *Let $\gamma > 0$ and σ as described in (4.73). Denote by $\tilde{\mathbf{a}}_\mathbf{0}$ the Laplace transform of σ, $\mathcal{L}\sigma$, and consider the Hankel operator $\tilde{\mathbf{H}} : L^2(\mathbb{R}_+^d) \to L^2(\mathbb{R}_+^d)$, with*

$$\big(\tilde{\mathbf{H}} f\big)(x_1, \dots, x_d) = \int_0^{+\infty}\!\!\int_0^{+\infty} \tilde{\mathbf{a}}_\mathbf{0}\Big(\sum_{i=1}^{d}(x_i + y_i)\Big) f(y_1, \dots, y_d)\, \mathrm{d}y_1 \dots \mathrm{d}y_d\,,\ \forall f \in L^2(\mathbb{R}_+^d).$$

Then $\tilde{\mathbf{H}}$ is unitarily equivalent to the pseudo-differential operator $\Psi = \mathcal{M}_\beta \alpha(D) \mathcal{M}_\beta$, where

$$\alpha(x) = \begin{cases} \frac{b_0}{2^d (d-1)!} x^{-\gamma}\big(1 + o(1)\big), & x \to +\infty \\ \\ \frac{b_\infty}{2^d (d-1)!} |x|^{-\gamma}\big(1 + o(1)\big), & x \to -\infty \end{cases}, \tag{4.75}$$

and

$$\beta = \sqrt{\mathcal{F}^{-1}\left(\cosh\left(\frac{\cdot}{2}\right)\right)^{-d}} \tag{4.76}$$

Proof. First, notice that β in (4.76) is well-defined. The square root is allowed since $\mathcal{F}^{-1}\left(\cosh\left(\frac{\cdot}{2}\right)\right)^{-d}$ is positive on $\mathbb{R}$. Indeed, it is enough to observe that $\mathcal{F}^{-1}\left(\cosh\left(\frac{\cdot}{2}\right)\right)^{-1} = 2\pi \left(\cosh(2\pi^2 \cdot)\right)^{-1}$, which is a positive function. Then the result is obtained by noticing that the convolution of positive functions is positive.

Now let $f, g \in L^2(\mathbb{R}_+^d)$, then

$$(\tilde{\mathbf{H}} f, g) = \int_0^{+\infty} \left(\int_{\mathbb{R}_+} \!\!\cdots\! \int_{\mathbb{R}_+} \sqrt{\sigma(\lambda)} e^{-\lambda \sum_{i=1}^d y_i} f(y_1, \dots, y_d)\, \mathrm{d}y_1 \dots \mathrm{d}y_d \right) \times$$

$$\times \overline{\left(\int_{\mathbb{R}_+} \!\!\cdots\! \int_{\mathbb{R}_+} \sqrt{\sigma(\lambda)} e^{-\lambda \sum_{i=1}^d x_i} g(x_1, \dots, x_d)\, \mathrm{d}x_1 \dots \mathrm{d}x_d \right)}\, \mathrm{d}\lambda\,.$$

So, if $L : L^2(\mathbb{R}_+^d) \to L^2(\mathbb{R}_+)$, such that

$$\big(Lf\big)(\lambda) = \sqrt{\sigma(\lambda)} \int_{\mathbb{R}_+} \!\!\cdots\! \int_{\mathbb{R}_+} e^{-\lambda \sum_{i=1}^d x_i} f(x_1, \dots, x_d)\, \mathrm{d}x_1 \dots \mathrm{d}x_d\,,\ \forall f \in L^2(\mathbb{R}_+^d),\ \forall \lambda > 0,$$

then $\tilde{\mathbf{H}} = L^*L$. Then, if we set $S := LL^*$, Lemma 4.1 implies that the non-zero parts of $\tilde{\mathbf{H}}$ and S are equivalent. Notice that, for any $f \in L^2(\mathbb{R}_+^d)$ and $g \in L^2(\mathbb{R}_+)$,

$$\big(Lf, g\big) = \int\limits_{\mathbb{R}_+} \cdots \int\limits_{\mathbb{R}_+} f(x_1, \dots, x_d) \overline{\int\limits_0^{+\infty} \sqrt{\sigma(\lambda)} e^{-\lambda \sum_{i=1}^d x_i} g(\lambda)\, \mathrm{d}\lambda}\, \mathrm{d}x_1 \dots \mathrm{d}x_d\,,$$

and consequently,

$$\big(L^* g\big)(x_1, \dots, x_d) = \big(\mathcal{L}\sqrt{\sigma} g\big)(x_1 + \cdots + x_d),\ \forall g \in L^2(\mathbb{R}_+),\ \forall (x_1, \dots, x_d) \in \mathbb{R}_+^d.$$

Therefore, for any $f \in L^2(\mathbb{R}_+)$ and $\lambda > 0$,

$$\big(Sf\big)(\lambda) = \sqrt{\sigma(\lambda)} \int\limits_{\mathbb{R}_+} \cdots \int\limits_{\mathbb{R}_+} \int\limits_0^{+\infty} \sqrt{\sigma(\mu)} f(\mu) e^{-\mu \sum_{i=1}^d x_i} e^{-\lambda \sum_{i=1}^d x_i}\, \mathrm{d}\mu\, \mathrm{d}x_1 \dots \mathrm{d}x_d\,.$$

Notice that

$$\int_0^{+\infty} e^{-(\lambda+\mu)x}\, \mathrm{d}x = \frac{1}{\lambda + \mu},$$

so

$$\big(Sf\big)(\lambda) = \int_0^{+\infty} \sqrt{\sigma(\lambda)} \frac{f(\mu)}{(\lambda+\mu)^d} \sqrt{\sigma(\mu)}\, \mathrm{d}\mu\,.$$

Now, with the help of the unitary transformation $U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$, where

$$\big(Uf\big)(x) = e^{\frac{x}{2}} f(e^x),$$

we can see that $S = U^* \mathcal{M}_\alpha^{\frac{1}{2}} T \mathcal{M}_\alpha^{\frac{1}{2}} U$, where

$$\alpha(x) = 2^{-d} e^{-(d-1)x} \sigma(e^x),\ \forall x \in \mathbb{R}, \tag{4.77}$$

and $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, with

$$\big(Tf\big)(x) = \int_{\mathbb{R}} \frac{1}{\cosh^d\big(\frac{x-y}{2}\big)} f(y)\, \mathrm{d}y\,,\ \forall f \in L^2(\mathbb{R}),\ \forall x \in \mathbb{R}.$$

Ensuing, let β be as defined in (4.76). Then $\mathcal{F}\beta^2 = \big(\cosh\big(\frac{\cdot}{2}\big)\big)^{-d}$ and as a result, S is unitarily equivalent to the pseudo-differential operator $\mathcal{M}_\beta \alpha(D) \mathcal{M}_\beta$. Finally, it is not difficult to see that α, as it is defined in (4.77), satisfies indeed (4.75). □

Lemma 4.24 (Eigenvalue asymptotics of the model operator)**.** *Let $\tilde{\mathbf{H}}$ be the model operator, as it is also described in Lemma 4.23. Then $\tilde{\mathbf{H}}$ is compact and its eigenvalues follow the asymptotic formula below:*

$$\lambda_n^{\pm}(\tilde{\mathbf{H}}) = C^{\pm} n^{-\gamma} + o(n^{-\gamma}),\ n \to +\infty, \tag{4.78}$$

where the constant $C^{\pm}$ is given by (3.17).

