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Aspects of Wilson loops in 3- and 4-dimensional gauge theories

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Aspects of Wilson loops in 3- and 4-dimensional gauge theories

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Abstract

In this thesis we study aspects of Wilson loops in some three- and four-dimensional gauge theories. In three dimensions, we construct the classically 1/2 BPS Wilson loops in $\mathcal{N} = 4$ superconformal Chern-Simons matter theories. We consider a circular or linear quiver with Chern-Simons levels k , $-k$ and 0. In the cases with no vanishing Chern-Simons levels, we find a pair of Wilson loops for each pair of adjacent nodes on the quiver connected by a hypermultiplet. This classical pairwise degeneracy has since been found to be lifted by quantum corrections. We also show that these Wilson loops are cohomologically equivalent to certain combinations of 1/4 BPS Wilson loops whose expectation value can be evaluated by the appropriate localized matrix model. We analyze the M-theory duals of those Wilson loops and comment on their degeneracy.

In four dimensions, we study Wilson loops in planar $\mathcal{N} = 4$ super Yang-Mills theory. We derive the Lax operator for a very large family of classical minimal surface solutions in $\mathbb{H}_3$ describing Wilson loops in $\mathcal{N} = 4$ SYM theory. These solutions, constructed in [1], are associated with a hyperelliptic Riemann surface of odd genus. The procedure which we use to construct the Lax operator was introduced in [2]. Here we apply the procedure to the very large family of minimal surfaces found in [1]. We verify that the algebraic curve derived from the Lax operator is none other than this hyperelliptic surface.

We also study operator insertions into the 1/2 BPS straight Wilson line in $\mathcal{N} = 4$ SYM. Deformations of this straight line operator may be mapped to certain operator insertions into the straight line. The operator insertions are classified according to the $OSp(2, 2|4)$ symmetry group preserved by the straight line. We study the multiplet structure of these insertions and present a method to compute the anomalous dimensions and three-point function structure constants of these insertions.

Contents

Abstract	2
Acknowledgements	7
Contents of this thesis	8
1 Introduction and summary	9
1.1 Supersymmetric Wilson loops	11
1.1.1 $\mathcal{N} = 4$ super Yang-Mills theory	11
1.1.2 ABJ(M) theory	13
1.1.3 $\mathcal{N} = 4$ Chern-Simons matter theories	14
1.2 BPS Wilson loops and operator insertions	20
1.2.1 Operator insertions into the straight Wilson line	21
1.3 Holographic description of Wilson loops	24
1.3.1 Four dimensions	25
1.3.2 Three dimensions	26
1.4 Strings in $\mathbb{H}_3$ and the spectral curve	27
2 $1/2$ BPS Wilson loops in $\mathcal{N} = 4$ Chern-Simons matter theories	33
2.1 The straight line	33
2.1.1 Alternating CS levels	34
2.1.2 Linear quivers	38
2.1.3 Vanishing CS levels: repeated untwisted hypermultiplets	40
2.1.4 Vanishing CS levels: repeated twisted hypermultiplets	46
2.2 The circle	46
2.2.1 Alternating levels	46
2.3 Localization	50
2.3.1 The straight line	51
2.3.2 The circle	55
2.4 Holographic description	57
2.4.1 Parameterizations of the 7-sphere	59
2.4.2 Supersymmetry analysis	61
2.4.3 The M2-brane embedding	62

2.5	Summary	63
3	Operator insertions and wavy Wilson lines	65
3.1	Operator insertions in Wilson loops	66
3.2	Equivalence of operator insertions and deformations	67
3.2.1	The operator expansion	67
3.2.2	Deformations of the straight line and explicit form of the operators	69
3.3	Operators representation and multiplets	70
3.3.1	Operators of classical dimension one	71
3.3.2	Operators of classical dimension two	71
3.3.3	Operators of classical dimension three	72
3.3.4	Operators of classical dimension four	73
3.4	Extracting the CFT data	73
3.4.1	The one- and two-loop expectation values for general contours	75
3.4.2	Order ϵ^2 : cutoff	76
3.4.3	Order ϵ^2 : ‘masses’	80
3.4.4	Order ϵ^3	81
3.4.5	Order ϵ^4	82
3.4.6	Order ϵ^4 : the results and scheme dependence	86
3.5	Summary	87
4	The algebraic curve of minimal surfaces in $\mathbb{H}_3$	89
4.1	The $\mathbb{H}_3$ sigma-model	89
4.2	Riemann theta functions	91
4.2.1	The zeros of $\hat{\theta}(2 \int_0^\lambda)$	94
4.3	The Lax operator and algebraic curve	96
4.3.1	The Lax operator for the minimal surfaces of Ishizeki <i>et al.</i> [1]	97
4.3.2	A meromorphic function	98
4.3.3	L_{11}	100
4.3.4	L_{12}	100
4.3.5	L_{21}	100
4.4	The genus-one case	101
4.5	Summary	103
A	Notation and conventions	105
A.1	$\mathcal{N} = 4$ Chern-Simons matter theories	105
B	Derivation of the Killing spinor (2.123)	108

C	Taylor expansion of theta functions	111
D	An alternative approach for genus-one curves	112
D.1	Equivalence sigma-model and Pohlmeyer reduced Lax operators . . .	112
D.2	Factorized string method for genus-one	113

List of Figures

1.1	ABJ(M) quiver diagram	13
1.2	$\mathcal{N} = 4$ quiver with alternating CS levels	15
1.3	$\mathcal{N} = 4$ quiver containing vanishing CS levels	17
1.4	Operator insertion Feynman diagram	22
2.1	$\mathcal{N} = 4$ quiver with alternating CS levels	34
2.2	$\mathcal{N} = 4$ quiver with vanishing CS levels	40
2.3	$\mathcal{N} = 4$ quiver with vanishing CS levels and repeated twisted hyper- multiplets	46
3.1	Supermultiplet structure of the operator insertions	74

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Contents of this thesis

This thesis presents work published in [3,4], corresponding to Chapter 4 and Chapter 2 in the main text, respectively. Chapter 3 is based on unpublished work of the author with Amit Dekel and Nadav Drukker. Large sections of the main text overlap with that of [3,4].

Chapter 1

Introduction and summary

Quantum field theory (QFT) is thus far the most successful framework with which to describe Nature. There has been remarkable success matching the predictions of The Standard Model and other QFTs with experimental data. However, beyond the most trivial examples, solving QFTs is a daunting task. Historically, with the notable exception of lattice field theory, calculations in QFTs have been restricted to the weak coupling regime where perturbative methods have been employed. While these methods have broadened our understanding of the theories and proved extremely successful in matching experimental results, they do not capture the full dynamics of the theories.

In more recent years, this situation has improved for certain theories with a very high degree of symmetry. A number of techniques which exploit these symmetries have been developed, allowing one to probe beyond the weak coupling regime and in some cases obtain exact results. Examples of such techniques are localization in supersymmetric theories [5], and integrability-based techniques, such as the spectral curve [6], in integrable theories. Throughout this thesis we consider such theories with a very high degree of symmetry. In Chapter 3 and Chapter 4 we study $\mathcal{N} = 4$ super Yang Mills theory (SYM) in four dimensions. This is the maximally supersymmetric theory in four dimensions and exactly conformal at the quantum level. Additionally in the planar limit, where the rank of the $SU(N)$ gauge group becomes infinite, the theory is believed to be integrable. We adopt this limit in both Chapter 3 and Chapter 4. In Chapter 2 we study Chern-Simons (CS) matter theories in three dimensions, which possess $\mathcal{N} = 4$ superconformal symmetry.

In these theories we study Wilson loop operators. Wilson loops are ubiquitous in gauge theories and natural observables to consider; they are non-local gauge invariant operators defined by a spacetime path $\mathcal{C}$ and a representation $\mathcal{R}$ of the gauge group

$$W[\mathcal{C}] = \frac{1}{N} \text{tr}_{\mathcal{R}} \vec{\mathcal{P}} \exp \left(-i \int_{\mathcal{C}} ds \dot{x}^{\mu}(s) A_{\mu}(x(s)) \right), \quad (1.1)$$

where s parameterizes the path, x^{μ} are the spacetime coordinates, $\text{tr}_{\mathcal{R}}$ is a trace over

the representation $\mathcal{R}$, N is the rank of the gauge group, A_μ is the gauge connection and $\overrightarrow{\mathcal{P}}$ is path-ordering (increasing from left to right). In this thesis, we shall consider supersymmetric theories for which the above definition may be augmented - the gauge connection may be replaced by a more general connection including coupling to matter fields and the spacetime path $\mathcal{C}$ may be generalized to include path-dependent couplings to the matter fields.

Wilson loops were originally introduced to study confinement in quantum chromodynamics (QCD). Aside from such physical motivation, in the theories we are studying there exist Wilson loops which are amenable to exact computations. These are loops which preserve some of the additional symmetries discussed above. We study supersymmetric (BPS) Wilson loops in Chapter 2, operator insertions into such loops in Chapter 3 and integrable Wilson loops in Chapter 4.

A further key feature of all of the theories which we consider in this thesis is that they admit holographic duals. Holography relates certain QFTs to a corresponding string theory or M-theory. The most studied example of holography is that of [7].¹ This states the equivalence of $\mathcal{N} = 4$ SYM in four dimensions with type IIB string theory on $AdS_5 \times S^5$. In particular, it relates the strong coupling regime of the field theory to the weak coupling regime of the string theory. This strong/weak coupling nature of the duality is characteristic of holographic dualities. Exact results can thus lead to non-trivial checks of dualities.

For each duality there is a dictionary translating between objects on one side of the duality and corresponding objects on the other side. For the theories which we study, Wilson loops are mapped to open string solutions in the corresponding string theory or to M2-branes in M-theory. Holography thus gives us a new paradigm through which to view Wilson loops in the field theory. Furthermore, it offers us relatively easy access to the strong coupling regime of the field theory.

We now review in Section 1.1 the construction of maximally supersymmetric Wilson loops in $\mathcal{N} = 4$ SYM and exact results for these loops, before outlining the corresponding construction for some superconformal Chern-Simons theories in three dimensions. In Section 1.2 we discuss near-BPS Wilson loops in SYM; in particular their interpretation as operator insertions into BPS loops and the correlation functions of these operator insertions. The open strings corresponding to Wilson loops in the holographic duals of Wilson loops in the aforementioned theories are discussed in Section 1.3. In Section 1.4 we review the integrability-based spectral curve description for closed strings in planar $\mathcal{N} = 4$ SYM. We contrast this with the case for open strings and, in this context, we study a family of open string solutions dual to Wilson loops.

¹Holography had previously been studied in [8, 9].

Original work

The construction of the 1/2 BPS Wilson loops in $\mathcal{N} = 4$ CS matter theories in Chapter 2 and its review in Section 1.1.3 is based on the work of the author and collaborators in [4]. As is the construction of their dual M2-brane description in Chapter 2 and Section 1.3.2,

The study of operator insertions into the 1/2 BPS straight line in $\mathcal{N} = 4$ SYM reviewed in Section 1.2 and described in detail in Chapter 3 is based on unpublished work of the author and collaborators.

The description in Section 1.4 of the construction of the Lax operator and algebraic curve for the family of open string solutions constructed in [1] is based on work of the author and collaborator in [3]. The same is true for the contents of Chapter 4.

1.1 Supersymmetric Wilson loops

Supersymmetric (BPS) operators are operators preserving some degree of the supersymmetry of the theory. Due to the symmetry they preserve, they are amenable to exact computations.

Local BPS operators reside in shortened multiplets of the theory and are often protected operators, free from the quantum corrections which afflict more generic operators. As a result, these operators are much less daunting to study. As we shall now review, BPS Wilson loops are no exception to this, and provide a rich field of study.

1.1.1 $\mathcal{N} = 4$ super Yang-Mills theory

$\mathcal{N} = 4$ super Yang-Mills theory [10] has played a central role in the advancement of high energy physics this century. This is principally due to the remarkable degree of symmetry which it possesses and its role in the AdS/CFT duality [7]. The latter follows from the realization of the theory as the low-energy worldvolume theory of a stack of N coincident D3-branes in type IIB string theory. It is the maximally supersymmetric theory in four dimensions and exactly conformal at the quantum level, having a vanishing beta function. Furthermore, in the planar limit it is believed to be an integrable theory, having an infinite number of conserved charges. We shall discuss integrable aspects of the theory in Section 1.4, and in the meantime shall focus on supersymmetric aspects of the theory.

In $\mathbb{R}^{1,3}$, the conformal group is $SO(2,4) \simeq SU(2,2)$. This is unbroken by the theory and combines with the $\mathcal{N} = 4$ Poincaré supersymmetries to form the full $PSU(2,2|4)$ superconformal symmetry of the theory. The field content comprises a gauge field A_μ , four spinors $\lambda_{a\alpha}$ ($\bar{\lambda}_{\dot{\alpha}}^a$) and six scalars Φ^I . Here μ is the four-dimensional spacetime index of the $SO(1,3)$ Lorentz group, α and $\dot{\alpha}$ label the **2** and

$\bar{\mathbf{2}}$ spinor representations of the $SL(2; \mathbb{C})$ double cover, and a and I label the $\mathbf{4}$ and $\mathbf{6}$ representations of the $SU(4) \simeq SO(6)$ R-symmetry group.

The theory has two parameters: a dimensionless coupling, g_{YM} , and the rank, N , of the $SU(N)$ gauge group. It is also useful to define the 't Hooft coupling $\lambda := g_{YM}^2 N$. In the planar limit [11], *i.e.* in the limit where the rank of the $SU(N)$ gauge group becomes infinite, while λ is kept finite. λ thus serves as the coupling in the planar theory.

We are interested in BPS Wilson loops; in particular we shall discuss the maximally supersymmetric 1/2 BPS loops, *i.e.* Wilson loops preserving half of the supersymmetries of the theory. For non-supersymmetric gauge theories, Wilson loops are defined by (1.1), with the connection $\mathcal{L}$ given by the gauge field one-form $A_\mu dx^\mu$. In [12, 13] it was argued that for $\mathcal{N} = 4$ SYM the connection should be augmented to include a scalar coupling. Indeed it is necessary to construct supersymmetric loops. Concretely

$$\mathcal{L} := (A_\mu(x(s))\dot{x}^\mu(s) + i|\dot{x}(s)|\theta^I(s)\Phi^I(x(s))) ds, \quad (1.2)$$

where $\dot{x}(s) := dx(s)/ds$ and the scalar coupling $\theta^I(s)$ parameterizes the five-sphere. We note that, with the inclusion of the factor of i in the scalar coupling, we are implicitly assuming a spacelike path; for a timelike path this factor should be removed.

The 1/2 BPS loops are defined by a spacetime path that is either an infinite straight line or a circle and constant scalar coupling. Taking δ to be the supersymmetric variation with respect to the choice of half of the sixteen supercharges, the 1/2 BPS connection $\mathcal{L}$ satisfies

$$\delta\mathcal{L} = 0. \quad (1.3)$$

From the definition of the Wilson loop it is apparent that the invariance of the loop with respect to these supersymmetries follows from (1.3). As we shall see when we consider theories in three dimensions, (1.3) is not the most general condition one may impose in order to ensure a loop is supersymmetric.

In [14] the 1/2 BPS circular loop was studied in the planar limit at weak coupling to two-loop order and at strong coupling via the AdS/CFT duality. They found that only the ladder Feynman diagrams contribute to this order at weak coupling. They also conjectured that this holds to all orders in the coupling and verified that this matches the behaviour at strong coupling, as dictated by AdS/CFT. The resulting expectation value of the loop is given by a Gaussian matrix model integral. In [15] it was conjectured, and in [5] verified using localization, that the expectation value away from the planar limit is also given by a Gaussian matrix model integral. In the planar limit the result is given by the elegant expression:

$$\langle W_{\text{circle}} \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}), \quad (1.4)$$

where I_n is the modified Bessel function.

1.1.2 ABJ(M) theory

As mentioned in Section 1.1.1, $\mathcal{N} = 4$ SYM describes the low energy worldvolume theory of a stack of coincident D3-branes in type IIB string theory. It is natural to search for the field theories describing the worldvolume theories of other brane configurations in string theories and M-theory. Much focus was placed on finding the three-dimensional theory describing the worldvolume theory of coincident M2-branes. In [16–18] (see also [19]) a theory was proposed giving the worldvolume theory of two coincident M2-branes. This was generalized in [20] to the worldvolume theory of N M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity, known as ABJM theory. In [21] this was further generalized to ABJ theory, considering $N_1 - N_2$ fractional M2-branes and N_2 M2-branes probing this singularity. By considering the near-horizon geometries of this brane configuration the holographic dual for the theory was also found; that is type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ (or more generally M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$)². The coset sigma-model description of the Green-Schwarz action on $AdS_4 \times \mathbb{CP}^3$ has been shown to be classically integrable [23, 24].