Proof. In Lemma 4.23 we saw that $\tilde{\mathbf{H}}$ is unitarily equivalent (modulo null-spaces) to a compact pseudo-differential operator $\Psi = \mathcal{M}_\beta \alpha(D)\mathcal{M}_\beta$, where α and β are described by (4.75) and (4.76), respectively. Thus, it only remains to retrieve the eigenvalue asymptotics of Ψ. This will be achieved by using Lemma D.1.

We start by proving that the conditions of Lemma D.1 are satisfied. Indeed, α is already given by (4.75) in the right form. Finally, by differentiating, we can see that $\big(\cosh\big(\frac{\cdot}{2}\big)\big)^{-d} \in \mathcal{S}(\mathbb{R})$ and consequently, $\beta^2 \in \mathcal{S}(\mathbb{R})$, too. As a result, all the conditions of Lemma D.1 are satisfied and therefore, the eigenvalue asymptotics of Ψ are given by (D.7), for $A(+\infty) = \frac{b_0}{2^d(d-1)!}$ and $A(-\infty) = \frac{b_\infty}{2^d(d-1)!}$. Therefore, the eigenvalue asymptotic formula for $\tilde{H}$ is described by (4.78). □

Proof of Theorem 3.8. The proof is based on an application of Lemma 4.2, but first we need to reduce to the case of $\kappa = \mathbf{1}$, where $\mathbf{1} = (1,1,\dots,1)$. Indeed, For $f \in L^2(\mathbb{R}^d_+)$,

$$\begin{aligned}
\big(\mathbf{H}_\mathbf{a} f\big)(\mathbf{x}) &= \int\limits_{\mathbb{R}^d_+} \mathbf{a}(\mathbf{x}+\mathbf{y}) f(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= \int\limits_{\mathbb{R}_+} \cdots \int\limits_{\mathbb{R}_+} \mathbf{a_0}\big(\kappa \cdot (x_1+y_1,\dots,x_d+y_d)\big) f(y_1,\dots,y_d)\,\mathrm{d}y_1 \dots \mathrm{d}y_d \\
&= (\kappa_1 \dots \kappa_d)^{-1} \int\limits_{\mathbb{R}_+} \cdots \int\limits_{\mathbb{R}_+} \mathbf{a_0}\Big(\sum_{i=1}^d (\kappa_i x_i + y_i')\Big) f\Big(\frac{y_1'}{\kappa_1},\dots,\frac{y_d'}{\kappa_d}\Big)\,\mathrm{d}y_1' \dots \mathrm{d}y_d',
\end{aligned} \tag{4.79}$$

by making the change of variables $y_i' = \kappa_i y_i$, for $i = 1,\dots,d$. Now we define the operator $T : L^2(\mathbb{R}^d_+) \to L^2(\mathbb{R}^d_+)$ by

$$\big(Tf\big)(x_1,\dots,x_d) := \frac{1}{\sqrt{\kappa_1 \dots \kappa_d}} f\left(\frac{x_1}{\kappa_1},\dots \frac{x_d}{\kappa_d}\right),\ \forall f \in L^2(\mathbb{R}^d_+),\ \forall (x_1,\dots,x_d) \in \mathbb{R}^d_+,$$

and we notice that it is an isometric isomorphism. Getting back to (4.79) and setting $x_i' = \kappa_i x_i$, for $i = 1,\dots,d$, we have:

$$\sqrt{\kappa_1 \dots \kappa_d}\big(\mathbf{H}_\mathbf{a} f\big)\Big(\frac{x_1'}{\kappa_1},\dots,\frac{x_d'}{\kappa_d}\Big) = \int\limits_{\mathbb{R}_+} \cdots \int\limits_{\mathbb{R}_+} \mathbf{a_0}\Big(\sum_{i=1}^d (x_i'+y_i')\Big) \big(Tf\big)(y_1',\dots,y_d')\,\mathrm{d}y_1' \dots \mathrm{d}y_d'$$

and equivalently,

$$\kappa_1 \dots \kappa_d \big(T\mathbf{H}_\mathbf{a} f\big)(x_1,\dots,x_d) = \big(\mathbf{H_1} Tf\big)(x_1,\dots,x_d),\ \forall f \in L^2(\mathbb{R}^d_+),\ \forall (x_1,\dots,x_d) \in \mathbb{R}^2_+.$$

where $\mathbf{H_1}$ is the integral Hankel operator with kernel

$$\mathbf{a^1}(\mathbf{x}) := \mathbf{a_0}(x_1 + x_2 + \dots + x_d),\ \forall \mathbf{x} = (x_1,x_2,\dots,x_d) \in \mathbb{R}^d_+.$$

Thus,

$$\mathbf{H_a} \sim \frac{1}{\kappa_1 \dots \kappa_d} \mathbf{H_1} \tag{4.80}$$

and it remains to find the eigenvalue asymptotics of $\mathbf{H_1}$.

First of all, the hypotheses of the Theorem enable us to express the function $\mathbf{a_0}$ as

$$\mathbf{a_0}(t) = b_0 t^{-d} |\log t|^{-\gamma} \chi_0(t) + b_\infty t^{-d} |\log t|^{-\gamma} \chi_\infty(t) + \mathbf{g}(t),\ \forall t > 0,$$

where the functions χ_0 and χ_∞ are defined in (4.72). Now express the operator $\mathbf{H_1}$ as

$$\mathbf{H_1} = \tilde{\mathbf{H}} + (\mathbf{H_1} - \tilde{\mathbf{H}}),$$

where $\tilde{\mathbf{H}}$ is the model operator which was constructed in Lemma 4.24. In terms of kernels, this corresponds to

$$\mathbf{a^1} = \tilde{\mathbf{a}} + (\mathbf{g} - \tilde{\mathbf{g}}),$$

where $\tilde{\mathbf{a}}(\mathbf{x}) = \tilde{\mathbf{a}}_\mathbf{0}(\mathbf{1} \cdot \mathbf{x})$. Observe that the asymptotic formula for the eigenvalues of $\tilde{\mathbf{H}}$ is already known by Lemma 4.24. Therefore, in order to apply Lemma 4.2 we need to prove that the spectral contribution of $\mathbf{H_1} - \tilde{\mathbf{H}}$ is negligible. To see this, notice that the function $\mathbf{g}$ satisfies the "little o" conditions in relations (3.14) and (3.15) and, by Lemma 4.22, same does $\tilde{\mathbf{g}}$. Thus, the function $\mathbf{g} - \tilde{\mathbf{g}}$ satisfies the assumptions of Lemma 3.7 and as a result,

$$s_n(\mathbf{H}_1 - \tilde{\mathbf{H}}) = o(n^{-\gamma}), \ n \to +\infty.$$

Now it is readily seen that Lemma 4.2 can be applied and it implies that

$$\lambda_n^\pm(\mathbf{H_1}) = \lambda_n^\pm(\tilde{\mathbf{H}}), \ \forall n \in \mathbb{N}.$$

As a result, relation (4.80) and the eigenvalue asymptotics of $\tilde{\mathbf{H}}$ (Lemma 4.24) yield (3.16). □

Chapter 5

Null-spaces of multi-variable Hankel operators

Our main results aim to provide asymptotics for the non-zero eigenvalues of the investigated Hankel operators. It is natural to ask though what happens with the zero eigenvalues or equivalently, with null-spaces of Hankel operators. We recall that, for any operator $T: X \to Y$, its null-space $\mathrm{Null}(T)$ is defined as

$$\mathrm{Null}(T) := \{x \in X : \; Tx = 0\}.$$

This section is devoted to the dimension of null-spaces of multidimensional Hankel operators. To begin with, let us consider the example of a Hankel operator $H_a : \ell^2(\mathbb{N}_0^d) \to \ell^2(\mathbb{N}_0^d)$, where $a(j) = a_0(|j|)$ and a_0 is a complex valued sequence defined on $\mathbb{N}_0$. It is easy to check that the null-space of H_a is of infinite dimension. This could be done by considering elements $x(j)$ of $\ell^2(\mathbb{N}_0^d)$ such that they sum to zero on the hyperplanes $|j| = n, \; \forall n \in \mathbb{N}$, and $x(0, \dots, 0) = 0$. For example, for every $n \in \mathbb{N}$, define

$$x_n(j_1, j_2, \dots, j_d) = \begin{cases} 1, & \text{if } j_1 = n \text{ and } j_d = 0, \\ -1, & \text{if } j_1 = 0 \text{ and } j_d = n, \\ 0, & \text{otherwise} \end{cases},$$

and observe that

$$\big(H_a x_n\big)(i) = \sum_{j \in \mathbb{N}_0^d} a_0(|i| + |j|) x_n(j) = 0, \; \forall i \in \mathbb{N}_0^d,$$

so that $H_a x_n = 0, \; \forall n \in \mathbb{N}$. Finally, notice that the sequence $\{x_n\}_{n\in\mathbb{N}}$ comprises linearly independent elements of $\ell^2(\mathbb{N}_0^d)$ and therefore $\mathrm{Null}(H_a)$ is infinite dimensional.

It is natural to ask if this is a common property of multi-dimensional Hankel operators or we can find a multi-dimensional Hankel operator with trivial null-space. We remind that in the one dimensional case what holds is that every Hankel operator has either trivial or infinite dimensional null-space. This can be proved by using Beurling's theorem (cf. [21, §1.2]) and it is intriguing to see whether a generalisation holds in several dimensions.

The answer to the previous question is affirmative. For consider an arbitrary Hankel operator $H_a : \ell^2(\mathbb{N}_0^d) \to \ell^2(\mathbb{N}_0^d)$ and define the operators L and R, acting on $\ell^2(\mathbb{N}_0^d)$, as follows: for any $x \in \ell^2(\mathbb{N}_0^d)$,

$$\big(Lx\big)(i_1, i_2, \dots, i_d) = x(i_1 + 1, i_2, \dots, i_d), \; \forall (i_1, i_2, \dots, i_d) \in \mathbb{N}_0^d;$$

and

$$\big(Rx\big)(i_1, i_2, \dots, i_d) = \begin{cases} 0, & \text{if } i_1 = 0 \\ x(i_1 - 1, i_2, \dots, i_d), & \text{otherwise} \end{cases}, \ \forall (i_1, i_2, \dots, i_d) \in \mathbb{N}_0^d.$$

Then, for any $y \in \ell^2(\mathbb{N}_0^d)$, there exists a (non-unique) $\tilde{y} \in \ell^2(\mathbb{N}_0^d)$, such that $L\tilde{y} = y$. Indeed, define

$$\tilde{y} = Ry. \tag{5.1}$$

Ensuing, assume that $x \in \text{Null}(H_a) \setminus \{0\}$ or equivalently, that $(H_a x, y) = 0$, $\forall y \in \ell^2(\mathbb{N}_0^d)$, where $x \neq 0$. Let $y \in \ell^2(\mathbb{N}_0^d)$ and observe that

$$\begin{aligned}
(H_a Rx, y) &= (H_a Rx, L\tilde{y}) \ \ \text{(where } \tilde{y} \text{ is defined in (5.1))} \\
&= \sum_{i,j \in \mathbb{N}_0^d} a(i+j)\big(Rx\big)(j)\overline{\big(L\tilde{y}\big)}(i) \\
&= \sum_{i \in \mathbb{N}_0^d} \sum_{\substack{j_1 \geq 1 \\ j_2, \dots, j_d \geq 0}} a(i+j)x(j_1 - 1, j_2, \dots, j_d)\overline{\tilde{y}}(i_1 + 1, i_2, \dots, i_d) \\
&= \sum_{\substack{i_1 \geq 1 \\ i_2, \dots, i_d \geq 0}} \sum_{j \in \mathbb{N}_0^d} a(i+j)x(j)\overline{\tilde{y}}(i) \\
&= \sum_{i,j \in \mathbb{N}_0^d} a(i+j)x(j)\overline{\tilde{y}}(i), \ \ \text{since } \tilde{y}(0, n) = 0, \ \forall n \in \mathbb{N}_0^{d-1} \\
&= (H_a x, \tilde{y}) = 0.
\end{aligned}$$

Thus, $Rx \in \text{Null}(H_a)$ and inductively, it can be proved that $R^n x \in \text{Null}(H_a)$, $\forall n \in \mathbb{N}$. Now it remains to show that the vectors $x, Rx, \dots, R^n x, \dots$ form a sequence of linearly independent elements of $\text{Null}(H_a)$. To this end, let $N \in \mathbb{N}$ and assume that there are complex constants $\{\kappa_j\}_{j=0}^N$, such that

$$\sum_{j=0}^N \kappa_j R^j x = 0 \ \text{ and } \ \sum_{j=0}^N |\kappa_j|^2 > 0.$$

Then, for any $i_1 \geq N$,

$$\sum_{j=0}^N \kappa_j x(i_1 - j, i_2, \dots, i_d) = 0, \ \forall i_2, \dots, i_d \geq 0. \tag{5.2}$$

Similarly, for any $i_1 \in \{0, 1, \dots, N-1\}$,

$$\sum_{j=0}^{i_1} \kappa_j x(i_1 - j, i_2, \dots, i_d) = 0, \ \forall i_2, \dots, i_d \geq 0.$$

Thus, if we assume that $\kappa_0 \neq 0$, taking $i_1 = 0$ yields that

$$x(0, i_2, \dots, i_d) = 0, \ \forall i_2, \dots, i_d \geq 0.$$

Repeating for $i_1 = 1, 2, \dots, N-1$ results that

$$x(i_1, i_2, \dots, i_d) = 0, \ \forall i_1 \in \{0, 1, \dots, N-1\}, \ \forall i_2, \dots, i_d \geq 0,$$

and therefore, (5.2) eventually gives that

$$x(i_1, i_2, \dots, i_d) = 0,\ \forall (i_1, \dots, i_d) \in \mathbb{N}_0^d \Leftrightarrow x = 0;$$

though the latter is a contradiction and as a result, $\kappa_0 = 0$. Then (5.2) is reduced to

$$\sum_{j=1}^{N} \kappa_j x(i_1 - j, i_2, \dots, i_d) = 0,\ \forall i_1 \geq N,\ \forall i_2, \dots, i_d \geq 0.$$

Moreover, we have that for any $i_1 \in \{1, 2, \dots, N-1\}$

$$\sum_{j=1}^{i_1} \kappa_j x(i_1 - j, i_2, \dots, i_d) = 0,\ \forall i_2, \dots, i_d \geq 0.$$

Thus, if we assume that $\kappa_1 \neq 0$ and pick $i_1 = 1$, then, a repetition of the previous arguments will lead to a contradiction. Thus, we prove inductively that $\kappa_j = 0$, for all $j = 0, 1, \dots, N$, and consequently, $\{R^j x\}_{j \in \mathbb{N}_0}$ is indeed a sequence of linearly independent vectors of $\mathrm{Null}(H_a)$, which shows that the null-space of H_a is infinite dimensional.