ABJ(M) theory is a three-dimensional Chern-Simons matter theory with $\mathcal{N} = 6$ supersymmetry and $U(N_1) \times U(N_2)$ gauge group. The theory may be represented by a quiver diagram (see Figure 1.1). Each node represents a $U(N_i)$ vector multiplet

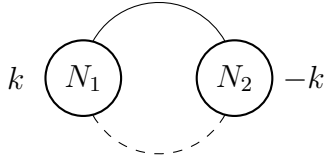


Figure 1.1: **ABJ(M) quiver diagram:** the two nodes represent vector multiplets of rank N_1 and N_2 with CS levels k and $-k$. They are joined by two hypermultiplets represented by the links (the dashed line denotes a ‘twisted hypermultiplet’ which we shall expand upon when discussing $\mathcal{N} = 4$ theories in the next section).

(with opposite CS levels) comprising a gauge field A_μ . These levels are quantized, *i.e.* $k \in \mathbb{Z}$, and act as the coupling in the theory. In particular the ’t Hooft couplings are given by $\lambda_i := N_i/k$. The two links in the quiver denote $\mathcal{N} = 4$ hypermultiplets. These hypermultiplets combine to give four fermions ψ_A^α and four scalars q^A , where α is the **2** spinor index of $SL(2; \mathbb{R})$ and A is the **4** index of the $SO(6) \simeq SU(4)$ R-symmetry group. All matter transforms in the bifundamental representation of the $U(N_1) \times U(N_2)$ gauge group.

In ABJ(M) theory, there is no 1/2 BPS Wilson loop with only scalar couplings; the most supersymmetric loops with such couplings are 1/6 BPS [25, 26]. For a time

²A type IIB string theory dual has also been constructed, see [22]

this was something of a puzzle as, from studying the dual M-theory, it was known that a 1/2 BPS loop existed in the theory [25, 26]. In [27] the 1/2 BPS loops were finally constructed; they require a $U(N_1|N_2)$ superconnection $\mathcal{L}$ including fermionic couplings. As noted in [28], the superconnection of the 1/2 BPS loop in ABJ(M) theory does not satisfy (1.3), but rather satisfies

$$\delta\mathcal{L} = \mathfrak{D}_s G := \partial_s G - i\{\mathcal{L}, G\}, \quad (1.5)$$

where $\mathfrak{D}_s$ is the supercovariant derivative along the direction of the path $\mathcal{C}$, defined with respect to the superconnection $\mathcal{L}$ and G is a $U(N_1|N_2)$ supermatrix. The type of commutator is determined by the Graßmann parity of $\mathcal{L}$ and G . This condition implies that taking the variation of the Wilson loop is equivalent to the insertion of a total (supercovariant) derivative into the loop. One must further take care with the boundary conditions to ensure supersymmetry invariance.

In [29] the evaluation of 1/2 BPS loops in $\mathcal{N} = 2$ Chern-Simons matter theories was reduced to a non-Gaussian matrix model integral using localization. It was also shown there that the 1/2 BPS $\mathcal{N} = 2$ loop enhances to the 1/6 BPS loop of ABJ(M) theory. This was exploited in [27] where it was demonstrated that the 1/2 BPS loop in ABJ(M) theory is cohomologically equivalent, with respect to the localizing supercharge, to a combination of 1/6 BPS loops. This implies that the expectation values are equal and allows one to use the results of [29] to reduce the evaluation of the 1/2 BPS loop in ABJ(M) theory to a matrix model integral. This matrix model was solved in the planar limit in [30] and $1/N$ corrections computed in [31, 32]. In the planar limit, the result is elegantly expressed in terms of matrix model observables. Expanding at weak coupling, the result is

$$\langle W_{\text{circle}} \rangle = 1 + i\pi(\lambda_1 - \lambda_2) - \frac{\pi^2}{3}(2\lambda_1^2 - 5\lambda_1\lambda_2 + 2\lambda_2^2) + \mathcal{O}(\lambda^3). \quad (1.6)$$

1.1.3 $\mathcal{N} = 4$ Chern-Simons matter theories

The previous sections have reviewed 1/2 BPS Wilson loops in $\mathcal{N} = 4$ SYM and ABJ(M) theory. We now outline work done by the author and collaborators in [4], concerning 1/2 BPS Wilson loops in a class of $\mathcal{N} = 4$ Chern-Simons matter theories. The details of this work appear in Chapter 2.

Prior to the discovery of ABJ(M) theory, supersymmetric Chern-Simons matter theories had already been the focus of much research. For a time it was believed that $\mathcal{N} = 3$ theories were the maximally supersymmetric extension of Chern-Simons theory which can be coupled to matter - see [33, 34]. In [35], however, $\mathcal{N} = 4$ theories were indeed constructed. These are linear quiver theories comprising two nodes (vector multiplets) connected by a link (hypermultiplet). The gauge groups are either $U(N_1)$ and $U(N_2)$ or $O(N_1)$ and $Sp(2N_2)$, with opposite Chern-Simons levels.

The hypermultiplet contains two scalars $q^{\bar{A}}$ and two fermions $\psi_{\underline{A}}$ transforming in the bifundamental of the gauge groups. The scalars transform in the $(\mathbf{2}, \mathbf{1})$ and the fermions in the $(\mathbf{1}, \mathbf{2})$ of the $SO(4) = SU(2)_A \times SU(2)_B$ R-symmetry group, reflected by the overlined and underlined doublet indices.

In [36] these theories were extended to include twisted hypermultiplets, and in [37] to include nodes with vanishing CS level. The twisted hypermultiplets comprise two scalars $q^{\bar{A}}$ and two fermions $\psi_{\bar{A}}$ transforming in the $(\mathbf{1}, \mathbf{2})$ and the $(\mathbf{2}, \mathbf{1})$ of $SU(2)_A \times SU(2)_B$, respectively. These allow for quivers of arbitrary length with $U(N_i)$ gauge groups or alternating $Sp(2N_i)$ and $O(N_j)$ gauge groups; here and in the main text we shall consider only the case of $U(N_i)$ gauge groups. The quivers may be linear or circular, and the level k_I of the I th node in the quiver is determined by the condition

$$k_I = \frac{k}{2}(s_I - s_{I+1}), \quad s_I = \pm 1, \quad (1.7)$$

where $s_I = 1$ for a hypermultiplet at the I^{th} link and $s_I = -1$ for a twisted hypermultiplet.³ With these conditions, it is apparent that $k_I \in \{k, -k, 0\}$. In [38] it was shown that ABJ(M) is a special case within this class of theories, with enhanced $\mathcal{N} = 6$ supersymmetry.

These theories arise as the low-energy worldvolume theories of type IIB brane configurations - see, for example, [22, 39] for an overview. Specifically, they describe stacks of D3-branes stretching between NS5-branes and $(1, k)$ -branes. The number of D3-branes gives the rank of the corresponding $U(N_i)$ gauge group. Alternating NS5 and $(1, k)$ -branes lead to quivers with alternating Chern-Simons levels. Either of these branes repeated leads to a node with vanishing Chern-Simons level, as described in [37].

In Chapter 2, based on [4], we construct the 1/2 BPS loops these theories on $\mathbb{R}^3$. We now review this construction here. We firstly consider quivers with alternating, non-vanishing CS levels k and $-k$. For such quivers, it is enough to consider a pair of nodes in the quiver and the adjacent hypermultiplets, see Figure 1.2.

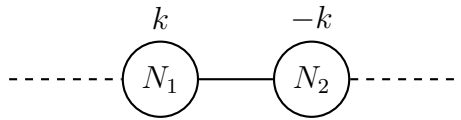


Figure 1.2: $\mathcal{N} = 4$ quiver segment with alternating CS levels: two $U(N_i)$ vector multiplets of opposite CS level connected by a hypermultiplet (solid links) and each with an additional adjacent twisted hypermultiplet (dashed links).

We consider loops preserving the $SU(2)_A$ of the R-symmetry group. We find that, as in ABJ(M) theory, the 1/2 BPS loops are defined with respect to a $U(N_1|N_2)$

³We adopt the convention that in quiver diagrams I increases from left to right.

superconnection $\mathcal{L}$ satisfying the supercovariant derivative condition (1.5). Surprisingly we find two such superconnections (preserving the same supersymmetries) for each segment of the quiver. Each loop couples to different fermions and we refer to these loops accordingly as ψ_1 - and ψ_2 -loops. Their explicit superconnections (for the spacelike straight line defined by $x^\mu = (s, 0, 0)^\mu$) are given by

$$\mathcal{L}^{\psi_1} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\underline{1}} - \frac{i}{k}\nu_{(1)} & \frac{1-i}{\sqrt{k}}\psi_{(1)\underline{1}}^+ \\ \frac{1-i}{\sqrt{k}}\bar{\psi}_{(1)+}^1 & A_{(2)1} - \frac{2i}{k}(\mu_{(2)})^1_{\underline{1}} - \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}, \quad (1.8)$$

and

$$\mathcal{L}^{\psi_2} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\underline{1}} + \frac{i}{k}\nu_{(1)} & \frac{1-i}{\sqrt{k}}\psi_{(1)\underline{2}}^- \\ \frac{1-i}{\sqrt{k}}\bar{\psi}_{(1)-}^2 & A_{(2)1} - \frac{2i}{k}(\mu_{(2)})^1_{\underline{1}} + \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}, \quad (1.9)$$

(see (2.18) and (2.19) in the main text). Here we have defined the bosonic bilinears

$$\begin{aligned} \nu_{(I)} &= q_{(I)}^{\bar{A}} \bar{q}_{(I)\bar{A}}, \quad \tilde{\nu}_{(I)} = \bar{q}_{(I)\bar{A}} q_{(I)}^{\bar{A}}, \\ (\mu_{(I)})^{\bar{A}}_{\bar{B}} &= q_{(I)}^{\bar{A}} \bar{q}_{(I)\bar{B}} - \frac{1}{2} \delta^{\bar{A}}_{\bar{B}} (\nu_{(I)})^{\bar{C}}_{\bar{C}}, \quad (\tilde{\mu}_{(I)})^{\bar{A}}_{\bar{B}} = \bar{q}_{(I)\bar{B}} q_{(I)}^{\bar{A}} - \frac{1}{2} \delta^{\bar{A}}_{\bar{B}} (\tilde{\nu}_{(I)})^{\bar{C}}_{\bar{C}}. \end{aligned} \quad (1.10)$$

The ‘(I)’ label specifies the node or link to which the fields correspond (the first link is to the right of the first node, and so on).

This is twice the number of 1/2 BPS loops one might expect. However, Wilson loops are composite operators and the interactions between these fields may lead to quantum effects. The supersymmetry analysis which we perform in Chapter 2, and for which we quote the results above, is classical in nature. For a unique operator with some set of quantum numbers (for a Wilson loop the analogs are the contour $\mathcal{C}$ and the representation $\mathcal{R}$), representation theory determines that a classical analysis suffices to determine the BPS operator. When there exists more than one operator with the same quantum numbers, one must consider quantum effects. While this is unprecedented for Wilson loops, it is quite familiar in the study of local operators. For example, in $\mathcal{N} = 4$ SYM, the single trace operators $\text{tr}(Z^2 X^2)$ and $\text{tr}(ZXZX)$ have the same classical scaling dimension and charge. Furthermore, they are both classically BPS. Once one-loop effects have been taken into consideration, however, only a single linear combination of the two remains BPS.

In [4] we predicted that the same would be true for the degenerate pair of loops which we found; in particular we proposed that the average of the two loops is truly BPS. In [40] and [41] these loops were studied at two- and three-loop order in the coupling. It was found that the degeneracy is lifted at three-loops. Indeed the 1/2 BPS loop is the average of the two which we found in [4]. We discuss the lifting of this degeneracy further in Section 1.1.3.

Quivers with vanishing Chern-Simons levels

In Section 2.1.3 we consider $\mathcal{N} = 4$ quivers including nodes with vanishing CS level; these are the theories constructed in [37]. We study the segment of the quiver depicted in Figure 1.3. Any nodes with vanishing CS levels in a quiver correspond

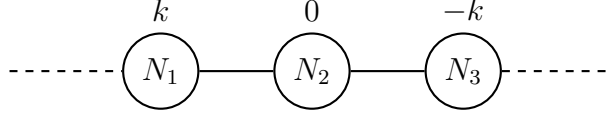


Figure 1.3: $\mathcal{N} = 4$ quiver containing vanishing CS levels

to ‘auxiliary vector multiplets’ in the language of [37]. There are no Chern-Simons terms corresponding to these nodes in the action, due to the vanishing of the CS level. As a result, all of the fields in the vector multiplet are auxiliary. They may be integrated out, as in [35, 42, 43], to obtain a non-linear theory. We follow the approach of [37], however, where these fields are not integrated out. The auxiliary vector multiplets comprise the gauge field A_μ , the gaugino $\lambda^{\bar{A}B}$ and an auxiliary scalar $\varphi^{\underline{A}}_{\underline{B}}$.

For the quiver segment in Figure 1.3 we find five 1/2 BPS loops. However, the connections do not satisfy the supercovariant derivative condition (1.5); rather, they satisfy an even weaker condition. This condition is

$$\delta\mathcal{L} = \mathfrak{D}_s G + \Delta. \quad (1.11)$$

Here Δ is a ‘remainder term’. We observe two types of remainder term, bosonic and fermionic. In the former case, the remainder term appears in the action of theory multiplied by an auxiliary field; integrating out the auxiliary field constrains the remainder term to vanish.

In the latter case, a fermionic field (in this case the gaugino λ) appears only linearly in the action and multiplying Δ . When we take the variation of the Wilson loop whose connection satisfies (1.11), we get an insertion of a total derivative into the loop from the first term and an insertion of the remainder term Δ from the second term. Taking the expectation value of the loop and using the form of the action described above, this is equivalent to the insertion of $\delta/\delta\lambda$ into the loop. This is a total Grassmannian derivative and must vanish. Thus the expectation value of the variation of the loop vanishes. We say that the Wilson loop is *weakly* 1/2 BPS.

Of the five weakly 1/2 BPS loops that we find for the segment in Figure 1.3, two are in representations of $U(N_1|N_2)$, two are in representations of $U(N_2|N_3)$ and the final one is in representations of $U(N_1 + N_3|2N_2)$. The explicit form of the latter

connection (for the straight line) is

$$\mathcal{L} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\underline{1}} & \bar{c}_{(1)}^1 \psi_{(1)\underline{1}}^+ & \bar{c}_{(1)}^2 \psi_{(1)\underline{2}}^- & \bar{d} \epsilon_{\bar{A}\bar{B}} q_{(1)}^{\bar{A}} q_{(2)}^{\bar{B}} \\ c_{(1)\underline{1}} \bar{\psi}_{(1)+}^1 & \mathcal{A}_{(2)} & 0 & \bar{c}_{(2)}^2 \psi_{(2)\underline{2}}^- \\ c_{(1)\underline{2}} \bar{\psi}_{(1)-}^2 & 0 & \mathcal{A}'_{(2)} & \bar{c}_{(2)}^1 \psi_{(2)\underline{1}}^+ \\ d \epsilon^{\bar{A}\bar{B}} \bar{q}_{(2)\bar{A}} q_{(1)\bar{B}} & c_{(2)\underline{2}} \bar{\psi}_{(2)-}^2 & c_{(2)\underline{1}} \bar{\psi}_{(2)+}^1 & A_{(3)1} - \frac{2i}{k}(\mu_{(3)})^1_{\underline{1}} \end{pmatrix}, \quad (1.12)$$

(see (2.38) in the main text). where $\mathcal{A}_{(2)} := A_{(2)1} + i(\varphi_{(2)})^1_{\underline{1}} + \frac{i}{2k}(\nu_{(2)} - \tilde{\nu}_{(1)})$ and $\mathcal{A}'_{(2)} := A_{(2)1} + i(\varphi_{(2)})^1_{\underline{1}} - \frac{i}{2k}(\nu_{(2)} - \tilde{\nu}_{(1)})$.

This concludes our search for 1/2 BPS loops in the theories. It would be interesting to also study the transformation of the loops under mirror symmetry, where we would expect them to become vortex loop operators [44]. This was studied in other 3d theories in [45, 46]. Note also that in the case of the ABJM theory there were 1/3 BPS vortices preserving 8 supercharges [44]. An analog Wilson loop was never found. Possibly a linear combination of two 1/2 BPS Wilson loops with different couplings (the analogs of $\psi_{\underline{1}}$ and $\psi_{\underline{2}}$) should be considered the appropriate dual.

It may also be interesting to look for 1/2 BPS loops in the non-linear sigma model of [42] which arise from integrating out all the fields of the $k = 0$ nodes.

Localization

As discussed earlier, for ABJ(M) theory it was shown in [27] that the 1/2 BPS loop is cohomologically equivalent to a combination of 1/6 BPS loops, with respect to the localizing supercharge. That is

$$W^{1/2} - W^{1/6} = QV, \quad (1.13)$$

where Q is the localizing supercharge and V is a $U(N_1|N_2)$ supermatrix. This equivalence implies that the computation of the expectation value of the 1/2 BPS loop reduces to the calculation of the expectation value of a combination of 1/6 BPS loops. This allows one to use the results of [29] to reduce the evaluation of the 1/2 BPS loop to a matrix model integral.

In Section 2.3, we demonstrate that the ψ_1 - and ψ_2 -loops are cohomologically equivalent to each other and to a combination of 1/4 BPS loops. This allows one to use the results of [29] to reduce the evaluation of the 1/2 BPS loop to a matrix model integral. In Section 2.3.1 we demonstrate that, for the case with vanishing levels discussed in the previous section, the loop defined in terms of $U(N_1 + N_3|2N_2)$ representations is cohomologically equivalent to a combination of 1/4 BPS loops.

In the case of ABJ(M) theory, the corresponding matrix model has been solved; however, the general matrix model for the $\mathcal{N} = 4$ theories which we consider has not yet been solved. We demonstrate the cohomological equivalence to the expectation

value of the 1/4 BPS loops but do not study the resulting matrix model integral.

The lifting of the degeneracy

We expect that a theory with p untwisted hypermultiplets and q twisted hypermultiplets possesses p independent 1/2 BPS Wilson loops (not accounting for possible representations) preserving the $SU(2)_A$ supersymmetries. There are q other Wilson loops preserving the supercharges with the underlined and overlined indices interchanged. They can be easily written explicitly, and one can also study them by looking at a quiver with the untwisted and twisted hypermultiplets exchanged.

In [40] a study was initiated of the quantum corrections to the ψ_1 - and ψ_2 -loops. They considered a circular quiver with alternating CS levels and the circular Wilson loop. The two-loop expectation values of the ψ_1 - and ψ_2 - loops were computed at weak coupling via Feynman diagram computation, and found to be identical. In [41] the study was extended to three-loops. There they restricted to quivers consisting of nodes with distinct gauge group ranks N_i ; this allows one to isolate a specific colour sector⁴ and to study the degeneracy of the loops for this sector. It was found that only the average of the ψ_1 - and ψ_2 -loops matches the localization/matrix model result for this colour sector at three-loops, which they also computed. That is, the 1/2 BPS loop is given by

$$W_{\text{BPS}} := \frac{1}{2} (W^{\psi_1} + W^{\psi_2}) . \quad (1.14)$$

This bears out the proposal in [4]. The full two-loop result (identical for the ψ_1 - and ψ_2 -loops) is given by [40]

$$\langle W_{\text{BPS}} \rangle = -\frac{1}{24k^2} \left[(N_1^2 + N_2^2 - N_1 N_2 - 1) - 3 \frac{N_0 N_1^2 + N_3 N_2^2}{N_1 + N_2} \right] + \mathcal{O}(1/k^3) . \quad (1.15)$$

The matrix model for this theory has not yet been solved to obtain an exact result. However, these loops should be the most amenable to compute of any in the theory, analogously to the 1/2 BPS loops of $\mathcal{N} = 4$ SYM and ABJ(M) theory discussed earlier.

It remains to extend this study to quivers with nodes with vanishing CS levels (presented in Section 2.1.3). There we found five possible superconnections. Their variation does not vanish and is not a total derivative, but we argued that the remainder cancels in every expectation value. Four of the connections involve a pair of nodes and the last one has a richer structure involving all three nodes (one of them doubled). This last one is a superconnection in $U(N_1 + N_3 | 2N_2)$. Indeed it would seem natural to also combine the other connections into four possible pairings

⁴They consider contributions proportional to the product $N_{I-1} N_I N_{I+1}$ to the expectation values.

of block-diagonal $U(N_1|N_2) \oplus U(N_2|N_3) \subset U(N_1 + N_3|2N_2)$ superconnections. The resulting Wilson loops are classified by representations of $U(N_1 + N_3|2N_2)$ and we again expect only one linear combination to be BPS. It is not clear actually why a Wilson loop involving only a pair of nodes would not be BPS. This remains an open question.

1.2 BPS Wilson loops and operator insertions

The discussion so far has concerned maximally supersymmetric Wilson loops. These preserve the highest degree of symmetry of the theories and are thus the most amenable to study. The natural Wilson loops to consider next are loops preserving a lower degree of symmetry. Indeed there is also a rich structure and many exact results for less supersymmetric loops [28, 47–49].

Another natural class of loops to study are small deformations of maximally supersymmetric operators; the study of which was initiated in [50–52]. In [52] a general small deformation to the 1/2 BPS straight line was studied, as an expansion in the deformation. We consider the 1/2 BPS straight line loop $\mathbb{R}^{1,3}$, defined by the path $\mathcal{C}_0$ given by $x^\mu(s) = (0, 0, 0, s)$:

$$W_{\text{line}} = \text{tr} \overleftarrow{\mathcal{P}} \exp \left(\oint ds i \left[\dot{x}^\mu A_\mu - \sqrt{\dot{x}^2} \Phi^1 \right] \right), \quad (1.16)$$

where $\dot{x}^2 = \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$. Note that here the path ordering increases from right to left. We work in the planar limit and adopt the conventions of [53–55] for $\mathcal{N} = 4$ SYM. In Section 3.3 we specify these conventions.

One may consider a general deformation of the loop (1.16), with the new path $\mathcal{C}$ given by $x^\mu = (\epsilon^i(s), s)$ where $i = 0, 1, 2$. We take $\epsilon := |\epsilon| = \sqrt{-\eta_{ij} \epsilon^i \epsilon^j} \ll 1$ and ϵ may serve as an expansion parameter. It was found in [52], via a Feynman diagram-based computation, that, to two-loop order at weak coupling, the expectation value of the deformed loop is given by

$$\langle W[\mathcal{C}] \rangle|_{\epsilon^2} = 1 + \left(\frac{\lambda}{16\pi^2} - \frac{\lambda^2}{384\pi^2} \right) \oint ds_1 ds_2 \frac{|\dot{\epsilon}_1 - \dot{\epsilon}_2|^2}{2s_{12}^2} + \mathcal{O}(\lambda^3), \quad (1.17)$$

where $\langle W[\mathcal{C}] \rangle|_{\epsilon^2}$ denotes the expectation value at second order in the deformation, $\epsilon_1 := \epsilon(s_1)$ and $s_{ij} := s_i - s_j$. We observe that, to quadratic order in the deformation, there is universal behaviour, *i.e.* the functional forms of the one- and two-loop expectation values are identical. By exploiting the AdS/CFT correspondence, this universality was found to extend to leading order at strong coupling [52].

A preliminary study of the equivalence between deformations of the line and operator insertions was also initiated in [52] (considering up to quadratic order in the deformation) and insertions into the straight line have also been discussed

elsewhere [56–58]. Here and in Chapter 3, we discuss operator insertions into the 1/2 BPS straight line, the equivalence with deformations of the line and how we may use this equivalence to extract information about the operator insertions. We do this up to and including the fourth order in the deformation. This is based on unpublished work of the author and collaborators.

1.2.1 Operator insertions into the straight Wilson line

We define the Wilson loop with insertions as

$$W [\mathcal{O}^{(1)}(x_1) \dots \mathcal{O}^{(n)}(x_n)] := \frac{1}{N} \text{tr} \overleftarrow{\mathcal{P}} \left[\mathcal{O}^{(1)}(x_1) \dots \mathcal{O}^{(n)}(x_n) e^{\int i(\dot{x}^\mu A_\mu(x(s)) - \sqrt{\dot{x}^2} \Phi^1(x(s))) ds} \right]. \quad (1.18)$$

where the integration is over the straight line. The expectation value of (1.18) can be viewed as the correlation functions of the operator insertions. Accordingly, we define

$$\langle\langle \mathcal{O}^{(1)}(x_1) \dots \mathcal{O}^{(n)}(x_n) \rangle\rangle := \langle W [\mathcal{O}^{(1)}(x_1) \dots \mathcal{O}^{(n)}(x_n)] \rangle / \langle W \rangle. \quad (1.19)$$

These correlation functions satisfy the axioms of a CFT.⁵ This is due to the residual symmetry preserved by the 1/2 BPS straight line, which includes the $SO(1, 2)_c$ group of conformal transformations along the line. This group comprises $\{D, K_3, P_3\}$, where D is the $\mathcal{N} = 4$ SYM dilatation operator, K_3 generates special conformal transformations along the line and P_3 generates translations along the line. The operator insertions can be classified by representations of this group (or by the full $OSp(2, 2|4)$ preserved by the line). Furthermore, one may define conformal primary operators with respect to the conformal group (and similarly superconformal primaries).

The correlation functions (1.19) are constrained in the same manner as in a usual CFT. For example, the Wilson loop with two scalar insertions satisfies the Ward identity for dilatation

$$iD \langle\langle \mathcal{O}^{(1)}(s) \mathcal{O}^{(2)}(0) \rangle\rangle = (\Delta^{(1)} + \Delta^{(2)} + s \partial_s) \langle\langle \mathcal{O}^{(1)}(s) \mathcal{O}^{(2)}(0) \rangle\rangle = 0, \quad (1.20)$$

which is identical in form to that of the two-point function of local operators, solved by

$$\langle\langle \mathcal{O}^{(1)}(s) \mathcal{O}^{(2)}(0) \rangle\rangle = \frac{a^{(12)}(\lambda)}{s^{\Delta^{(1)} + \Delta^{(2)}}}, \quad (1.21)$$

where $\Delta^{(n)} = \Delta^{(n)}(\lambda)$. In this way we may associate dimensions (classical and anomalous) to the insertions. We note that the two point function of primary operators vanishes unless they have the same dimension.

⁵More precisely, this is a defect CFT, and does not have an energy-momentum tensor that is decoupled from the bulk.

Above we have considered scalar primary operators. However, all of the above constraints generalize to descendant operators and to operators with tensor structure. The correlation function of a descendant operator is obtained from the correlation function of the primary operator of the multiplet by the application of the lowering operator, P_3 , of the residual conformal algebra (we discuss the above and higher point functions in Section 3.1).

In the planar theory, the anomalous dimensions can be determined, in principle, at all values of the gauge coupling by using the tools of integrability. Alternatively, one can use Feynman diagrams to evaluate any of the correlation functions. For example, Figure 1.4 is one Feynman diagram contributing to the two-point function of insertions at two-loops in the coupling. We note that the two-point function includes interactions with the 1/2 BPS line.

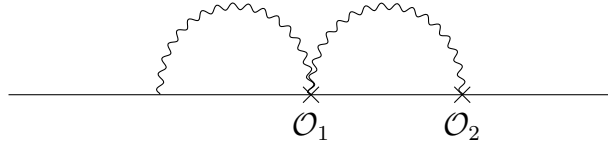


Figure 1.4: **Feynman diagram contributing to $\langle\langle \mathcal{O}_1 \mathcal{O}_2 \rangle\rangle$** The straight line represents the 1/2 BPS line. The wavy lines represent propagators in theory, the exact form of which depends on the operators $\mathcal{O}_1$ and $\mathcal{O}_2$. This figure was drawn using [59].

In Chapter 3, we adopt a third approach, which exploits an equivalence between certain operator insertions into the straight line and the Wilson loop defined by a small deformation of the straight line. We use known expressions for the one- and two-loop expectation values of a general smooth Wilson loop to extract information about the operator insertions. The work presented in Chapter 3 and reviewed here represents work-in-progress; there are unresolved issues which we discuss later and are currently being investigated.

Now, we study the deformation/operator insertions at second order in the deformation parameter (this is presented in more detail in Chapter 3). We describe how the computation of the expectation value of the deformed loop is determined by a set of coefficients; these coefficients contain physical information about operator insertions. We then describe how these coefficients may be determined, using known results for Wilson loops with a general smooth contour. The information which we obtain is not new, though the method is. In Chapter 3, we implement the same procedure up to and including the fourth order in the deformation; yielding new results.

Let us consider a Wilson loop (or line) along a contour $\mathcal{C}_0$, defined by $x^\mu(s)$ in $\mathbb{R}^{1,3}$. We consider a small deformation of the contour $\mathcal{C}_0$ to define a new contour $\mathcal{C}$, by adding a small deformation to $x^\mu(s)$ denoted by $\delta x^\mu(s)$. The Wilson loop $W[\mathcal{C}]$

is then given formally by the functional Taylor expansion

$$W[\mathcal{C}] = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n W[\mathcal{C}_0], \quad (1.22)$$

where δ corresponds to taking $x^\mu(s) \rightarrow x^\mu(s) + \delta x^\mu(s)$. We do not consider deformations of the scalar couplings, beyond the change of the magnitude of the Φ^1 coupling which is $|\dot{x}|$.

We apply this to the 1/2 BPS loop (1.16), with $\delta x^\mu = (\epsilon^i(s), 0)$ ($i = 0, 1, 2$). It is straightforward to see that the functional Taylor expansion gives rise to operator insertions into the straight line. At second order in the deformation, the expectation value of the deformed loop is mapped, by this functional Taylor expansion, to

$$\langle W[\mathcal{C}] \rangle = 1 + \oint_{s_1 > s_2} ds_1 ds_2 \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) \rangle\rangle \epsilon_1^i \epsilon_2^j + \mathcal{O}(\epsilon^3), \quad (1.23)$$

where $\mathbb{F}_{i3} := F_{i3} + iD_i\Phi^1$ and $\epsilon_n^i := \epsilon^i(s_n)$. As discussed above, conformal symmetry constrains the form of the operator insertion correlation functions. Above, we have used that one-point functions of operator insertions are constrained to vanish by conformal symmetry. Conformal symmetry also constrains the two-point function:

$$\begin{aligned} \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) \rangle\rangle &= \frac{a(\lambda)\eta_{ij}}{s_{12}^{2\Delta(\lambda)}} \\ &= \frac{(a_0 + \lambda[a_1 - 2a_0\gamma \log \frac{s_{12}}{\mu}])\eta_{ij}}{s_{12}^{2\Delta_0}} + \mathcal{O}(\lambda^2). \end{aligned} \quad (1.24)$$

In the second line we have expanded in the coupling and introduced a scale parameter μ . The constants a_0 , a_1 and γ are the tree-level and one-loop two-point function coefficients, and the anomalous dimension of $i\mathbb{F}_{i3}$, respectively.

If we can compute the left hand side of (1.23) at one- and two-loops for a number of explicit deformations ϵ^i , we may fix the coefficients in (1.24). That is, we may extract the tree-level and one-loop two point function coefficient as well as the anomalous dimension of $i\mathbb{F}_{i3}$. Equivalently, if we have an expression for $\langle W[\mathcal{C}] \rangle$ for an arbitrary deformation, we may manipulate this expression such that we may compare directly with (1.24) and read off the above information.

We do the latter in Chapter 3, using the well-known expression for the one-loop expectation value of a general smooth contour

$$\langle W[\mathcal{C}] \rangle_{1\text{-loop}} = -\frac{\lambda}{16\pi^2} \oint ds_1 ds_2 \frac{\dot{x}_1 \cdot \dot{x}_2 - \sqrt{\dot{x}_1^2} \sqrt{\dot{x}_2^2}}{x_{12}^2}. \quad (1.25)$$

At two-loops, we use an analagous expression derived in [60] (see Section 3.4 for details). Expanding the integrand in (1.25) to second order in ϵ , after some manipulations (see Section 3.4 for details), we find that the tree-level two-point function

coefficient of $i\mathbb{F}_{i3}$ is

$$a_0 = -\frac{3}{4\pi^2}. \quad (1.26)$$

Applying the same process to the two-loop result of [60], we find

$$a_1 = \frac{1}{32\pi^2}, \quad \gamma = 0. \quad (1.27)$$

It is already known that $i\mathbb{F}_{i3}$ is a protected operator [56]; it is a superconformal descendant of the protected Φ^A scalar ($A = 2, \dots, 5$). Our results are thus consistent with this. Returning to the universality observed in (1.17) (and at strong coupling), the universal behaviour may be understood simply as the statement that $i\mathbb{F}_{i3}$ is a protected operator.

In Chapter 3 we extend the discussion of operator insertions in the line to fourth order in the deformation. Our goal is to extract information about the two- and three-point functions of the operator insertions from known expressions for the one- and two-loop expectation values. At fourth order in the deformation, we demonstrate in Chapter 3 that there are operator insertions belonging to two different supermultiplets: the Φ^A supermultiplet and the Φ^1 supermultiplet. Here we have labelled the supermultiplets by their superconformal primary operator. The latter is not protected, and the anomalous dimension has been calculated using Feynman diagrams in [61].

Our analysis is consistent with the previously known results of [52] and [61]. It also generalizes the results of [52], in the sense that we consider not only the second order in the deformation parameter, but up to fourth order in this parameter. Our results also satisfy several consistency conditions imposed by the multiplet structure of the insertions.