For the continuous case, we can similarly define the operators $L : L^2(\mathbb{R}_+^d) \to L^2(\mathbb{R}_+^d)$, with

$$\big(Lf\big)(x_1, x) = f(x_1 + 1, x),\ \forall (x_1, x) \in \mathbb{R}_+ \times \mathbb{R}_+^{d-1},\ \forall f \in L^2(\mathbb{R}_+^d),$$

and $R : L^2(\mathbb{R}_+^d) \to L^2(\mathbb{R}_+^d)$, with

$$\big(Rf\big)(x_1, x) = \begin{cases} 0, & x_1 \in (0, 1] \\ f(x_1 - 1, x), & x_1 \in (1, +\infty) \end{cases},\ \forall (x_1, x) \in \mathbb{R}_+ \times \mathbb{R}_+^{d-1},\ \forall f \in L^2(\mathbb{R}_+^d).$$

By a similar reasoning, we can prove that if $f \in \mathrm{Null}(\mathbf{H}) \setminus \{0\}$, where $\mathbf{H}$ is an integral Hankel operator on $L^2(\mathbb{R}_+^d)$, then $\{R^j f\}_{j \in \mathbb{N}_0}$ forms a linear independent sequence of $\mathrm{Null}(\mathbf{H})$.

Finally, to the best of the author's knowledge, a multi-variable analogue of Beurling-Lax theorem is unknown. In particular, a classification or an explicit description of the shift invariant subspaces of $H^2(\mathbb{D}^d)$, i.e. closed subspaces S of $H^2(\mathbb{D}^d)$ such that $z_i S \subset S$, for any $i = 1, 2, \dots, d$, seems to be a very difficult problem (cf. [27, p. 78]). Useful information about the topic, as well as, a construction of a particular type of invariant spaces of $H^2(\mathbb{D}^d)$ can be found in [17].

Appendices

Appendix A

Interpolation

Let X_1 and X_2 be two topological vector spaces (eg. quasi-Banach spaces). If there exists a Hausdorff topological vector space V such that X_1 and X_2 are continuously embedded into it, then X_1 and X_2 are called *compatible* (or *compatible couple*). Moreover, $X_1 \cap X_2$ and $X_1 + X_2$ are subspaces of V. For sake of simplicity, let us assume that X_i and Y_i, where $i = 1, 2$, are two pairs of compatible quasi-Banach spaces. Denote by $X := (X_1, X_2)$ and $Y := (Y_1, Y_2)$ the quasi-Banach spaces with the following properties. For $A = X$ or Y, $A_1 \cap A_2$ is continuously embedded into A and A is continuously embedded into $A_1 + A_2$. Furthermore, if $T : X_1 + X_2 \to Y_1 + Y_2$ is a linear operator such that the restrictions $T_{X_i \to Y_i}$, $i = 1, 2$, are bounded linear operators, then the restriction $T_{X \to Y}$ is a bounded linear operator, too. Under those conditions, X and Y are called *interpolation spaces* with respect to X_1, X_2 and Y_1, Y_2, respectively. The main purpose of the various interpolation methods is to reduce the boundedness examination to simpler vector spaces, on which boundedness conditions are obtained more easily. Two of the most common techniques in interpolation are the *K-method* (a real interpolation method) and the *complex interpolation method*. More precisely, given a compatible couple of quasi-Banach spaces X_1 and X_2, the K-method generates the interpolation spaces

$$X_{\theta,q} := (X_1, X_2)_{\theta,q},$$

where either $\theta \in (0, 1)$ and $q \in [1, +\infty]$, or $\theta \in [0, 1]$ and $q = +\infty$; and the complex interpolation method the spaces

$$X_{[\theta]} := (X_1, X_2), \; \theta \in (0, 1).$$

The reiteration theorem below provides a way to produce intermediate spaces, when we interpolate between spaces that have been obtained by the K-method or by complex interpolation; see [3, §3.5 and §4.6], for the real and the complex method, respectively.

Theorem A.1 (Reiteration Theorem)**.** *Let X_{θ_0,q_0} and X_{θ_1,q_1} (resp. $X_{[\theta_1]}$, $X_{[\theta_2]}$) be two interpolation spaces created by the K-method (resp. complex method) from the compatible couple X_1, X_2. Then for any $q \in [1, +\infty]$*

$$(X_{\theta_0,q_0}, X_{\theta_1,q_1})_{\theta,q} = X_{\theta,q} \; \left(\textit{resp. } \left(X_{[\theta_0]}, X_{[\theta_1]}\right)_{[\theta]} = X_{[\theta]}\right), \; \textit{where } \; \theta = (1-t)\theta_0 + t\theta_1, \; t \in (0, 1).$$

Finally, another useful technique is the so called *retract argument*. Like the reiteration theorem, it also applies in both real and complex interpolation. If X and Y are two quasi-Banach spaces, then X is a *retract* of Y if there are bounded linear mappings $\mathcal{J} : X \to Y$ and $\mathcal{K} : Y \to X$ such that $\mathcal{K}\mathcal{J}$ is the identity map on X. Then we have the following theorem:

Theorem A.2 (Retract argument, [3], Theorem 6.4.2). *If X_i and Y_i, for $i = 1, 2$, are two compatible couples of quasi-Banach spaces such that X_i is a retract of Y_i, then the (quasi-Banach) spaces $X_{\theta,q}$ and $X_{[\theta]}$ are retracts of the (quasi-Banach) spaces $Y_{\theta,q}$ and $Y_{[\theta]}$, respectively.*

A.1 Complex interpolation

We present a bit more detailed description of some aspects of complex interpolation, since this method is also used for proving some of our lemmas.

Let $(X_0, \|\cdot\|^{(0)})$ and $(X_1, \|\cdot\|^{(1)})$ be a compatible couple of complex quasi-Banach spaces and define the space $X := X_0 + X_1$, which we endow with the norm

$$\|x\|_+ := \inf \left\{ \|x_0\|^{(0)} + \|x_1\|^{(1)}, \ x = x_0 + x_1 \right\}, \ \forall x \in X.$$

Then $(X, \|\cdot\|_+)$ is a quasi-Banach space, too.

Ensuing, we define the closed strip

$$S = \{z \in \mathbb{C} : \ 0 \leq \operatorname{Re} z \leq 1\} \tag{A.1}$$

and we denote by S° it interior. Let $\mathscr{F}(X_0, X_1)$ be the set of functions $f : S \to X$ which are continuous on S, analytic in S° and satisfy the following three conditions:

(i) $f(it) \in X_0$, for all $t \in \mathbb{R}$, and the map $t \mapsto f(it)$ is $\|\cdot\|^{(0)}$-continuous. Similarly, $f(1 + it) \in X_1$, for all $t \in \mathbb{R}$, and the map $t \mapsto f(1 + it)$ is $\|\cdot\|^{(1)}$-continuous.

(ii) $\sup_{z\in S} \|f(z)\|_+ < +\infty$.