However, we find extra terms from manipulating two-loop expression of [60] at fourth order in the deformation. These terms are seemingly inconsistent with the operator insertion description. The right hand side of (1.23) is UV divergent and requires regularization (see Section 3.4.1). Accordingly, we adopt two alternative regularization schemes in our procedure. The extra terms we observe may be an artifact of our regularization procedure. We also see that some of the two-point function coefficients do not fit with the multiplet structure which we derive in Section 3.3; this may be due to operator mixing and/or the regularization procedure. Chapter 3 reports work-in-progress; the extra terms and possible operator mixing is currently under investigation.

1.3 Holographic description of Wilson loops

It was proposed in [12, 13] that Wilson loop operators in certain large N field theories are dual to fundamental strings in a string theory (or M2-branes in M-theory). The

string is an open string ending on the boundary of AdS, where it describes a path $\mathcal{C}$ defining the Wilson loop in the dual field theory. Here we discuss the holographic duals of Wilson loops in the fundamental representation; discussions of the duals of loops in higher representations, involving Dp-branes may be found in [62–65].

1.3.1 Four dimensions

AdS/CFT postulates the equivalence of type IIB string theory on $AdS_5 \times S^5$ with $\mathcal{N} = 4$ SYM on $\mathbb{R}^{1,3}$, the boundary of AdS_5 [7]. A fundamental string carries electric charge and an open string ending at the boundary of AdS_5 is a natural candidate to describe a Wilson loop in type IIB string theory. Concretely, the expectation value of the loop is given by [12, 13]

$$\langle W[\mathcal{C}] \rangle = Z[\text{string ending on } \mathcal{C}] , \quad (1.28)$$

where Z is the type IIB partition function on $AdS_5 \times S^5$. We note that for non-trivial scalar couplings the string worldsheet extends along the internal S^5 . The right hand side is generally extremely hard to compute; however taking the planar limit and subsequently taking the 't Hooft coupling to be very large, suppressing string fluctuations, we may adopt the saddle-point approximation in (1.28). This gives

$$\langle W[\mathcal{C}] \rangle \simeq \exp \left(-\sqrt{\lambda} A \right) , \quad (1.29)$$

where A is the area of the minimal surface ending on $\mathcal{C}$. The area of this minimal surface is infinite due to the curvature of AdS. This divergence may be cancelled using boundary terms to extract the finite regularized area. Specifically, introducing a cutoff ϵ near the boundary the area is given by

$$A = \frac{L}{\epsilon} + A_f , \quad (1.30)$$

where L is the circumference of the loop and A_f is the regularized area.

The dual strings to 1/2 BPS Wilson loops must preserve half of the supersymmetries of the $AdS_5 \times S^5$ vacuum. In the Green-Schwarz formulation, kappa-symmetry must be imposed on the string worldsheet to ensure equal bosonic and fermionic degrees of freedom on the worldsheet. Given a string embedding in $AdS_5 \times S^5$ and the Killing spinor of the vacuum, one can check the number of supersymmetries preserved by the consistency condition

$$\Gamma \epsilon = \epsilon , \quad (1.31)$$

where ϵ is the Killing spinor of $AdS_5 \times S^5$ and Γ is the kappa-symmetry projector of the string. The condition (1.31) determines the number of supersymmetries of the

vacuum which are consistent with the kappa-symmetry projection, and hence how many supersymmetries are preserved by the string.

1.3.2 Three dimensions

If we restrict ourselves to the subset of the $\mathcal{N} = 4$ CS theories described in Section 1.1.3 which are described by circular quivers with vector multiplets of equal rank, then the theory is dual to M-theory on $AdS_4 \times M_{p,q,k}$ [66]. p and q correspond to the number of hypermultiplets and twisted hypermultiplets in the field theory; and k corresponds to the Chern-Simons level. $M_{p,q,k}$ is an orbifold of the seven-sphere

$$M_{p,q,k} = (S^7/(\mathbb{Z}_p \oplus \mathbb{Z}_q)) / \mathbb{Z}_k, \quad (1.32)$$

with radius $R^6 = 2^5 \pi^2 N k p q$ (in units of $\ell_s = 1$). We note that $p = q = 1$ corresponds to ABJ(M) theory.

Alternatively, one may consider the IIB holographic duals of these $\mathcal{N} = 4$ theories [67–70]. The IIB holographic dual of a circular quiver is a fibration over an annulus Σ , and the theories are characterized by two harmonic functions h_1 and h_2 defined on Σ . This mirrors the brane construction for these theories, with the positions of singularities of the harmonic functions indicating the positions of five-branes in the brane construction of the quiver theory.

To obtain the M-theory dual, one T-dualizes and uplifts to M-theory. The harmonic functions then no longer depend on the exact location of the five-branes. This may be viewed as a smearing of the five-branes [22]. As a result, the M-theory geometry is insensitive to arrangement of these hypermultiplets in the dual quiver and only depends on their number. The dependence of the M-theory duals on this order may be encoded in the torsion of the 3-form potential field C_3 , however [66]. In this thesis, we consider only the M-theory duals; though, as discussed above, to distinguish between the different quiver arrangements one should consider the IIB holographic duals.

We describe $M_{p,q,k}$ using complex embedding coordinates, which are subject to the constraint $\sum_{i=1}^4 |z_i|^2 = 1$. These coordinates are further subject to the identifications

$$(z_1, z_2, z_3, z_4) \sim (\omega_{kp}^m z_1, \omega_{kp}^m z_2, \omega_{kq}^{-m} z_3, \omega_{kq}^{-m} z_4), \quad \omega_r = e^{2\pi i/r}, \quad m \in \mathbb{Z}. \quad (1.33)$$

This (without the embedding coordinate constraint) defines the manifold

$$\mathcal{M}_{p,q,k} = (\mathbb{C}^2/\mathbb{Z}_p \otimes \mathbb{C}^2/\mathbb{Z}_q) / \mathbb{Z}_k. \quad (1.34)$$

Now the action of the R -symmetry is manifest: the $SU(2)_A$ subgroup acts on $\mathbb{C}^2/\mathbb{Z}_p$ and $SU(2)_B$ acts on $\mathbb{C}^2/\mathbb{Z}_q$.

In Section 2.4, we consider the M2-brane world-volume extending along an AdS_2

subspace of AdS_4 and wrapping an S^1 on the internal space (*i.e.* the M-theory circle). It sits at a fixed point in the other coordinates. The supersymmetries preserved by the ψ_1 - and ψ_2 -loops (1.8) and (1.9) are doublets of $SU(2)_A$, which is left unbroken. It is thus natural that the M2-brane embedding will be at a fixed point of the $SU(2)_A$ action. This is the tip of the $\mathbb{C}/\mathbb{Z}_p$ cone in $\mathcal{M}_{p,q,k}$. In Section 2.4.2, we verify that this is indeed the case by studying the kappa-symmetry projection (1.31).

In Section 2.4.2 we argue that the expectation value is given by

$$\langle W \rangle \simeq e^{\pi \sqrt{\frac{2Np}{kq}}} . \quad (1.35)$$

However, it is actually unlikely that this expression be valid for all the p different Wilson loops, as the answer should depend on the details of the resolution of the orbifold singularity (and fluxes, and fractional branes). This expression does at least hold in the simplest case where the Wilson loops are all equivalent, so theories with $p = q$ and alternating $\pm k$ levels. These are orbifolds of ABJ(M) theory.

As mentioned above, to answer the question of the expectation values more concretely one should study the type IIB duals of these theories [67–70]. This mirrors the brane construction of these theories [69, 70] and the possible ways to add fundamental strings to them [46]. The orbifold singularity in the M-theory description is resolved in these theories. These also allow to study linear quivers, while M-theory is only a good description for the circular quivers. It is also easier to specify the different ranks of the gauge groups and the location of the $k = 0$ nodes in the IIB language.

1.4 Strings in $\mathbb{H}_3$ and the spectral curve

We have thus far discussed supersymmetric aspects of Wilson loops. One of the motivations for studying such loops was that they are amenable to exact computation. Integrable theories also offer an opportunity to obtain exact results. An infinite-dimensional integrable theory may be defined as the existence of infinite number of conserved charges in the theory. $\mathcal{N} = 4$ SYM in the planar limit [71] is believed to be an example of an integrable theory. In the planar limit, the rank of the $SU(N)$ gauge group becomes infinite, while the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$ is kept finite. Via the AdS/CFT correspondence this limit corresponds to considering free strings ($g_s = 0$) in type IIB string theory. Integrability has led in recent years to great advances in the detailed understanding of the AdS/CFT correspondence between the two theories. The most well-studied sector is the spectrum of single-trace operators in the field theory and of the dual closed strings. Other advancements are in calculating scattering amplitudes, structure constants and open string states.

Studying open strings dual to Wilson loops is more involved than solving the usual

spectral problem for closed strings; at the classical level, the boundary conditions are not periodic (or Dirichlet/Neumann for fixed directions), but vary arbitrarily along the boundary. Solutions are not classified only by initial data, but by a rich structure of boundary conditions. If the Wilson loop is made of light-like segments, the regularized action on the surface (though not its shape) can be computed by solving a set of TBA-like equations [72]. By considering a continuum limit of such a null polygon, a set of TBA-like equations have been formulated for generic smooth contours [73]. Here we focus on the case where the Wilson loop is space-like and piece together different integrability approaches that have been used to study them, verifying their consistency.

The spectral curve approach [6] provides a framework to study classical closed strings by constructing the monodromy associated to a flat connection. This monodromy serves as a generating function of an infinite tower of conserved charges, including the energy. Solutions can be classified in terms of a spectral curve, which is naturally defined in terms of the monodromy matrix.

The analogous story for open strings dual to Wilson loops is far less well understood. There are no non-trivial cycles on the world-sheet, which makes it harder to construct a monodromy matrix. While it is more difficult to construct, it should be possible by either considering an open boundary-to-boundary monodromy, or by taking the monodromy including reflections from two boundary points. This latter approach was implemented in [74] (based on [75]) for open strings ending on D-branes in $AdS_5 \times S^5$. Previous work studying the integrable structure of open strings ending on D-branes include [76–82]. The analogous construction for open strings ending on the boundary has not been found, so there is no satisfactory monodromy matrix for the strings dual to Wilson loops.

Complimentary approaches to the study of such surfaces were proposed in [2] and [83]. In the theory of integrable systems, the monodromy matrix of a field theory plays an analogous role to that of the Lax operator in finite-dimensional systems. In particular one may extract a spectral curve from the Lax operator for a finite-dimensional system, as the spectral curve is extracted from the monodromy. As advocated in [2], one may construct a Lax operator with multiple time parameters for a field theory. We implement this approach; the resulting Lax operator is local in both worldsheet coordinates, in contrast to a monodromy. It is not unique and *à priori* neither is the resulting algebraic curve. Furthermore, one may not extract the conserved charges from the Lax operator, as one can from the monodromy. Nonetheless the constructions of [2, 83] give an algebraic curve based on a Lax operator for classical strings in AdS_3 , which appear to characterize the string solution. Here we outline how their construction may be extended from genus-one solutions to higher genus.

In Chapter 4, we implement this on a very large class of explicit solutions in

Euclidean AdS_3 ($\mathbb{H}_3$) which were constructed in [1] and [84] (see also [85], for a different approach to these solutions). These solutions are expressed in terms of theta functions associated with a hyperelliptic curve of arbitrary odd genus. Furthermore, they depend on a spectral parameter, such that each curve gives rise to a one-parameter family of classical solutions.⁶ The purpose of our study is to verify that the algebraic curve associated with these surfaces is indeed the hyperelliptic curve they were constructed from and elucidate the role of the spectral parameter.

We now briefly review the spectral curve construction for closed strings in $\mathbb{H}_3$. We contrast this with case of open strings dual to Wilson loops in the field theory. The $\mathbb{H}_3$ subsector of $AdS_5 \times S^5$ is classically integrable and may be described by a non-linear sigma model. The action is given by

$$S = \frac{1}{4\pi} \int d^2w \, \text{tr}(\mathcal{J} \bar{\mathcal{J}}), \quad (1.36)$$

where w and $\bar{w}$ are the complex worldsheet coordinates; $\mathcal{J}$ and $\bar{\mathcal{J}}$ are $\mathfrak{sl}(2; \mathbb{C})/\mathfrak{su}(2)$ Maurer-Cartan forms. These currents encode the embedding of the string worldsheet in the target space.

The sigma-model equations of motion and Virasoro constraints may be written in terms of these currents. In particular, as is typical for integrable systems, flat connection $J(x) := \frac{\mathcal{J}}{1+x}$ and $\bar{J}(x) := \frac{\bar{\mathcal{J}}}{1-x}$ may be constructed such that the equations of motion and Virasoro constraints are equivalent to the flatness condition

$$\bar{\partial}J - \partial\bar{J} + [\bar{J}, J] = 0. \quad (1.37)$$

Here $x \in \mathbb{C}$ is the spectral parameter. We may expand about $x = 0$ to obtain the equations of motion and Virasoro constraints.

Closed strings

When presented with a flat connection, as in (1.37), a natural next step is to construct a monodromy from this connection. For closed strings, this is straightforward - a monodromy is easily constructed by considering the parallel transportation of the connection about a closed path γ along the periodic worldsheet direction

$$\Omega(\tau; x) := \overleftarrow{\mathcal{P}} \exp \left(\int_{\gamma} d\sigma J(\tau, \sigma; x) \right), \quad (1.38)$$

where τ is the non-compact worldsheet direction.

As we have mentioned, an infinite number of conserved charges are characteristic of infinite-dimensional integrable systems. By taking the trace of powers of the

⁶There is a further discrete choice of possible boundaries for each curve and spectral parameter, see [1], but we shall not discuss it.

monodromy matrix and expanding in the spectral parameter, we generate an infinite number of conserved charges. In particular, the level-zero charges are the familiar Noether charges.

The characteristic polynomial of the monodromy matrix is given by

$$\det(y^2 - \Omega(\tau; x)) = 0. \quad (1.39)$$

The flatness condition ensures that this curve is worldsheet independent. In general this curve will have infinite genus and possibly essential singularities. It is typical to restrict to the so-called ‘finite-gap solutions’, *i.e.* solutions yielding a curve with finite genus. The singularities may be resolved by algebraic geometric methods.

The string solution is completely characterized by the spectral curve or the so-called quasi-momenta. The quasi-momenta are related to the eigenvalues α_i of the monodromy matrix by $\alpha_i(x) = e^{ip_i(x)}$. The quasi-momenta are functions of the spectral parameter x and typically possess branch cuts (unlike the monodromy or Lax connection). For $\mathbb{H}_3$ the quasi-momenta describe a two-sheeted Riemann surface and the quasi-momenta satisfy

$$p(x + i\epsilon) - p(x - i\epsilon) = 2\pi n_i, \quad x \in C_i, \quad (1.40)$$

at the cuts, where $n_i \in \mathbb{Z}$ and C_i denote the two sheets of the Riemann surface. The string solutions may be characterized by the analytic properties of the quasi-momenta. We note that we have discussed the $\mathbb{H}_3$ sector here. The procedure generalizes to the full $AdS_5 \times S^5$ sigma model [86]⁷.

Open strings

We have outlined above how the spectral curve provides an elegant and powerful description of finite-gap solutions. Central to this was the construction of the monodromy (1.38). This monodromy is invariant under smooth deformations of the path γ ; as a consequence, a contractible monodromy is trivial. For closed strings it is straight forward to define a non-contractible integration contour by considering a path along the periodic worldsheet direction. For open strings, such as those corresponding to Wilson loops in the boundary theory, the same construction is not possible as all such paths γ are contractible. However, one may define a non-trivial monodromy by considering an open path with fixed endpoints and integrability preserving boundary conditions (see [75] for examples of such boundary conditions). Indeed this was done for open strings ending on a $Y = 0$ brane [74]. For open string solutions holographically dual to Wilson loops, no such monodromy has, as yet, been constructed.

⁷Prior to this generalization, this had been done for various subsectors [6, 87, 88]. Backgrounds other than $AdS_5 \times S^5$ and its subsectors have also been studied [89–92].

In [2], a procedure was given to extract an algebraic curve from open string solutions corresponding to Wilson loops. This procedure circumvents the construction of a monodromy, taking the algebraic curve to be the characteristic polynomial of a Lax operator. This was performed for two explicit open string solutions corresponding to Wilson loops: the null cusp and the $q\bar{q}$ potential.

We now perform a digression to contrast finite- and infinite-dimensional systems. For finite-dimensional systems the Lax operator L satisfies

$$\frac{dL}{dt} = [L, M] , \quad (1.41)$$

where t is a time parameter and M completes the Lax pair (L, M) . The spectral curve may be extracted from the Lax operator:

$$\det(y^2 - L(t)) = 0 , \quad (1.42)$$

which is independent of the time t due to the Lax equation (1.41). In much the same way as conserved quantities may be extracted from the monodromy matrix for infinite-dimensional systems, conserved quantities in finite-dimensional integrable systems may be extracted from the Lax operator.