(iii) $|||f||| := \sup_{t\in\mathbb{R}} \left\{ \|f(it)\|^{(0)}, \|f(1 + it)\|^{(1)} \right\} < +\infty$.

Then $(\mathscr{F}(X_0, X_1), |||\cdot|||)$ is a Banach space. Furthermore, for any $t \in (0, 1)$, define the space

$$X_t := \{f(t), \ f \in \mathscr{F}(X_0, X_1)\}$$

with norm

$$\|x\|^{(t)} := \inf_{\{f\in\mathscr{F}(X_0,X_1): \ f(t)=x\}} |||f|||.$$

Then X_t is the interpolation space $(X_0, X_1)_{[t]}$.

We close this section by displaying a useful lemma that often occurs in complex interpolation.

Theorem A.3 (Hadamard's three line theorem). *Let $\phi : S \to \mathbb{C}$ be a continuous function which is analytic in S°, where S is defined in (A.1) In addition, assume that there exist some positive constants M_0 and M_1 such that*

$$|\phi(it)| \leq M_0 \ \ \textit{and} \ \ |\phi(1 + it)| \leq M_1, \ \forall t \in \mathbb{R}.$$

Then

$$|\phi(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}, \ \forall z \in S.$$

A.2 Lorentz and Schatten-Lorentz spaces

Some of the most famous applications of the K-method and the complex interpolation method yield interpolation spaces of Lorentz function spaces and Schatten-Lorentz ideals. These spaces occur in our survey so, for the reader's convenience, we briefly recall their definitions.

Let (X, ν) be an arbitrary measure space. For any complex valued function f on X, define the *decreasing rearrangement* f^* of f by

$$f^*(t) := \inf_{s>0} \left\{ \nu\big(\{x \in X : |f(x)| > s\} \big) \leq t \right\}, \ \forall t > 0.$$

Then, for any $p, q \in (0, +\infty)$, we define the *Lorentz space* $L^{p,q}(X, \nu)$ as the space of all ν-measurable functions of X such that

$$\int_0^{+\infty} \frac{1}{t} \left(t^{\frac{1}{p}} f^*(t) \right)^q \mathrm{d}t < +\infty.$$

We also define the Lorentz space $L^{p,\infty}(X, \nu)$, for $p \in (0, +\infty)$, by

$$f \in L^{p,\infty}(X, \nu) \Leftrightarrow \sup_{t>0} t^{\frac{1}{p}} f^*(t) < +\infty.$$

We note that the definition above is equivalent to that one of the usual weak $L^{p,\infty}$ (see [13, §1.4.2]).

For completeness, we recall the definition of the weak spaces $L^{p,\infty}(X, \nu)$. First, for any ν-measurable function f, we define the *distribution function* $d_f : (0, +\infty) \to [0, +\infty]$, with

$$d_f(t) := \nu \{x \in X : \ |f(x)| > t\}, \ \forall t > 0.$$

For any $p \in (0, +\infty)$, the space $L^{p,\infty}(X, \nu)$ is defined to be the space of all ν-measurable functions f such that

$$\|f\|_{p,\infty} := \sup_{t>0} t d_f^{\frac{1}{p}}(t) < +\infty.$$

Moreover, it is also possible to define the weighted version of the weak L^p. More precisely, for a ν-measurable function υ, and every $p > 0$, we define the space $L^p_\upsilon(X)$ to comprise all the ν-measurable functions f such that

$$\|f\|_{L^p_{\upsilon,\infty}} := \sup_{t>0} t \left(\int_{\{x \in X: \ |f(x)|>t\}} \upsilon(x) \, \mathrm{d}\nu(x) \right)^{\frac{1}{p}} < +\infty.$$

Finally, for $p, q \in (0, +\infty)$ we define the *Schatten-Lorentz* class $\mathbf{S}_{p,q}$ as the class which comprises all the bounded operators T such that

$$\sum_{n \in \mathbb{N}_0} (1+n)^{\frac{q}{p}-1} \big(s_n(T) \big)^q < +\infty.$$

For $q = \infty$, the Schatten-Lorentz class $\mathbf{S}_{p,\infty}$ is the usual weak Schatten class that was defined in the introduction.

Finally, for more topics on interpolation, as well as, on interpolation of Lorentz spaces or compact operator ideals could be found in [3], [12], [13], [16] and [21].

Appendix B

The Fourier transform

We briefly recall the definition of the Fourier transform on the unit circle $\mathbb{T}$ (which is identified with the interval $[0,1)$, under the mapping $t \mapsto e^{2\pi i t}$):

$$\big(\mathcal{F}f\big)(n) = \hat{f}(n) = \int_0^1 f(t)e^{-2\pi i n t}\,\mathrm{d}t\,,\ \forall n \in \mathbb{Z},\ \forall f \in L^1(\mathbb{T});$$

and on the real line $\mathbb{R}$:

$$\big(\mathcal{F}f\big)(x) = \hat{f}(x) = \int_{\mathbb{R}} f(y)e^{-2\pi i y x}\,\mathrm{d}y\,,\ \forall x \in \mathbb{R}. \tag{B.1}$$

For sake of accuracy, (B.1) defines the Fourier transform on a dense subset of $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ (for example, the space of Schwartz functions) and subsequently, it is extended to a unitary operator on the whole $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Therefore, we have that $\mathcal{F}^{-1} = \mathcal{F}^*$, where

$$\big(\mathcal{F}^{-1}f\big)(x) = \int_{\mathbb{R}} f(y)e^{2\pi i y x}\,\mathrm{d}y\,,\ \forall x \in \mathbb{R},\ \forall f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Regarding the Fourier transform on $\mathbb{T}$, notice that $\mathcal{F}$ is a unitary operator from $L^2(\mathbb{T})$ to $\ell^2(\mathbb{Z})$, with inverse

$$\big(\mathcal{F}^{-1}\{f_n\}_{n\in\mathbb{Z}}\big)(e^{it}) = \big(\mathcal{F}^*\{f_n\}_{n\in\mathbb{Z}}\big)(e^{it}) = \sum_{n\in\mathbb{Z}} f_n e^{2\pi i n t},\ \forall t \in [0,1),\ \forall \{f_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

The Fourier transform gives its name to a whole branch of mathematical analysis, the Fourier Analysis. Some of the concepts that one can meet there, and which we also make use of, are the Fourier multipliers (cf.[3], [13]) and the Paley-Wiener theory.

B.1 Fourier multipliers

For $p \in (1,+\infty)$, a sequence on $\mathbb{Z}$ ρ is a *Fourier Multiplier* from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$, denote $\rho \in \mathscr{M}_p(\mathbb{T})$ (or simply $\mathscr{M}_p$ when there is no danger for confusion), if and only if the mapping

$$f(\cdot) \longmapsto \sum_{n\in\mathbb{Z}} \rho(n)\hat{f}(n)e^{2\pi i n \cdot}$$

is a bounded linear operator on $L^p(\mathbb{T})$. We also define the space of Fourier multipliers on $\mathbb{R}$ in a similar way. More precisely, a function $\rho : \mathbb{R} \to \mathbb{C}$ is a *Fourier Multiplier* from

$L^p(\mathbb{R})$ to $L^p(\mathbb{R})$, denote $\rho \in \mathscr{M}_p(\mathbb{R})$ (or simply $\mathscr{M}_p$), if and only if the mapping

$$f(\cdot) \longmapsto \int_{\mathbb{R}} \rho(y)\hat{f}(y)e^{2\pi i y \cdot}\, \mathrm{d}y$$

is a bounded linear operator on $L^p(\mathbb{R})$. The space of Fourier multipliers $\mathscr{M}_p$ is a Banach algebra. Ensuing, we present some useful theorems concerning multipliers.