By introducing a spectral parameter $x \in \mathbb{C}$ one may generalize the above construction to consider infinite-dimensional systems. We then have $(L(x), M(x))$. For two-dimensional systems, such as the string worldsheet, we may package the Lax pair into a Lax connection J , satisfying

$$dJ(x) + J \wedge J(x) = 0 , \quad (1.43)$$

where the exterior derivative d is with respect to the worldsheet coordinates. This is the flatness condition we saw in (1.37) and this generalizes the Lax equation (1.41). For the closed string, we defined the monodromy (1.38). This satisfies

$$\frac{\partial}{\partial \tau} \Omega(\tau; x) = [J_\tau(\hat{\sigma}, \tau; x), \Omega(\tau; x)] , \quad (1.44)$$

which follows from the flatness condition and where $\hat{\sigma}$ is a fixed endpoint of the integration in the definition of the monodromy. From (1.44) it is apparent that the monodromy is the generalization of the Lax operator. However as discussed earlier, it is far from straightforward to construct a monodromy for open strings. As in [2] we consider a Lax operator satisfying

$$\frac{\partial L}{\partial \sigma^\alpha} = [J_\alpha, L] , \quad (1.45)$$

where σ^α are the worldsheet coordinates. This ‘generalizes’ the Lax equation (1.41) to a system with two ‘time’ parameters. While it is apparent that the appropri-

ate generalization is given by the monodromy matrix, a Lax operator L satisfying (1.45) characterizes the solution in the sense that the string solutions may be reconstructed from the corresponding algebraic curves in an analogous way to the finite-gap solutions [2]. We are denied the capability to extract conserved charges in this construction, however. Agreement was found between the curve found in [2] for the cusped Wilson line and the strong coupling limit of the Y-system solution on the gauge theory side [93].

In Chapter 4, we construct a Lax operator and extract an algebraic curve for the very large family of minimal surfaces presented in [1] (see (4.37) for the explicit Lax operator). These solutions are expressed in terms of Riemann theta functions. We devote Section 4.2 to studying these functions and their analytic properties, which are crucial in our derivation of the Lax operator and algebraic curve. The algebraic curve which we extract from the Lax operator is exactly that defining the Riemann surface associated with the theta functions (4.12). Though there are ambiguities in defining the Lax operator, it is still natural to expect the resulting curve to be unique, and not dependent on these ambiguities. This suggests that this curve plays the same role as that normally derived from the monodromy matrix. It would of course be interesting to rederive these results from a monodromy matrix on the open string surface (including appropriate reflections from the boundaries). Monodromies have been constructed for open strings with certain integrable D-brane boundary conditions [74], but no such construction has been successfully applied to strings ending on the boundary of AdS .

Aside from providing a large family of analytic minimal surface solutions, these solutions depend explicitly on a continuous parameter. The action of the parameter is a symmetry at strong coupling, *i.e.* the regularized area of the minimal surface is invariant under a change in the parameter. This action has since been understood as a level-changing operator of the Yangian [94]. It would be interesting to understand any connection between our results and the work there.

Chapter 2

1/2 BPS Wilson loops in $\mathcal{N} = 4$ Chern-Simons matter theories

In this chapter we study 1/2 BPS operators in $\mathcal{N} = 4$ Chern-Simons matter theories in three dimensions; presenting the results of the author and collaborators found in [4]. We consider a circular or linear quiver theory with Chern-Simons levels k , $-k$ and 0. In Section 2.1 we construct the 1/2 BPS Wilson loops with straight line geometry. We initially consider segments of the quiver with alternating k , $-k$ CS levels, and subsequently segments including vanishing CS levels. Surprisingly we find two 1/2 BPS loops with a given geometry (either the straight line or the circle) preserving the same supersymmetries for each segment of the quiver. In Section 2.2 we study the case of the circular Wilson loop.

We also discuss how the expectation values of these loops may be reduced to a matrix model integral via localization in Section 2.3. Specifically, we demonstrate classical cohomological equivalence, with respect to the localizing supercharge, to a combination of 1/4 BPS loops. The implication of this is that the expectation values of the 1/2 BPS loops are equivalent to the expectation value of a combination of 1/4 BPS loops. The matrix model integral giving the expectation value of 1/4 BPS loop is known [29], but the matrix model has not been solved; we neither present nor attempt to solve this matrix model.

Finally, in Section 2.4 we find the M2-brane embeddings which are holographically dual to the aforementioned Wilson loops, for circular quivers. We provide an argument, based on the connection with ABJ(M) theory, for the strong coupling expectation values of the corresponding loops.

2.1 The straight line

Wilson loops are traditionally defined as the holonomy of the gauge connection. It was however found in [27] that the 1/2 BPS Wilson loops in ABJ(M) theory must be expressed in terms of the holonomy of a *superconnection* $\mathcal{L}$, with modified

connections for the two $U(N_1)$ and $U(N_2)$ vector multiplets on the diagonal blocks and bifundamental Fermi fields in the off-diagonal blocks. The Wilson loop is then

$$W = \text{tr}_{\mathcal{R}} \vec{\mathcal{P}} \exp \left(-i \int ds \mathcal{L}(s) \right), \quad (2.1)$$

where $\mathcal{R}$ is a representation of the supergroup $U(N_1|N_2)$.

The supersymmetry variation of the superconnection does not vanish, rather it is a total differential. In [28] this was expressed in terms of a supercovariant derivative of a supermatrix valued in the superalgebra, *i.e.*,

$$\delta \mathcal{L} = \mathfrak{D}_s G := \partial_s G - i \{ \mathcal{L}, G \}, \quad (2.2)$$

with a commutator or an anti-commutator depending on the Grassmann parity of the supermatrices. One must furthermore ensure that the trace is such that boundary terms cancel. This is enough to guarantee that the Wilson loop is invariant under the corresponding supersymmetries. Our construction of 1/2 BPS loops in $\mathcal{N} = 4$ theories will be based on similar principles.

In this section, we begin by considering an infinite spacelike line in the x^1 -direction in $\mathbb{R}^3$ parametrized by

$$x^1 = s, \quad x^2 = x^3 = 0. \quad (2.3)$$

2.1.1 Alternating CS levels

The theories we consider have a number of vector multiplets coupled by p hypermultiplets and q twisted hypermultiplets forming a circular or linear quiver. We study different possible sections of the quiver and find the Wilson loops supported on the relevant nodes.

We start by considering a segment of the quiver of the form where a solid link

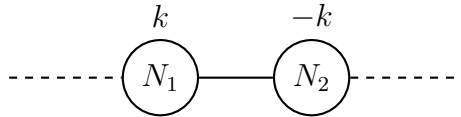


Figure 2.1: $\mathcal{N} = 4$ **quiver segment with alternating CS levels:** two $U(N_i)$ vector multiplets of opposite CS level connected by a hypermultiplet (solid links) and each with an additional adjacent twisted hypermultiplet (dashed links).

corresponds to a hypermultiplet and a dashed link to a twisted hypermultiplet. Each node represents a $U(N_I)$ vector multiplet with Chern-Simons level k_I (we label our nodes by the index I). This CS level is fixed by the condition

$$k_I = \frac{k}{2}(s_I - s_{I+1}), \quad s_I = \pm 1, \quad (2.4)$$

where $s_I = 1$ for a hypermultiplet at the I^{th} link and $s_I = -1$ for a twisted hypermultiplet.¹ With these conditions, it is apparent that $k_I \in \{k, -k, 0\}$.

The hypermultiplet between the first and second nodes consists of a scalar $q_{(1)}^{\bar{A}}$ and a fermion $\psi_{(1)\underline{B}}$. They transform in the bifundamental representation of the $U(N_1) \times U(N_2)$ gauge group. The scalar and fermion are doublets of the $SU(2)_A$ and $SU(2)_B$ factors of the $SO(4) = SU(2)_A \times SU(2)_B$ R-symmetry group, respectively. The over- and under-lined indices reflect this (see Appendix A.1 for our notation and conventions). We also have the Hermitian conjugate fields $\bar{q}_{(1)\bar{A}}$ and $\bar{\psi}_{(1)}^{\underline{B}}$.

The twisted hypermultiplets to the left of the first node and to the right of the second node consist of $\{q_{(0)}^{\bar{A}}, \psi_{(0)\bar{B}}\}$ and $\{q_{(2)}^{\bar{A}}, \psi_{(2)\bar{B}}\}$, respectively. We note the R-symmetry representations, contrasting with those of the hypermultiplet. For vector multiplets with non-vanishing CS level, the gaugino and auxiliary scalar may be integrated out. We are left with the gauge connection $A_{(I)\mu}$.

Before proceeding we define, following [37], the useful bilinears

$$\begin{aligned} \nu_{(I)} &= q_{(I)}^{\bar{A}} \bar{q}_{(I)\bar{A}}, \quad \tilde{\nu}_{(I)} = \bar{q}_{(I)\bar{A}} q_{(I)}^{\bar{A}}, \\ (\mu_{(I)})^{\bar{A}}{}_{\bar{B}} &= q_{(I)}^{\bar{A}} \bar{q}_{(I)\bar{B}} - \frac{1}{2} \delta^{\bar{A}}{}_{\bar{B}} (\nu_{(I)})^{\bar{C}}{}_{\bar{C}}, \quad (\tilde{\mu}_{(I)})^{\bar{A}}{}_{\bar{B}} = \bar{q}_{(I)\bar{B}} q_{(I)}^{\bar{A}} - \frac{1}{2} \delta^{\bar{A}}{}_{\bar{B}} (\tilde{\nu}_{(I)})^{\bar{C}}{}_{\bar{C}}, \\ j_{(I)}^{\bar{A}B a} &= \sqrt{2} q_{(I)}^{\bar{A}} \bar{\psi}_{(I)}^{B a} - \sqrt{2} \epsilon^{\bar{A}\bar{C}} \epsilon^{BD} \psi_{(I)\underline{D}}^a \bar{q}_{(I)\bar{C}}, \\ \tilde{j}_{(I)}^{\bar{A}B a} &= \sqrt{2} \bar{\psi}_{(I)}^{B a} q_{(I)}^{\bar{A}} - \sqrt{2} \epsilon^{\bar{A}\bar{C}} \epsilon^{BD} \bar{q}_{(I)\bar{C}} \psi_{(I)\underline{D}}^a. \end{aligned} \quad (2.5)$$

The currents j and $\tilde{j}$ are descendents of the moment maps μ and $\tilde{\mu}$.

We seek loops preserving the supersymmetries generated by the four parameters

$$\xi_{\bar{A}\underline{1}}^+, \quad \xi_{\bar{A}\underline{2}}^-, \quad (2.6)$$

where $\pm$ is the chirality. We begin our analysis by considering the variation² of the gauge connection of the first node with respect to the above supersymmetry parameters:

$$\delta A_{(1)1} = \frac{1}{k} \xi_{\bar{A}\underline{1}}^+ \left(j_{(1)+}^{\bar{A}\underline{1}} - \tilde{j}_{(0)+}^{\bar{A}\underline{1}} \right) - \frac{1}{k} \xi_{\bar{A}\underline{2}}^- \left(j_{(1)-}^{\bar{A}\underline{2}} - \tilde{j}_{(0)-}^{\bar{A}\underline{2}} \right). \quad (2.7)$$

In CS-matter theories, it is natural to allow for a bilinear of the scalars in the connection [25–27, 34, 95]. We can eliminate all the terms in the variation of $A_{(1)}$ (2.7) that depend on the twisted hypermultiplets by taking the linear combination

$$\delta \left[A_{(1)1} - \frac{2i}{k} (\tilde{\mu}_{(0)})^{\underline{1}}{}_{\underline{1}} \right] = \frac{1}{k} \xi_{\bar{A}\underline{1}}^+ j_{(1)+}^{\bar{A}\underline{1}} - \frac{1}{k} \xi_{\bar{A}\underline{2}}^- j_{(1)-}^{\bar{A}\underline{2}}. \quad (2.8)$$

Since $\tilde{\mu}$ is traceless, this is the same as adding $\frac{2i}{k} (\tilde{\mu}_{(0)})^{\underline{2}}{}_{\underline{2}}$.

¹We adopt the convention that in quiver diagrams I increases from left to right.

²See Appendix A.1 for the supersymmetric variations of the component fields.

We can also include terms from the scalars in the untwisted hypermultiplets. A term proportional to their moment map will lead to a connection invariant under half of the supercharges in (2.6), that is a 1/4 BPS Wilson loop. Expanding the moment maps and currents in terms of component fields, we find that adding and subtracting $\nu_{(1)}$ to/from the modified connection (2.8) gives

$$\delta \left[A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\underline{1}} - \frac{i}{k}\nu_{(1)} \right] = \frac{2\sqrt{2}}{k} \left[\xi_{\underline{A}\underline{1}}^+ q_{(1)}^{\bar{A}} \bar{\psi}_{(1)+}^1 + \epsilon^{\bar{A}\bar{B}} \xi_{\underline{B}\underline{2}}^- \psi_{(1)\underline{1}-} \bar{q}_{(1)\bar{A}} \right], \quad (2.9a)$$

$$\delta \left[A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\underline{1}} + \frac{i}{k}\nu_{(1)} \right] = -\frac{2\sqrt{2}}{k} \left[\xi_{\underline{A}\underline{2}}^- q_{(1)}^{\bar{A}} \bar{\psi}_{(1)-}^2 - \epsilon^{\bar{A}\bar{B}} \xi_{\underline{B}\underline{1}}^+ \psi_{(1)\underline{2}+} \bar{q}_{(1)\bar{A}} \right], \quad (2.9b)$$

respectively.

In each case we find a non-vanishing variation, which we will now show may be packaged into supercovariant derivatives. The fermionic couplings are $\bar{\psi}_{(1)+}^1$ and $\psi_{(1)\underline{1}-}$ for the first sign choice, and $\bar{\psi}_{(1)-}^2$ and $\psi_{(1)\underline{2}+}$ in the second case. We refer in the following to the two respective loops as “ ψ_1 -loops” and “ ψ_2 -loops”.

ψ_1 -loop

Following [27, 28], we introduce a $U(N_1|N_2)$ super-connection whose top left block is given by the bosonic connection appearing in (2.9a). On dimensional and Graßmann odd/even grounds, it can be seen that the (1,2) and (2,1) component of the superconnection will be of the form, $\bar{c}_a^A \psi_{(1)\underline{A}}^a$ and $c_{\underline{A}}^a \bar{\psi}_{(1)a}^A$, where the spinor couplings $\bar{c}_a^A$ and $c_{\underline{A}}^a$ are Graßmann even.

The supersymmetry condition (2.2) imposes that the variation of the fermionic entries be given by the covariant derivative of some bosonic field; with the covariant derivative defined with respect to the bosonic connections appearing in the diagonal blocks of $\mathcal{L}$. By assumption, we take the connection appearing in (2.9a) as the first block on the diagonal; we fix the second block by studying the variation of the fermionic entries of $\mathcal{L}$.

In order for the covariant derivative in the variation of the fermions to be projected along the x^1 direction, we fix $c_{\underline{1}}^- = c_{\underline{2}}^+ = \bar{c}_{\underline{1}}^+ = \bar{c}_{\underline{2}}^+ = 0$. Motivated by the absence of ψ_2 and $\bar{\psi}^2$ in (2.9a), we further assume that $c_{\underline{2}}^- = \bar{c}_{\underline{2}}^- = 0$. We thus write the superconnection as

$$\mathcal{L}^{\psi_1} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\underline{1}} - \frac{i}{k}\nu_{(1)} & \bar{c} \psi_{(1)\underline{1}}^+ \\ c \bar{\psi}_{(1)+}^1 & \star \end{pmatrix}, \quad (2.10)$$

where c , $\bar{c}$ and ‘ $\star$ ’ remain to be fixed.