Theorem B.1 (Mikhlin's Multiplier Theorem; [3], Theorem 6.1.6)**.** *Let $\rho : \mathbb{R} \to \mathbb{C}$ be a function which satisfies*

$$|\rho^{(n)}(x)| \le A \langle x \rangle^{-n}, \ \forall x \in \mathbb{R}, \ n = 0, 1,$$

where $\langle x \rangle = \sqrt{1 + x^2}$. Then $\rho \in \mathscr{M}_p(\mathbb{R})$, for every $p \in (1, +\infty)$, and, more precisely, there exists a positive constant C_p which depends only on p such that

$$\|\rho\|_{\mathscr{M}_p} \le C_p A.$$

The next two Theorems concern the invariance of multipliers' norm after scaling.

Theorem B.2 ([3], Theorem 6.1.3)**.** *Let $\rho : \mathbb{R} \to \mathbb{C}$ belong to $\mathscr{M}_p(\mathbb{R})$. Then, for any $t \in \mathbb{R} \setminus \{0\}$, the function $\rho_t : \mathbb{R} \to \mathbb{C}$ which maps x to $\rho(tx)$ belongs to $\mathscr{M}_p(\mathbb{R})$ with $\|\rho_t\|_{\mathscr{M}_p} \le \|\rho\|_{\mathscr{M}_p}$.*

Theorem B.3 ([13], Theorem 4.3.7)**.** *Let $\rho : \mathbb{R} \to \mathbb{C}$ be a continuous function such that $\rho \in \mathscr{M}_p(\mathbb{R})$, for some $p \in (1, +\infty)$. Then, for any $t > 0$, the sequence $\rho_t = \{\rho(tn)\}_{n \in \mathbb{Z}}$ belongs to $\mathscr{M}_p(\mathbb{T})$ and moreover,*

$$\sup_{t>0} \|\rho_t\|_{\mathscr{M}_p(\mathbb{T})} \le \|\rho\|_{\mathscr{M}_p(\mathbb{R})}.$$

B.2 Paley-Wiener theory

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Then f is called *of exponential type A* if

$$|f(z)| = O(e^{A|z|}), \quad \text{when} \quad |z| \to +\infty.$$

Theorem B.4 (Paley-Wiener, [32], Theorem 7.2.1)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function supported on $[-A, A]$, for some positive number A, such that $f \in L^2([-A, A])$. Then the Fourier transform*

$$\hat{f}(z) = \int_{\mathbb{R}} f(x)e^{-2\pi i x z}\, \mathrm{d}x, \ \forall z \in \mathbb{C},$$

is an entire function of exponential type A.

Theorem B.5 (Plancherel-Polya, [22], Theorem no. 31)**.** *Let f be an entire function of exponential type A. Then, for any $p > 0$, there exists a constant $C = C(p, A)$ such that*

$$\sum_{m \in \mathbb{Z}} |f(m)|^p \le C \|f\|^p_{L^p(\mathbb{R})}.$$

Appendix C

Hardy spaces, BMO and VMO

C.1 The Hardy space H^p

C.1.1 The Hardy space on the disk

Let $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ and $\mathbb{T} = \partial\mathbb{D}$, which is identified with the interval $[0,1)$, under the action of the map $t \mapsto e^{2\pi i t}$. We define the set of holomorphic (or analytic) functions on $\mathbb{D}$,

$$Hol(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} : \ f \text{ is holomorphic}\}$$

Let $p \in (0,+\infty]$. Then, for any $f \in Hol(\mathbb{D})$, we define the means $M_p(r,f)$ and $M_\infty(r,f)$, for $p \in (0,+\infty)$ and $p = +\infty$, respectively, as follows:

$$M_p(r,f) := \left(\int_0^1 \left|f(re^{2\pi i t})\right|^p \mathrm{d}t\right)^{\frac{1}{p}}, \ \forall r \in [0,1),$$

and

$$M_\infty(r,f) := \max_{0 \le t < 1} \left|f(re^{2\pi i t})\right|, \ \forall r \in [0,1).$$

These means are increasing functions of the radius r, for a fixed analytic function f, and this fact prompts the following definition:

$$H^p(\mathbb{D}) := \left\{ f \in Hol(\mathbb{D}) : \ \|f\|_{H^p(\mathbb{D})} := \sup_{0 \le r < 1} M_p(r,f) < +\infty \right\}, \ \forall p \in (0,+\infty].$$

It can be proved though ([7, Theorem 2.2]) that, for any $p \in (0,+\infty]$, if $f \in H^p(\mathbb{D})$, then f has an almost everywhere non-tangential limit $\tilde{f} \in L^p(\mathbb{T})$; i.e. there exists a function $\tilde{f} \in L^p(\mathbb{T})$, such that

$$\lim_{\sphericalangle z \to e^{2\pi i t}} f(z) = \tilde{f}(e^{2\pi i t}), \quad \text{for almost every } t \in [0,1).$$

Due to this observation, we can extend every $f \in H^p(\mathbb{D})$ on $\mathbb{T}$, by setting

$$f(e^{2\pi i t}) := \tilde{f}(e^{2\pi i t}), \ \forall t \in [0,1). \tag{C.1}$$

This previous observation also leads to the following definition. For any $p \in (0,+\infty]$,

$$H^p(\mathbb{T}) := \left\{ \tilde{f} \in L^p(\mathbb{T}) : \ \exists f \in H^p(\mathbb{D}) \text{ with } \lim_{\sphericalangle z \to e^{2\pi i t}} f(z) = \tilde{f}(e^{2\pi i t}), \ \text{for a.e. } t \in [0,1) \right\}. \tag{C.2}$$

We also endow the space $H^p(\mathbb{T})$ with the $L^p(\mathbb{T})$ norm. Obviously, the $H^p(\mathbb{T})$ space contains all the trigonometric polynomials

$$P_N(t) = \sum_{n=0}^{N} p_n e^{2\pi int}, \ \forall t \in [0,1),$$

for any arbitrary $N \in \mathbb{N}_0$. Moreover, it is proved ([7, Theorem 3.3]) that the space of these polynomials is dense in $H^p(\mathbb{T})$, for any $p \in (0,+\infty)$. Therefore, we can give the following, equivalent to (C.2), definition:

$$H^p(\mathbb{T}) := \left\{ f \in L^p(\mathbb{T}) : \ f(z) = \sum_{n\geq 0} f_n z^n, \ \forall z \in \mathbb{T} \right\}, \ \forall p \in (0,+\infty). \tag{C.3}$$

For $p \in [1,+\infty]$, it can be proved ([7, Theorem 3.4]) that the space $H^p(\mathbb{T})$ can be identified with the space of $L^p(\mathbb{T})$ functions with whose Fourier coefficients vanish on negative integers. Thus, the definition (C.3) can be extended for $p = +\infty$, too. Besides, due to (C.1), we can identify the spaces $H^p(\mathbb{D})$ and $\mathbb{H}^p(\mathbb{T})$, so that they can equally be called as *Hardy spaces* and any switch between these two notations should not cause any confusion. Finally, it can be proved that the Hardy space, H^p, is Banach when $p \in [1,+\infty]$, and quasi-Banach, for $p \in (0,1)$; see [7, Corollary 1 and 2], as well as, the discussion between them.