As the variation of the bosonic connection contains no derivatives, the supermatrix

G of (2.2) is necessarily of the form

$$G^{\psi_1} = \begin{pmatrix} 0 & g \\ \bar{g} & \star \end{pmatrix}. \quad (2.11)$$

We fix g , $\bar{g}$ and the ‘ $\star$ ’s by examining the variation of the fermions. We have

$$\begin{aligned} \delta\psi_{(1)\underline{1}}^+ &= \sqrt{2}(\not{D})^+_{+q_{(1)}^{\bar{A}}} \xi_{\bar{A}\underline{1}}^+ - \frac{\sqrt{2}}{k} \xi_{\bar{A}\underline{1}}^+ \left(\nu_{(1)} q_{(1)}^{\bar{A}} - q_{(1)}^{\bar{A}} \tilde{\nu}_{(1)} \right) \\ &\quad - \frac{2\sqrt{2}}{k} \xi_{\bar{A}\underline{1}}^+ (\tilde{\mu}_{(0)})^{\underline{1}}_{\underline{1}} q_{(1)}^{\bar{A}} + \frac{2\sqrt{2}}{k} \xi_{\bar{A}\underline{1}}^+ q_{(1)}^{\bar{A}} (\mu_{(2)})^{\underline{1}}_{\underline{1}}, \end{aligned} \quad (2.12)$$

$$\begin{aligned} \delta\bar{\psi}_{(1)+}^1 &= \sqrt{2}(\not{D})_{+q_{(1)}^{\bar{A}}} \epsilon^{\bar{A}\bar{B}} \epsilon^{12} \xi_{\bar{B}\underline{2}}^- + \frac{\sqrt{2}}{k} \epsilon^{\bar{A}\bar{B}} \epsilon^{12} \xi_{\bar{B}\underline{2}+} \left[\tilde{\nu}_{(1)} \bar{q}_{(1)\bar{A}} - \bar{q}_{(1)\bar{A}} \nu_{(1)} \right] \\ &\quad - \frac{2\sqrt{2}}{k} \epsilon^{\bar{A}\bar{B}} \epsilon^{12} \xi_{\bar{B}\underline{2}+} \bar{q}_{(1)\bar{A}} (\tilde{\mu}_{(0)})^{\underline{1}}_{\underline{1}} + \frac{2\sqrt{2}}{k} \epsilon^{\bar{A}\bar{B}} \epsilon^{12} \xi_{\bar{B}\underline{2}+} (\mu_{(2)})^{\underline{1}}_{\underline{1}} \bar{q}_{(1)\bar{A}}. \end{aligned} \quad (2.13)$$

Thus, we may write

$$\bar{c} \delta\psi_{(1)\underline{1}}^+ = \mathcal{D}_s g, \quad (2.14)$$

with

$$\begin{aligned} g &:= \sqrt{2} \bar{c} q_{(1)}^{\bar{A}} (\xi_{\bar{A}\underline{1}})^+, \\ \mathcal{D}_s g &:= D_s g + i \left(\frac{i}{k} \nu_{(1)} + \frac{2i}{k} (\tilde{\mu}_{(0)})^{\underline{1}}_{\underline{1}} \right) g - i g \left(\frac{i}{k} \tilde{\nu}_{(1)} + \frac{2i}{k} (\mu_{(2)})^{\underline{1}}_{\underline{1}} \right), \\ D_s g &:= \partial_s g - i A_{(1)1} g + i g A_{(2)1}. \end{aligned} \quad (2.15)$$

Firstly, this is consistent with the choice of bosonic connection in (2.10) in the context of (2.2). Secondly, (2.2) imposes that the bottom right block of $\mathcal{L}^{\psi_1}$ be $A_{(2)} - \frac{2i}{k} (\mu_{(2)})^{\underline{1}}_{\underline{1}} - \frac{i}{k} \tilde{\nu}_{(1)}$. It follows that the bottom right block of G^{ψ_2} vanishes. Similarly, studying the variation of $\bar{\psi}_{(1)+}^1$ fixes

$$\bar{g} := -\sqrt{2} c \bar{q}_{(1)}^{\bar{A}} \epsilon^{\bar{A}\bar{B}} \xi_{\bar{B}\underline{2}+}, \quad (2.16)$$

and is consistent with the choices of bosonic connections.

It remains to check the variations of the bosonic connections satisfy the conditions coming from (2.2), *i.e.*

$$\delta \left[A_{(1)1} - \frac{2i}{k} (\tilde{\mu}_{(0)})^{\underline{1}}_{\underline{1}} - \frac{i}{k} \nu_{(1)} \right] = -i \bar{c} (\psi_{(1)\underline{1}})^+ \bar{g} + i g c (\bar{\psi}_{(1)}^1)_+, \quad (2.17a)$$

$$\delta \left[A_{(2)1} - \frac{2i}{k} (\mu_{(2)})^{\underline{1}}_{\underline{1}} - \frac{i}{k} \tilde{\nu}_{(1)} \right] = -i c (\bar{\psi}_{(1)}^1)_+ g + i \bar{g} \bar{c} (\psi_{(1)\underline{1}})^+. \quad (2.17b)$$

A simple calculation shows that these conditions hold, provided $c\bar{c} = -\frac{2i}{k}$. The value of c itself is immaterial, as it always will appear in the combination $c\bar{c}$ in the trace of the superconnection. We choose therefore $c = \bar{c} = \frac{1-i}{\sqrt{k}}$. In summary, we find that

the loop with super-connection

$$\mathcal{L}^{\psi_1} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})_{\underline{1}} - \frac{i}{k}\nu_{(1)} & \frac{1-i}{\sqrt{k}}\psi_{(1)\underline{1}}^+ \\ \frac{1-i}{\sqrt{k}}\bar{\psi}_{(1)+}^{\underline{1}} & A_{(2)1} - \frac{2i}{k}(\mu_{(2)})_{\underline{1}} - \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix} \quad (2.18)$$

preserves the supersymmetries (2.6).

ψ_2 -loop

We now consider the second modified bosonic connection (2.9b) $A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})_{\underline{1}} + \frac{i}{k}\nu_{(1)}$ and proceed similarly to the previous section. We find that the super-connection of the ψ_2 -loop is given by

$$\mathcal{L}^{\psi_2} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})_{\underline{1}} + \frac{i}{k}\nu_{(1)} & \frac{1-i}{\sqrt{k}}\psi_{(1)\underline{2}}^- \\ \frac{1-i}{\sqrt{k}}\bar{\psi}_{(1)-}^{\underline{2}} & A_{(2)1} - \frac{2i}{k}(\mu_{(2)})_{\underline{1}} + \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}. \quad (2.19)$$

Examining

$$\begin{aligned} \delta\psi_{(1)\underline{2}}^- &= \sqrt{2}(\not{D})^- - q_{(1)}^{\bar{A}}\xi_{\bar{A}\underline{2}}^- - \frac{\sqrt{2}}{k}\xi_{\bar{A}\underline{2}}^- \left(\nu_{(1)}q_{(1)}^{\bar{A}} - q_{(1)}^{\bar{A}}\tilde{\nu}_{(1)} \right) \\ &\quad - \frac{2\sqrt{2}}{k}\xi_{\bar{A}\underline{2}}^- (\tilde{\mu}_{(0)})_{\underline{2}}^{\bar{A}} q_{(1)}^{\bar{A}} + \frac{2\sqrt{2}}{k}\xi_{\bar{A}\underline{2}}^- q_{(1)}^{\bar{A}} (\mu_{(2)})_{\underline{2}}^{\bar{A}}, \end{aligned} \quad (2.20)$$

and the variation of the bosonic connections we find that the supermatrix appearing in the covariant derivative (2.2) is

$$G^{\psi_2} := \begin{pmatrix} 0 & -\frac{\sqrt{2}(1-i)}{\sqrt{k}}q_{(1)}^{\bar{A}}\xi_{\bar{A}\underline{2}}^- \\ -\frac{\sqrt{2}(1-i)}{\sqrt{k}}\bar{q}_{(1)\bar{A}}\epsilon^{\bar{A}\bar{B}}\xi_{\bar{B}\underline{1}}^- & 0 \end{pmatrix}. \quad (2.21)$$

In summary, we find a pair of superconnections preserving the supersymmetries (2.6) for each hypermultiplet in a quiver connecting nodes with CS levels k to $-k$. From either we may construct a Wilson loop in an arbitrary representation of the supergroup $U(N_1|N_2)$. This is twice the number one would expect; as discussed in Section 1.1.3, this degeneracy has been shown to be lifted at three loop order [41].

2.1.2 Linear quivers

Before proceeding to the case of quivers with nodes with vanishing CS levels, let us comment on linear quivers. We may construct any linear quiver by starting from a circular quiver and removing a hypermultiplet or a twisted hypermultiplet (or a pair and the intermediate vector multiplet). The corresponding loops are obtained from those constructed above by erasing the coupling to the fields of the removed multiplets.

Consider the segment of the quiver in Figure 2.1. The loops we constructed couple to scalars and fermions of the hypermultiplet between the nodes and to the scalars of the two adjacent twisted hypermultiplets.

Consider removing the twisted hypermultiplet to the right of the second node. Erasing it from the ψ_1 -loop (2.18) gives

$$\mathcal{L}^{\psi_1} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\perp} - \frac{i}{k}\nu_{(1)} & \frac{1-i}{\sqrt{k}}\psi_{(1)\perp}^+ \\ \frac{1-i}{\sqrt{k}}\bar{\psi}_{(1)+}^1 & A_{(2)1} - \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}, \quad (2.22)$$

and it is easy to check that this is indeed 1/2 BPS, with the appropriate supersymmetry transformations of the linear quiver in Appendix A. Similarly, the ψ_2 -loop (2.19) becomes

$$\mathcal{L}^{\psi_2} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\perp} + \frac{i}{k}\nu_{(1)} & \frac{1-i}{\sqrt{k}}\psi_{(1)\perp}^- \\ \frac{1-i}{\sqrt{k}}\bar{\psi}_{(1)-}^2 & A_{(2)1} + \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}. \quad (2.23)$$

Likewise we can remove the twisted hypermultiplet from the left, which will remove the coupling to the $\tilde{\mu}_{(0)}$ moment map from the above.

If we instead remove the hypermultiplet connecting the N_1 and N_2 nodes we expect to lose the structure of the superconnection. In that case we find two independent 1/2 BPS Wilson loops, each with a gauge connection and a coupling to the twisted moment maps

$$\mathcal{L}_{(1)} = A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\perp}, \quad \mathcal{L}_{(2)} = A_{(2)1} - \frac{2i}{k}(\mu_{(2)})^1_{\perp}. \quad (2.24)$$

Note that both ψ_1 - and ψ_2 -loops give the same loops via this process, so we lose the two-fold degeneracy. These loops are the same as the usual 1/4 BPS loops one gets from a modified connection including the scalar φ of the vector multiplet. In this formulation this scalar has been integrated out giving rise to the coupling to the moment maps $\tilde{\mu}_{(0)}$ and $\mu_{(2)}$. Those loops are 1/2 BPS in theories with $\mathcal{N} = 2$ supersymmetry and in this particular case get enhanced to preserve four supercharges, rather than two.

A similar story holds when considering linear quivers that end on nodes with vanishing CS levels. The case with vanishing levels inside a quiver are studied in the following two sections. Again, one can remove a hypermultiplet or twisted hypermultiplet to open up the quiver and the Wilson loops with those fields removed would remain 1/2 BPS. We do not list all the examples or repeat the algebra for all of them.

2.1.3 Vanishing CS levels: repeated untwisted hypermultiplets

We have thus far considered Chern-Simons matter quivers with alternating $\pm k$ Chern-Simons levels. Here we study the Wilson loops in the case with extra nodes of vanishing CS levels [35, 37] that preserve $\mathcal{N} = 4$ supersymmetry.

Let us consider the quiver with a node with vanishing CS coupling to a *pair* of untwisted hypermultiplets. We have removed for convenience the twisted hypermul-

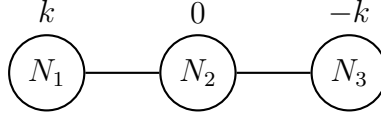


Figure 2.2: $\mathcal{N} = 4$ quiver segment with vanishing CS levels

tiplets from the left and the right, but they may be incorporated by adding the moment maps to the gauge connection.

We start by mimicking the construction of superconnections in the alternating levels case and find several different possible connections involving pairs of nodes. Then in Section 2.1.3 we find a new structure of a superconnection which couples to all the different fields of the *three* nodes. We expect this structure to be more fundamental than the ones involving only pairs of nodes.

ψ_1 -loop for the first link

We begin by constructing the ψ_1 -loop for the first link. The analysis proceeds as in Section 2.1.1: We take the same connection for the first node, since it couples to the same fields as previously. We then continue to study the fermions and the connection on the second node.

Our starting point is³

$$\mathcal{L}_{(1)}^{\psi_1} = \begin{pmatrix} A_{(1)1} - \frac{i}{k}\nu_{(1)} & \bar{c}\psi_{(1)1}^+ \\ c\bar{\psi}_{(1)+}^1 & \star \end{pmatrix}, \quad (2.25)$$

where it remains to fix ‘ $\star$ ’ and the fermionic couplings c and $\bar{c}$.

We fix the form of the bosonic connection of the second node by studying the

³ The ‘(1)’ subscript for $\mathcal{L}$ indicates that we are attempting to construct a $U(N_1|N_2)$ valued connection.

variation of the fermionic terms

$$\begin{aligned}\delta\psi_{(1)\underline{1}}^+ &= \sqrt{2}D_s q_{(1)}^{\bar{A}} \xi_{\bar{A}\underline{1}}^+ - \frac{\sqrt{2}}{k} \xi_{\bar{A}\underline{1}}^+ \left[\nu_{(1)} q_{(1)}^{\bar{A}} - q_{(1)}^{\bar{A}} \tilde{\nu}_{(1)} \right] \\ &\quad - \frac{2\sqrt{2}}{k} \xi_{\bar{A}\underline{1}}^+ (\tilde{\mu}_{(0)})_{\underline{1}} q_{(1)}^{\bar{A}} - \sqrt{2} \xi_{\bar{A}\underline{1}}^+ q_{(1)}^{\bar{A}} (\varphi_{(2)})_{\underline{1}}^{\underline{1}}.\end{aligned}\quad (2.26)$$

Imposing that the variation of the fermion is a total derivative fixes ‘ $\star$ ’ to be $A_{(2)1} + i(\varphi_{(2)})_{\underline{1}}^{\underline{1}} - \frac{i}{k} \tilde{\nu}_{(1)}$. Furthermore, it fixes

$$g_{(1)} = \sqrt{2} \bar{c} q_{(1)}^{\bar{A}} \xi_{\bar{A}\underline{1}}^+. \quad (2.27)$$

It is also in agreement with the bosonic connection of the first node given above. The variation of the conjugate fermion is consistent with the above bosonic connections and fixes

$$\bar{g}_{(1)} = -\sqrt{2} c \bar{q}_{(1)\bar{A}} \epsilon^{\bar{A}\bar{B}} \xi_{\bar{B}\underline{2}+}. \quad (2.28)$$

This suggests that the form of the supermatrix $G_{(1)}$ is identical to that of the alternating level case.

Combining the above results, we see that if we take

$$\mathcal{L}_{(1)}^{\psi_1} = \begin{pmatrix} A_{(1)1} - \frac{i}{k} \nu_{(1)} & \frac{1-i}{\sqrt{k}} \psi_{(1)\underline{1}}^+ \\ \frac{1-i}{\sqrt{k}} \bar{\psi}_{(1)+}^{\underline{1}} & A_{(2)1} + i(\varphi_{(2)})_{\underline{1}}^{\underline{1}} - \frac{i}{k} \tilde{\nu}_{(1)} \end{pmatrix}, \quad (2.29)$$

then the variation of all but the bottom right block are total covariant superderivatives. The variation of that block is

$$\begin{aligned}\delta \left[A_{(2)1} + i(\varphi_{(2)})_{\underline{1}}^{\underline{1}} - \frac{i}{k} \tilde{\nu}_{(1)} \right] &= \frac{2\sqrt{2}}{k} \left[\xi_{\bar{A}\underline{1}}^+ \bar{\psi}_{(1)+}^{\underline{1}} q_{(1)}^{\bar{A}} + \xi^{\bar{A}\underline{1}-} \bar{q}_{(1)\bar{A}} \psi_{(1)\underline{1}-} \right] \\ &\quad + \frac{1}{2k} \left(\xi_{\bar{A}\underline{1}}^+ \left[j_{(2)+}^{\bar{A}\underline{1}} - \tilde{j}_{(1)+}^{\bar{A}\underline{1}} \right] - \xi_{\bar{A}\underline{2}}^- \left[j_{(2)-}^{\bar{A}\underline{2}} - \tilde{j}_{(1)-}^{\bar{A}\underline{2}} \right] \right).\end{aligned}\quad (2.30)$$

The first two terms are exactly as for the variation of the bosonic connection of the second node in the alternating levels case. Thus we can write

$$\begin{aligned}\delta \mathcal{L}_{(1)}^{\psi_1} &\equiv \mathfrak{D}_{(1)s}^{\psi_1} G_{(1)}^{\psi_1} + \Delta J \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ \Delta J &= \frac{1}{2k} \left(\xi_{\bar{A}\underline{1}}^+ \left[j_{(2)+}^{\bar{A}\underline{1}} - \tilde{j}_{(1)+}^{\bar{A}\underline{1}} \right] - \xi_{\bar{A}\underline{2}}^- \left[j_{(2)-}^{\bar{A}\underline{2}} - \tilde{j}_{(1)-}^{\bar{A}\underline{2}} \right] \right),\end{aligned}\quad (2.31)$$

where $\mathfrak{D}_{(1)s}^{\psi_1}$ is a supercovariant derivative with respect the superconnection $\mathcal{L}_{(1)}^{\psi_1}$.