C.1.2 The Hardy space on the upper half-plane

In complete analogy with the theory of the Hardy spaces on the unit disk, is developed the respective theory of Hardy spaces on the (complex) upper half-plane. The basic steps that lead to the main definitions do not differ a lot from those for Hardy spaces on $\mathbb{D}$ so, we only give the necessary definitions, with fewer details this time. For a more analytic approach though, we refer to [7, Chapter 11].

Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \ \operatorname{Im} z > 0\}$ and define the set of holomorphic (analytic) functions on $\mathbb{C}_+$,

$$Hol(\mathbb{C}_+) := \{f : \mathbb{C}_+ \to \mathbb{C} : \ f \text{ is holomorphic}\}.$$

Then, for any $p \in (0,+\infty)$, define the space

$$H^p(\mathbb{C}_+) := \left\{ f \in Hol(\mathbb{C}_+) : \ \|f\|_{H^p(\mathbb{C}_+)} := \sup_{y>0} \left(\int_{\mathbb{R}} |f(x+iy)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} < +\infty \right\};$$

and for $p = +\infty$,

$$H^\infty(\mathbb{C}_+) := \left\{ f \in Hol(\mathbb{C}_+) : \ \|f\|_{H^\infty(\mathbb{C}_+)} := \sup_{y>0} |f(x+iy)| < +\infty \right\}.$$

As it happens in the unit disk, we can again consider the boundary values of $H^p(\mathbb{C}_+)$, almost everywhere on $\mathbb{R}$ (Corollary of [7, Theorem 11.1]). Then, for any $p \in (0,+\infty]$, we define

$$H^p(\mathbb{R}) := \left\{ \tilde{f} \in L^p(\mathbb{R}) : \ \exists f \in H^p(\mathbb{C}_+) \text{ with} \lim_{z\to x} f(z) = \tilde{f}(x), \ \text{for a.e. } x \in \mathbb{R} \right\},$$

which we endow with the $L^p(\mathbb{R})$ norm. As it happens in the case of the unit disk, it can be proved ([7, Theorem 11.4]) that the spaces $H^p(\mathbb{C}_+)$ and $H^p(\mathbb{R})$ can be essentially identified. In other words, a function $f \in Hol(\mathbb{C}_+)$ belongs to the Hardy space $H^p(\mathbb{C}_+)$ if and only if it can be extended to an $L^p(\mathbb{R})$ function. Moreover, for any $p \in [1,2]$, the Hardy space $H^p(\mathbb{C}_+)$ (or equivalently, $H^p(\mathbb{R})$) can be identified with the space of $L^p(\mathbb{R})$ functions whose Fourier transform vanishes on the negative semi-line ([7, Theorem 11.10]).

C.1.3 Multipliers

In section B.1 we presented some results on the theory of Fourier multipliers. Those results concern the case of $\mathscr{M}_p$, for $p > 1$. When $p \in (0,1]$, we need some further smoothness conditions which are expressed in terms of Hardy spaces. So now it is time to complete the aforementioned theory.

The analogue of Mikhlin's Theorem (see Theorem B.1) for the case of $p \in (0,1]$ is due to E. Stein (cf. [31], Théorème 1). Notice that this gives a sufficient condition for multipliers on a smaller class of functions, compared to the $p > 1$ case. More precisely, we have the following theorem:

Theorem C.1. *Let $p \in (0,1]$ and consider the Hardy space $H^p(\mathbb{R})$. Let $k \in \mathbb{N}$ such that $k^{-1} < p$ and $\rho : \mathbb{R}_+ \to \mathbb{C}$ which satisfies the following conditions:*

(i) $|\rho(t)| \leq A,\ \forall t \in \mathbb{R}_+$;

(ii) $\rho \in C^k(\mathbb{R}_+)$ *and*

$$\int_R^{2R} |\rho^{(l)}(t)|^2 \, \mathrm{d}t \leq A R^{-2l+1}, \quad \forall R > 0,\ \forall l = 1, \ldots, k;$$

where A is a positive constant. Then ρ is a multiplier on $H^p(\mathbb{R})$.

Finally, the analogue of Theorem B.3 is the following:

Theorem C.2 ([5])**.** *Let $\rho : \mathbb{R} \to \mathbb{C}$ be a continuous function that gives rise to a multiplier on the Hardy space $H^p(\mathbb{R})$, for some $p \in (0,1]$. Then, for any $t > 0$, the sequence $\rho_t = \{\rho(tn)\}_{n\in\mathbb{Z}}$ is a multiplier on the Hardy space $H^p(\mathbb{T})$ and furthermore, the multiplier norm $\|\rho_t\|_{\mathscr{M}(H^p(\mathbb{T}))}$ is uniformly bounded with respect to t.*

C.2 BMO and VMO

In this section we define two classes of spaces, the BMO and VMO, which may look irrelevant to aforementioned theory of Hardy spaces but, they are actually closely related to it. For example, the BMO space arises when investigating the duals of Hardy spaces.

We begin by defining the notion of the mean oscillation of a function when it is defined either on $\mathbb{T}$ or on $\mathbb{R}$. For let $f \in L^1(\mathbb{T})$ and I be an arbitrary arc of $\mathbb{T}$. We define the *mean value* of f on I, f_I, by

$$f_I := \frac{1}{|I|} \int_I f\left(e^{2\pi i t}\right) \mathrm{d}t\,,$$

where by $|I|$ we denote the (normalised) Lebesgue measure on $\mathbb{T}$ of I. Respectively, let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ and I be a bounded interval. Then the *mean value* of f on I is given by

$$f_I := \frac{1}{|I|} \int_I f(t) \, \mathrm{d}t\,,$$

where $|I|$ is the Lebesgue measure on $\mathbb{R}$ of I.

Next we define the mean oscillation of a function f. In the case of the unit circle, if $f \in L^1(\mathbb{T})$, we define the *mean oscillation* of f over an arc I as

$$\langle f \rangle_I := \frac{1}{|I|} \int_I \left| (f - f_I) \left(e^{2\pi i t} \right) \right| \mathrm{d}t \,.$$

Similarly, for the case of the real line, we define the *mean oscillation* of an $L^1_{\text{loc}}(\mathbb{R})$ function f, over a bounded interval I as

$$\langle f \rangle_I := \frac{1}{|I|} \int_I |(f - f_I)(t)| \, \mathrm{d}t \,.$$

Then the BMO spaces on the unit circle and on the real line are defined as follows:

$$\mathrm{BMO}(\mathbb{T}) := \left\{ f \in L^1(\mathbb{T}) : \ \|f\|_{\mathrm{BMO}(\mathbb{T})} := \sup_I \langle f \rangle_I < +\infty \right\};$$

and similarly,

$$\mathrm{BMO}(\mathbb{R}) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}) : \ \|f\|_{\mathrm{BMO}(\mathbb{R})} := \sup_I \langle f \rangle_I < +\infty \right\}.$$

So that the BMO space is actually the space of functions with *bounded mean oscillation.*

Notice that the BMO space arises naturally in the theory of Hankel operators. We have already seen that it provides a sufficient condition for boundedness (see Theorems 1.2 and 1.7). By a closer look to the proofs of these theorems (cf. [21]), it is observed that the kernel of a Hankel operator is essentially linked with the analytic projection of a function that acts as a linear functional on the Hardy space H^1. More precisely, by considering for simplicity the discrete case, H_a is a bounded Hankel operator on $\ell^2(\mathbb{N}_0)$, with kernel $a = \{a(n)\}_{n \in \mathbb{N}_0}$, if and only if there is a function $\phi : \mathbb{T} \to \mathbb{C}$ such that

$$\hat{\phi}(n) = a(n), \ \forall n \in \mathbb{N}_0,$$

and ϕ acts as a linear functional on $H^1(\mathbb{T})$, or equivalently, $\phi \in \left(H^1(\mathbb{T})\right)^*$. But the dual space of $H^1(\mathbb{T})$, $\left(H^1(\mathbb{T})\right)^*$, is identified with the analytic functions of the $\mathrm{BMO}(\mathbb{T})$ space (aka BMOA). Respectively, for the continuous case, $\left(H^1(\mathbb{R})\right)^* = \mathrm{BMO}(\mathbb{R})$. For more on the connection between the Hardy and the BMO spaces, we refer to the fundamental work of C. Fefferman and E. M. Stein in [8].

Finally, we proceed to the definition of the VMO spaces; i.e. the space of functions with *vanishing mean oscillation*:

$$\mathrm{VMO}(\mathbb{T}) := \left\{ f \in \mathrm{BMO}(\mathbb{T}) : \ \limsup_{|I| \to 0} \langle f \rangle_I = 0 \right\};$$

and similarly,

$$\mathrm{VMO}(\mathbb{R}) := \left\{ f \in \mathrm{BMO}(\mathbb{R}) : \ \limsup_{|I| \to 0} \langle f \rangle_I = 0 \right\}.$$

For a detailed introduction to the theory of Hardy, BMO and VMO spaces we refer the reader to [7], [9] and [8].

Appendix D

Pseudo-differential operators

Formally, a pseudo-differential operator Ψ on $L^2(\mathbb{R})$ can be described by

$$\big(\Psi f\big)(x) = \int_{\mathbb{R}} a(x,y)\hat{f}(y)e^{2\pi i x y}\,\mathrm{d}y\,,\ \forall x\in\mathbb{R},\ \forall f\in L^2(\mathbb{R}). \tag{D.1}$$

The function $a:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is the *symbol* of Ψ and it should belong in some symbol class.

More precisely, for any $m\in\mathbb{R}$ we define the *symbol class* S^m as the set of smooth functions $a:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that, for any $k,l\in\mathbb{N}$, there exists a positive constant $C_{k,l}$ such that

$$\left|\partial_x^k\partial_y^l a(x,y)\right| \le C_{k,l}\,\langle y\rangle^{m-l}\,,\ \forall x,y\in\mathbb{R}.$$

Then we define the class of *pseudo-differential operators* Ψ^m to comprise all the operators Ψ that are described by (D.1), with symbol $a\in S^m$. For every $\Psi\in\Psi^m$ with symbol a, we write $\Psi := a(X,D)$. Therefore, (D.1) becomes

$$\left[a(X,D)f\right](x) = \int_{\mathbb{R}} a(x,y)\hat{f}(y)e^{2\pi i x y}\,\mathrm{d}y\,,\ \forall x\in\mathbb{R},\ \forall f\in L^2(\mathbb{R}). \tag{D.2}$$

Observe that if a is a symbol independent of x, then (D.2) gives the pseudo-differential operator

$$\left[a(D)f\right](x) = \int_{\mathbb{R}} a(y)\hat{f}(y)e^{2\pi i x y}\,\mathrm{d}y\,,\ \forall x\in\mathbb{R},\ \forall f\in L^2(\mathbb{R}). \tag{D.3}$$

In the context of the thesis, we deal with operators of the form $\mathcal{M}_\beta\alpha(D)\mathcal{M}_\beta$. Thus, according to (D.3), we formally obtain

$$\begin{aligned}\left[\mathcal{M}_\beta\alpha(D)\mathcal{M}_\beta f\right](x) &= \int_{\mathbb{R}} \beta(x)\alpha(y)\widehat{\beta f}(y)e^{2\pi i y x}\,\mathrm{d}y\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \beta(x)\alpha(y)\hat{\beta}(y-\xi)\hat{f}(\xi)e^{2\pi i y x}\,\mathrm{d}\xi\,\mathrm{d}y\,.\end{aligned} \tag{D.4}$$

Notice that

$$\int_{\mathbb{R}} \beta(x)\alpha(y)\hat{\beta}(y-\xi)e^{2\pi i y x}\,\mathrm{d}y = e^{2\pi i \xi x}\int_{\mathbb{R}} \beta(x)\alpha(y+\xi)\hat{\beta}(y)e^{2\pi i y x}\,\mathrm{d}y\,,$$

and set

$$a(x,\xi) := \int_{\mathbb{R}} \beta(x)\alpha(y+\xi)\hat{\beta}(y)e^{2\pi i y x}\,\mathrm{d}y\,,\ \forall x,\xi\in\mathbb{R}. \tag{D.5}$$

Therefore, by going back to (D.4),

$$\left[\mathcal{M}_\beta \alpha(D)\mathcal{M}_\beta f\right](x) = \int_{\mathbb{R}} a(x,\xi)\hat{f}(\xi)e^{2\pi i\xi x}\,\mathrm{d}\xi\,,\ \forall x\in\mathbb{R},$$

and consequently, $\mathcal{M}_\beta \alpha(D)\mathcal{M}_\beta$ can be regarded as a pseudo-differential operator $a(X,D)$, where a is given by (D.5). For this class of pseudo-differential operators we can obtain Weyl type eigenvalue asymptotics. More precisely, we have the following theorem (cf. [23, Theorem 2.4]):

Lemma D.1. *Let α denote a real valued function in $C^\infty(\mathbb{R})$, such that*

$$\alpha(x) = \begin{cases} A(+\infty)x^{-\gamma} + o(x^{-\gamma}), & x\to+\infty \\ A(-\infty)|x|^{-\gamma} + o(x^{-\gamma}), & x\to-\infty, \end{cases}$$

for some real constants $A(+\infty)$, $A(-\infty)$ and $\gamma > 0$. Now let β be a real valued function on $\mathbb{R}$ such that

$$|\beta(x)| \le C\,\langle x\rangle^{-\rho}\,,\ \forall x\in\mathbb{R}, \tag{D.6}$$

where $\rho > \frac{\gamma}{2}$ and C a non-negative constant. Finally, we determine the pseudo-differential operator $\Psi = \mathcal{M}_\beta\alpha(D)\mathcal{M}_\beta$ on $L^2(\mathbb{R})$. Then Ψ is compact and we have the following eigenvalue asymptotic formula:

$$\lambda_n^{\pm}(\Psi) = C^{\pm}n^{-\gamma} + o(n^{-\gamma}),\ n\to+\infty, \tag{D.7}$$

where

$$C^{\pm} = \left[\left(A(+\infty)_{\pm}^{\frac{1}{\gamma}} + A(-\infty)_{\pm}^{\frac{1}{\gamma}}\right)\int_{\mathbb{R}}|\beta(x)|^{\frac{2}{\gamma}}\,\mathrm{d}x\right]^{\gamma}.$$

Finally, for more details on the theory of pseudo-differential operators, we refer the reader to [28] and [30].

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