The variation of this superconnection is neither vanishing nor a total supercovariant derivative. Rather it is a supercovariant derivative plus a remainder term. We shall come back to discuss this ΔJ remainder term in Section 2.1.3. For now we examine more possibilities for superconnections.

ψ_2 -loop for the first link

We can also take the superconnection

$$\mathcal{L}_{(1)}^{\psi_2} = \begin{pmatrix} A_{(1)1} + \frac{i}{k}\nu_{(1)} & \bar{c}\psi_{(1)2}^- \\ c\bar{\psi}_{(1)-}^2 & A_{(2)1} + i(\varphi_{(2)})_{\underline{1}} + \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}, \quad (2.32)$$

and find

$$\delta\mathcal{L}_{(1)}^{\psi_2} \equiv \mathfrak{D}_{(1)s}^{\psi_2} G_{(1)}^{\psi_2} + \Delta J \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.33)$$

where $G_{(1)}^{\psi_2}$ is the supermatrix for the alternating level case.

Superconnections in the second link

We can also consider superconnections involving the vector fields $A_{(2)}$ and $A_{(3)}$ and the hypermultiplet connecting them. By reflection it is clear that the resulting loops would have superconnections

$$\mathcal{L}_{(1)}^{\psi_1} = \begin{pmatrix} A_{(2)1} + i(\varphi_{(2)})_{\underline{1}} - \frac{i}{k}\nu_{(2)} & \frac{1-i}{\sqrt{k}}\psi_{(2)\underline{1}}^+ \\ \frac{1-i}{\sqrt{k}}\bar{\psi}_{(2)+}^1 & A_{(3)1} - \frac{i}{k}\tilde{\nu}_{(2)} \end{pmatrix}, \quad (2.34)$$

satisfying

$$\delta\mathcal{L}_{(2)}^{\psi_1} = \mathfrak{D}_{(2)s}^{\psi_1} G_{(2)}^{\psi_1} - \Delta J \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.35)$$

where $G_{(2)}^{\psi_1}$ is the supermatrix found in the alternating level case.

Likewise

$$\mathcal{L}_{(2)}^{\psi_2} = \begin{pmatrix} A_{(2)1} + i(\varphi_{(2)})_{\underline{1}} + \frac{i}{k}\nu_{(2)} & \frac{1-i}{\sqrt{k}}\psi_{(2)\underline{2}}^- \\ \frac{1-i}{\sqrt{k}}\bar{\psi}_{(2)-}^2 & A_{(3)1} + \frac{i}{k}\tilde{\nu}_{(2)} \end{pmatrix}, \quad (2.36)$$

satisfies

$$\delta\mathcal{L}_{(2)}^{\psi_2} = \mathfrak{D}_{(2)s}^{\psi_2} G_{(2)}^{\psi_2} - \Delta J \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.37)$$

where $G_{(2)}^{\psi_2}$ is the supermatrix found in the alternating level case.

A 4×4 construction

Thus far we have found four superconnections involving pairs of vector fields and the relevant hypermultiplets. We may construct from them Wilson loops in arbitrary representations of $U(N_1|N_2)$ and $U(N_2|N_3)$. We may also combine pairs of them together, to form block diagonal superconnections with a 4×4 block structure, with the central node represented twice. Those will allow to write Wilson loops in

arbitrary representations of $U(N_1 + N_3 | 2N_2)$.

In fact, this 4×4 ansatz allows for a more general superconnection, which is not block diagonal. In addition to bosonic entries transforming in the adjoints of the individual nodes, and fermions in the bifundamentals of adjacent nodes, we have two bosonic entries transforming in bifundamentals of $U(N_1) \times U(N_3)$. On dimensional and representation grounds, they have to take the form $\bar{d}_{\bar{A}\bar{B}} q_{(1)}^{\bar{A}} q_{(2)}^{\bar{B}}$ and $d^{\bar{A}\bar{B}} \bar{q}_{(2)\bar{A}} q_{(1)\bar{B}}$, respectively. We consider the antisymmetric couplings $\bar{d}_{\bar{A}\bar{B}} = \bar{d}\epsilon_{\bar{A}\bar{B}}$ and $d^{\bar{A}\bar{B}} = d\epsilon^{\bar{A}\bar{B}}$.

Starting from a general ansatz (with the above assumed form of the couplings) we have been able to show that the structure of the superconnection has to be either the block diagonal ones made of pairs of superconnections discussed above or the superconnection

$$\mathcal{L} = \begin{pmatrix} A_{(1)1} & \bar{c}_{(1)}^1 \psi_{(1)\underline{1}}^+ & \bar{c}_{(1)}^2 \psi_{(1)\underline{2}}^- & \bar{d}\epsilon_{\bar{A}\bar{B}} q_{(1)}^{\bar{A}} q_{(2)}^{\bar{B}} \\ c_{(1)\underline{1}} \bar{\psi}_{(1)+}^1 & \mathcal{A}_{(2)} & 0 & \bar{c}_{(2)}^2 \psi_{(2)\underline{2}}^- \\ c_{(1)\underline{2}} \bar{\psi}_{(1)-}^2 & 0 & \mathcal{A}'_{(2)} & \bar{c}_{(2)}^1 \psi_{(2)\underline{1}}^+ \\ d\epsilon^{\bar{A}\bar{B}} \bar{q}_{(2)\bar{A}} q_{(1)\bar{B}} & c_{(2)\underline{2}} \bar{\psi}_{(2)-}^2 & c_{(2)\underline{1}} \bar{\psi}_{(2)+}^1 & A_{(3)1} \end{pmatrix}, \quad (2.38)$$

with $\mathcal{A}_{(2)} := A_{(2)1} + i(\varphi_{(2)})_{\underline{1}}^1 + \frac{i}{2k}(\nu_{(2)} - \tilde{\nu}_{(1)})$ and $\mathcal{A}'_{(2)} := A_{(2)1} + i(\varphi_{(2)})_{\underline{1}}^1 - \frac{i}{2k}(\nu_{(2)} - \tilde{\nu}_{(1)})$. The connection in the top left and bottom right corners involve only the gauge fields (and one could include the appropriate moment maps, if coupling to extra twisted hypermultiplets). It is rather interesting that in the case of alternating levels we had to augment all gauge connections by a coupling to ν with a coefficient $\pm i/k$. Indeed that is also what we found in the loops associated to the first and second link. But here we have the third option of not including this coupling for the external nodes.⁴

The supermatrix G is given by

$$G = \begin{pmatrix} 0 & g_{(1)\underline{1}} & g_{(1)\underline{2}} & 0 \\ \bar{g}_{(1)}^1 & 0 & 0 & g_{(2)\underline{2}} \\ \bar{g}_{(1)}^2 & 0 & 0 & g_{(2)\underline{1}} \\ 0 & \bar{g}_{(2)}^2 & \bar{g}_{(2)}^1 & 0 \end{pmatrix}, \quad (2.39)$$

where

$$\begin{aligned} g_{(1)\underline{1}} &= \sqrt{2}\bar{c}_{(1)}^1 q_{(1)}^{\bar{A}} \xi_{\bar{A}\underline{1}}^+, & g_{(1)\underline{2}} &= -\sqrt{2}\bar{c}_{(1)}^2 q_{(1)}^{\bar{A}} \xi_{\bar{A}\underline{2}}^-, \\ g_{(2)\underline{1}} &= \sqrt{2}\bar{c}_{(2)}^1 q_{(2)}^{\bar{A}} \xi_{\bar{A}\underline{1}}^+, & g_{(2)\underline{2}} &= -\sqrt{2}\bar{c}_{(2)}^2 q_{(2)}^{\bar{A}} \xi_{\bar{A}\underline{2}}^-, \\ \bar{g}_{(1)}^1 &= -\sqrt{2}c_{(1)\underline{1}} \bar{q}_{(1)\bar{A}} \epsilon^{\bar{A}\bar{B}} \epsilon^{12} \xi_{\bar{B}\underline{2}+}, & \bar{g}_{(1)}^2 &= \sqrt{2}c_{(1)\underline{2}} \bar{q}_{(1)\bar{A}} \epsilon^{\bar{A}\bar{B}} \epsilon^{21} \xi_{\bar{B}\underline{1}-}, \\ \bar{g}_{(2)}^1 &= -\sqrt{2}c_{(2)\underline{1}} \bar{q}_{(2)\bar{A}} \epsilon^{\bar{A}\bar{B}} \epsilon^{12} \xi_{\bar{B}\underline{2}+}, & \bar{g}_{(2)}^2 &= \sqrt{2}c_{(2)\underline{2}} \bar{q}_{(2)\bar{A}} \epsilon^{\bar{A}\bar{B}} \epsilon^{21} \xi_{\bar{B}\underline{1}-}. \end{aligned} \quad (2.40)$$

⁴We were not able to find BPS Wilson loops with any other values for the coupling to ν .

We firstly consider the variations of the bosonic components, listing below the corresponding conditions which must be imposed for supersymmetry.

$$\begin{aligned}
11 \text{ entry: } & c_{(1)}^1 \bar{c}_{(1)\underline{1}} = c_{(1)}^2 \bar{c}_{(1)\underline{2}} = -\frac{i}{k}, \\
44 \text{ entry: } & c_{(2)}^1 \bar{c}_{(2)\underline{1}} = c_{(2)}^2 \bar{c}_{(2)\underline{2}} = -\frac{i}{k}, \\
14 \text{ entry: } & -\bar{c}_{(1)}^1 \bar{c}_{(2)}^2 = \bar{c}_{(1)}^2 \bar{c}_{(2)}^1 = \bar{d}, \\
41 \text{ entry: } & -c_{(1)\underline{1}} c_{(2)\underline{2}} = c_{(1)\underline{2}} c_{(2)\underline{1}} = d.
\end{aligned} \tag{2.41}$$

The supersymmetry conditions corresponding to the 22 and 33 entries are then automatically satisfied.

For the central 23 and 32 entries in (2.38) the difference between the supersymmetry variation and the super-covariant derivative of G gives

$$\begin{aligned}
(\delta\mathcal{L} - \mathfrak{D}_s G)_{23} &= -i \left(c_{(1)\underline{1}} \bar{c}_{(1)}^2 j_{(1)+}^{\bar{A}1} + c_{(2)\underline{1}} \bar{c}_{(2)}^2 j_{(2)+}^{\bar{A}1} \right) \xi_{\bar{A}2}^-, \\
(\delta\mathcal{L} - \mathfrak{D}_s G)_{32} &= i \left(c_{(1)\underline{2}} \bar{c}_{(1)}^1 j_{(1)-}^{\bar{A}2} + c_{(2)\underline{2}} \bar{c}_{(2)}^1 j_{(2)-}^{\bar{A}2} \right) \xi_{\bar{A}\underline{1}}^+,
\end{aligned} \tag{2.42}$$

where $\mathfrak{D}_s$ is the super-covariant derivative with respect to the super-connection $\mathcal{L}$. Imposing the bilinear constraints (2.41), we see that $(\delta\mathcal{L} - \mathfrak{D}_s G)_{23}$ and $(\delta\mathcal{L} - \mathfrak{D}_s G)_{32}$ are proportional to $j_{(2)+}^{\bar{A}1} - \tilde{j}_{(1)+}^{\bar{A}1}$ and $j_{(2)-}^{\bar{A}2} - \tilde{j}_{(1)-}^{\bar{A}2}$, respectively. This is similar to the 2×2 cases discussed above.

With the constraints (2.41), the supersymmetry conditions for the fermionic entries of (2.38) are satisfied, up to a remainder term, which is proportional to $\Delta\mu \equiv (\mu_{(2)})^{\bar{A}}_{\bar{B}} - (\tilde{\mu}_{(1)})^{\bar{A}}_{\bar{B}}$. We present the example of the 12 entry. Consider

$$(\mathfrak{D}_s G)_{12} = \partial_s g_{(1)\underline{1}} - i A_{(1)1} g_{(1)\underline{1}} + i g_{(1)\underline{1}} \mathcal{A}_{(2)} - i \bar{d} \epsilon_{\bar{A}\bar{B}} q_{(1)}^{\bar{A}} q_{(2)}^{\bar{B}} \bar{g}_{(2)}^2. \tag{2.43}$$

With some manipulation it may be seen that

$$\begin{aligned}
\epsilon_{\bar{A}\bar{B}} q_{(1)}^{\bar{A}} q_{(2)}^{\bar{B}} \bar{g}_{(2)}^2 &= \sqrt{2} c_{(2)\underline{2}} \left(\nu_{(1)} q_{(1)}^{\bar{A}} \xi_{\bar{A}\underline{1}}^+ - \frac{1}{2} q_{(1)}^{\bar{A}} (\tilde{\nu}_{(1)} + \nu_{(2)}) \xi_{\bar{A}\underline{1}}^+ \right. \\
&\quad \left. - \epsilon_{\bar{A}\bar{B}} q_{(1)}^{\bar{A}} \left((\mu_{(2)})^{\bar{B}}_{\bar{C}} - (\tilde{\mu}_{(1)})^{\bar{B}}_{\bar{C}} \right) \epsilon^{\bar{C}\bar{D}} \xi_{\bar{D}\underline{1}-} \right).
\end{aligned} \tag{2.44}$$

This identity, coupled with (2.41) gives us that

$$(\delta\mathcal{L} - \mathfrak{D}_s G)_{12} = -i \bar{d} \epsilon_{\bar{A}\bar{B}} q_{(1)}^{\bar{A}} \left((\mu_{(2)})^{\bar{B}}_{\bar{C}} - (\tilde{\mu}_{(1)})^{\bar{B}}_{\bar{C}} \right) \epsilon^{\bar{C}\bar{D}} \xi_{\bar{D}\underline{1}-}. \tag{2.45}$$

Indeed all fermionic entries have such a remainder term, proportional to $(\mu_{(2)})^{\bar{B}}_{\bar{C}} - (\tilde{\mu}_{(1)})^{\bar{B}}_{\bar{C}}$. In summary, we find that the variation of $\mathcal{L}$ is given by a supercovariant derivative plus remainder terms proportional to ΔJ and $\Delta\mu$. We discuss these remainder terms in the next section.

We have also tried to construct Wilson loops with 3×3 and 6×6 block structure,

but could not find any useful ones. We expect then that the Wilson loops will be given by this connection, or that made of a pair of the previous 2×2 connections, or a linear combination thereof. In any case the Wilson loops will be classified by representations of $U(N_1 + N_3 | 2N_2)$.

The story for longer segments of the quiver with vanishing CS levels should be similar. The even entries will be along the diagonal or in the bifundamental of $U(N_I) \times U(N_{I+2})$ and made of a bilinear of the scalars. In the 4-node case the resulting connection would be a $U(N_1 + 2N_3 | 2N_2 + N_4)$ matrix, and the generalization to longer quivers is obvious.

On ΔJ and $\Delta\mu$

In the alternating level case we found a pair of superconnections whose variation is a total derivative, mimicking the previously known case of ABJ(M) theory as outlined in Section 1.1. In the case with a single $k = 0$ node we found five possible superconnections but in all cases we found an extra term in the variation proportional to ΔJ (2.31). One can add more hypermultiplets and more nodes with vanishing CS levels and the same construction leads to superconnections whose variation includes terms proportional to ΔJ terms on the different nodes with $k = 0$. For the construction with the 4×4 connection we also found the remainder $\Delta\mu$ term (2.45). In this case, the variation of the Wilson loop built out of any of the five superconnections will not vanish, but will rather give the insertion of the integral of ΔJ (and $\Delta\mu$ in the 4×4 case) into the Wilson loop.

Still, we expect there to be a BPS Wilson loop associated with this segment of the quiver. Indeed, examining the action [37] for the vector multiplets with $k = 0$ one notices that the gaugino λ appears only linearly and multiplying ΔJ . Thus the variation of the 2×2 Wilson loops are equivalent to the insertion of $\frac{\delta}{\delta\lambda}$ into the Wilson loop and the path integral over λ of any observable will be a total Grassmann derivative and will therefore vanish. We conclude that the variation of the Wilson loop is zero in the weak sense — all correlation functions with it vanish. BPS Wilson loops satisfying this condition, but not the stronger conditions discussed in Section 1.1, have not previously been found.

For the 4×4 superconnection, we also have a remainder term proportional to $\Delta\mu$ (2.45). Again the action of the vector multiplet includes this combination multiplied by an auxiliary field. Integrating it out identifies $(\mu_{(2)})^{\bar{B}}_{\bar{C}} = (\tilde{\mu}_{(1)})^{\bar{B}}_{\bar{C}}$, so this term does not obstruct the supersymmetry analysis, and the 4×4 loops are thus also supersymmetric in the weak sense.

2.1.4 Vanishing CS levels: repeated twisted hypermultiplets

One last simple case to consider is a segment of the quiver of the form

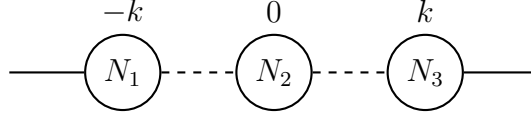


Figure 2.3: $\mathcal{N} = 4$ quiver segment with vanishing CS levels and repeated twisted hypermultiplets

the left and on the right will couple to extra nodes through the hypermultiplets (solid lines) and form superconnections with them (or if we remove the hypermultiplets and consider a linear quiver, will have 1/2 BPS connections like in (2.24)). In either case we can construct a 1/2 BPS Wilson loop coupling to the central node alone

$$\mathcal{L}_{(2)} = A_{(2)1} + i(\varphi_{(2)})^{\underline{1}}_{\underline{1}} - \frac{i}{k} ((\mu_{(2)})^{\underline{1}}_{\underline{1}} + (\tilde{\mu}_{(1)})^{\underline{1}}_{\underline{1}}). \quad (2.46)$$

2.2 The circle

For every straight BPS Wilson loop we expect to also find a circular one which will have a finite expectation value calculable using the localization matrix model. We study those here. The circle is given by

$$x^1 = \cos s, \quad x^2 = \sin s, \quad x^3 = 0. \quad (2.47)$$

Whereas the straight line preserved half of the Poincaré supersymmetries and half of the superconformal ones, we expect the circular loop to preserve eight linear combinations of the two. Concretely, we look for Wilson loops that preserve supersymmetries where the superconformal variation parameters η are related to the superpoincaré ones ξ by

$$\eta_{\underline{A}\underline{1}}^a = i(\sigma^2)^a_b \xi_{\underline{A}\underline{1}}^b, \quad \eta_{\underline{A}\underline{2}}^a = -i(\sigma^2)^a_b \xi_{\underline{A}\underline{2}}^b. \quad (2.48)$$

The superconformal transformations of the fields are given in the usual way [96] by replacing $(\xi_{\underline{A}\underline{B}})^a \rightarrow x^\mu (\gamma_\mu)^a_b (\eta_{\underline{A}\underline{B}})^b$, except for the variations of the fermions which pick up an extra term (see Appendix A).

2.2.1 Alternating levels

Let us consider the same segment as for the straight line case in Section 2.1.1, *i.e.* the segment in Figure 2.1. As in the straight line case, contribution to the variation

of the gauge field from the adjoining twisted hypermultiplet may be cancelled by considering the combination $A_{(1)1} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\underline{1}}$. Furthermore, we may consider the two combinations

$$\begin{aligned}
& \delta \left[\dot{x}^\mu A_{(1)\mu} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\underline{1}} - \frac{i}{k}\nu_{(1)} \right] \\
&= \frac{2\sqrt{2}}{k} \left\{ \left[(1 - \sin s)\xi_{\bar{A}1}^+ + \cos s \xi_{\bar{A}1}^- \right] q_{(1)}^{\bar{A}} \bar{\psi}_{(1)+}^1 \right. \\
&\quad + \left[\cos s \xi_{\bar{A}1}^+ + (1 + \sin s)\xi_{\bar{A}1}^- \right] q_{(1)}^{\bar{A}} \bar{\psi}_{(1)-}^1 \\
&\quad + \left[(1 + \sin s)\epsilon^{\bar{A}\bar{B}} \xi_{\bar{B}2}^+ - \cos s \epsilon^{\bar{A}\bar{B}} \xi_{\bar{B}2}^- \right] \psi_{(1)1+} \bar{q}_{(1)}^{\bar{A}} \\
&\quad \left. + \left[-\cos s \epsilon^{\bar{A}\bar{B}} \xi_{\bar{B}2}^+ + (1 - \sin s)\epsilon^{\bar{A}\bar{B}} \xi_{\bar{B}2}^- \right] \psi_{(1)1-} \bar{q}_{(1)}^{\bar{A}} \right\}, \tag{2.49}
\end{aligned}$$

and

$$\begin{aligned}
& \delta \left[\dot{x}^\mu A_{(1)\mu} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\underline{1}} + \frac{i}{k}\nu_{(1)} \right] \\
&= -\frac{2\sqrt{2}}{k} \left\{ \left[(1 + \sin s)\xi_{\bar{A}2}^+ - \cos s \xi_{\bar{A}2}^- \right] q_{(1)}^{\bar{A}} \bar{\psi}_{(1)+}^2 \right. \\
&\quad + \left[-\cos s \xi_{\bar{A}2}^+ + (1 - \sin s)\xi_{\bar{A}2}^- \right] q_{(1)}^{\bar{A}} \bar{\psi}_{(1)-}^2 \\
&\quad + \left[-(1 - \sin s)\epsilon^{\bar{A}\bar{B}} \xi_{\bar{B}1}^+ - \cos s \epsilon^{\bar{A}\bar{B}} \xi_{\bar{B}1}^- \right] \psi_{(1)2+} \bar{q}_{(1)}^{\bar{A}} \\
&\quad \left. + \left[-\cos s \epsilon^{\bar{A}\bar{B}} \xi_{\bar{B}1}^+ - (1 + \sin s)\epsilon^{\bar{A}\bar{B}} \xi_{\bar{B}1}^- \right] \psi_{(1)2-} \bar{q}_{(1)}^{\bar{A}} \right\}. \tag{2.50}
\end{aligned}$$

From the straight line case, we expect that these two choices correspond to the ψ_1 - and ψ_2 -loops respectively. We assume the form of the superconnections and verify that they are indeed supersymmetric.

ψ_1 -loop

We take the superconnection to be of the form

$$\mathcal{L}^{\psi_1} = \begin{pmatrix} \dot{x}^\mu A_{(1)\mu} - \frac{2i}{k}(\tilde{\mu}_{(0)})^1_{\underline{1}} - \frac{i}{k}\nu_{(1)} & \bar{c}_a \psi_{(1)1}^a \\ c^a \bar{\psi}_{(1)}^{1a} & \dot{x}^\mu A_{(2)\mu} - \frac{2i}{k}(\mu_{(2)})^1_{\underline{1}} - \frac{i}{k}\tilde{\nu}_{(1)} \end{pmatrix}, \tag{2.51}$$

and the supermatrix G to be of the form

$$G = \begin{pmatrix} 0 & g \\ \bar{g} & 0 \end{pmatrix}. \tag{2.52}$$

As in [27], we consider the projector

$$\mathcal{P}^+ := \delta^a_b + \dot{x}^\mu (\gamma_\mu)^a_b = \begin{pmatrix} 1 - \sin s & \cos s \\ \cos s & 1 + \sin s \end{pmatrix}, \tag{2.53}$$

and demand that the fermionic couplings be eigenstates of this projector. In particular, we choose

$$c^a = (\cos s, 1 + \sin s)^a c(s), \quad \bar{c}_a = \begin{pmatrix} 1 - \sin s \\ \cos s \end{pmatrix}_a \bar{c}(s), \quad (2.54)$$

where $c(s)$ and $\bar{c}(s)$ are functions of s . The variation is given by

$$\begin{aligned} \delta [\bar{c}_a \psi_{(1)\underline{1}}^a] &= 2\sqrt{2} D_s q^{\bar{A}} \bar{c}_a \xi_{\bar{A}\underline{1}}^a - \frac{2\sqrt{2}}{k} \left[(\nu_{(1)})^{\bar{B}} q_{(1)}^{\bar{A}} - q_{(1)}^{\bar{A}} (\tilde{\nu}_{(1)})^{\bar{B}} \right] \bar{c}_a \xi_{\bar{A}\underline{1}}^a \\ &\quad + i\sqrt{2} \bar{c}_a (\sigma^2)^a_b \xi_{\bar{A}\underline{1}}^b q_{(1)}^{\bar{A}}. \end{aligned} \quad (2.55)$$

We wish to write this as a covariant derivative. We may impose

$$\partial_s \bar{c}_a = \frac{i}{2} \bar{c}_b (\sigma^2)^b_a, \quad (2.56)$$

which tells us that

$$\bar{c}(s) = \frac{\bar{C}}{\cos \frac{s}{2} - \sin \frac{s}{2}}, \quad (2.57)$$

where $\bar{C}$ is a constant. With this choice of $\bar{c}(s)$, we have

$$\delta [\bar{c}_a (\psi_{(1)\underline{1}})^a] = \mathcal{D}_s \left(2\sqrt{2} q^{\bar{A}} \bar{c}_a \xi_{\bar{A}\underline{1}}^a \right), \quad (2.58)$$

where $\mathcal{D}_s q_{(1)}^{\bar{A}} \equiv D_s q^{\bar{A}} + \frac{1}{k} \nu_{(1)} q^{\bar{A}} - \frac{1}{k} q^{\bar{A}} \tilde{\nu}_{(1)}$. The supersymmetry conditions demand

$$g = 2\sqrt{2} q_{(1)}^{\bar{A}} \bar{c}_a \xi_{\bar{A}\underline{1}}^a. \quad (2.59)$$

Similarly, we find

$$\delta [c^a (\bar{\psi}_{(1)})_a] = \mathcal{D}_s \left(-2\sqrt{2} \bar{q}_{(1)\bar{A}} c^a \epsilon^{\bar{A}\bar{B}} \epsilon^{\underline{12}} \xi_{\bar{B}2a} \right), \quad (2.60)$$

where we have imposed that

$$\partial_s c^a = -\frac{i}{2} c^b (\sigma^2)^a_b, \quad (2.61)$$

which is solved by

$$c(s) = \frac{C}{\cos \frac{s}{2} + \sin \frac{s}{2}}, \quad (2.62)$$

where C is a constant. We thus have

$$\bar{g} = -2\sqrt{2} \bar{q}_{(1)\bar{A}} c^a \epsilon^{\bar{A}\bar{B}} \epsilon^{\underline{12}} \xi_{\bar{B}2a}. \quad (2.63)$$

A simple calculation, as in the straight line case in Section 2.1.3 gives that $C\bar{C} = -\frac{i}{k}$.

The story for the segment of the quiver with vanishing CS level is very similar,

where for example the circular analog of the straight line presented in Section 2.1.3 is the same as (2.51) with the lower right corner replaced by $\dot{x}^\mu A_{(2)\mu} + i(\varphi_{(2)})^\perp_{\underline{1}} - \frac{i}{k} \tilde{\nu}_{(1)}$, which is essentially the same as in (2.29). The variation of the super-connection is then

$$\delta \mathcal{L}^{\psi_1} \equiv \mathfrak{D}_s^{\psi_1} G + \Delta J \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.64)$$

with G as in (2.52) and ΔJ is as defined in (2.31). The other loops for this case may be similarly obtained in analogy with the straight line case.

ψ_2 -loop

To construct a loop coupling to ψ_2 , we take the fermionic couplings to be eigenstates of the projector $\mathcal{P}^- = \delta^a_b - \dot{x}^\mu (\gamma_\mu)^a_b$, given explicitly by

$$\mathcal{P}^- = \begin{pmatrix} 1 + \sin s & -\cos s \\ -\cos s & 1 - \sin s \end{pmatrix}. \quad (2.65)$$

We take the fermionic couplings to be

$$c^a = c(s) \begin{pmatrix} -\cos s & 1 - \sin s \end{pmatrix}^a, \quad \bar{c}_a = \bar{c}(s) \begin{pmatrix} 1 + \sin s \\ -\cos s \end{pmatrix}_a, \quad (2.66)$$

with the superconnection given by

$$\mathcal{L}^{\psi_2} = \begin{pmatrix} \dot{x}^\mu A_{(1)\mu} - \frac{2i}{k} (\tilde{\mu}_{(0)})^\perp_{\underline{1}} + \frac{i}{k} \nu_{(1)} & \bar{c}_a \psi_{(1)2}^- \\ c^a \bar{\psi}_{(1)-}^2 & \dot{x}^\mu A_{(2)\mu} - \frac{2i}{k} (\mu_{(2)})^\perp_{\underline{1}} + \frac{i}{k} \tilde{\nu}_{(1)} \end{pmatrix}. \quad (2.67)$$

Proceeding as in the previous section, studying the variations of the fermions gives us

$$\begin{aligned} g &= -2\sqrt{2} q^{\bar{A}} \bar{c}_a (\xi_{\bar{A}2})^a, & \bar{c}(s) &= \frac{\bar{C}}{\cos \frac{s}{2} + \sin \frac{s}{2}}, \\ \bar{g} &= 2\sqrt{2} \bar{q}_{\bar{A}} c^a (\xi^{\bar{A}2})_a, & c(s) &= \frac{C}{\cos \frac{s}{2} - \sin \frac{s}{2}}, \end{aligned} \quad (2.68)$$

as well as compatibility with the conjectured bosonic connection of the two nodes. An examination of the variation of the bosonic connections gives $C\bar{C} = \frac{i}{k}$.

The boundary conditions

We have shown that the supersymmetric variation of $\mathcal{L}$ may be written as a super-covariant derivative. In order for the circle, which is compact, to be invariant under supersymmetry we must ensure that there are no boundary terms when integrating this total derivative term. We follow the analysis in [28].

We define $\mathfrak{W}$ to be the untraced Wilson loop and analyze its transformation by the total derivative viewed as a super gauge transformation. Consider the matrix G (for either the ψ_1 and ψ_2 -loops) found above, then we find

$$G(2\pi) = -G(0). \quad (2.69)$$

This can be written as

$$G(2\pi) = \mathcal{T}G(0)\mathcal{T}^{-1}, \quad \text{with} \quad \mathcal{T} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (2.70)$$

This implies that the finite gauge transformation $U = \exp(iG)$ also satisfies

$$U(2\pi) = \mathcal{T}U(0)\mathcal{T}^{-1}. \quad (2.71)$$

Now we consider the transformation of $\mathfrak{WT}$. It is invariant under the supersymmetry/super-gauge transformations.

$$\text{str}(\mathfrak{WT}) \rightarrow \text{str}(U^{-1}(0)\mathfrak{W}U(2\pi)\mathcal{T}) = \text{str}(U^{-1}(0)\mathfrak{WT}U(0)) = \text{str}(\mathfrak{WT}). \quad (2.72)$$

We thus should consider the supertrace of $\mathfrak{WT}$ rather than of $\mathfrak{W}$. Equivalently, examining that form of $\mathcal{T}$, we see that the trace of $\mathfrak{W}$ (rather than the supertrace) is a supersymmetric operator. This is similar to the result for ABJM in [27].

2.3 Localization

We would now like to show that we can reduce the computation of the expectation value of the 1/2 BPS Wilson loops to a matrix model integral using localization. The idea, as in [27], is to show that the 1/2 BPS loops are cohomologically equivalent, under the localization supercharge, to certain combinations of 1/4 BPS loops for which the localization calculation has already been done [29]. This reduces the calculation of the Wilson loop to that of an observable in a matrix model, as in [30–32, 97].

Recall that localization modifies the path integral as follows:

$$Z[\Phi] = \int \mathcal{D}[\Phi] e^{-S[\Phi]} \rightarrow Z_t[\Phi] = \int \mathcal{D}[\Phi] e^{-S[\Phi] - tQV[\Phi]}, \quad (2.73)$$

where t is a continuous real parameter, Φ represents the fields of the theory, and Q is a supercharge with respect to which the action is invariant. If Q^2 is a bosonic symmetry of both the action and V , then

$$\frac{d}{dt} Z_t[\Phi] = 0. \quad (2.74)$$

This allows one to set $t \rightarrow \infty$ and consider the modified path integral Z_∞ ; from (2.74), this is equivalent to considering the unmodified path integral. The same procedure may be applied when computing the expectation value of an observable in the theory which is invariant with respect to the localizing supercharge above (such as the 1/4 or 1/2 BPS loops we are considering).

Now we show that the expectation value (with respect to the modified path integral) of a Q exact quantity vanishes. We have

$$\langle QW \rangle = \int \mathcal{D}[\Phi] QW e^{-S[\Phi] - tQV[\Phi]}. \quad (2.75)$$

As the action is invariant under Q and Q^2 is a bosonic symmetry of both the action and V , we may write (2.75) as

$$\langle QW \rangle = \int \mathcal{D}[\Phi] Q \left(W e^{-S[\Phi] - tQV[\Phi]} \right), \quad (2.76)$$

which we may interpret as a total derivative on the field space, and thus (assuming that there are no boundary terms)

$$\langle QW \rangle = 0. \quad (2.77)$$

Therefore, if we can show that the expectation value of the 1/2 BPS loops are cohomologically equivalent (*i.e.* their difference is Q exact) to the expectation value of a combination of 1/4 BPS Wilson loops (for which the resulting matrix model integral is already known [29]), then we may apply the results of [29] for the 1/4 BPS loops to the 1/2 BPS loops in order to obtain a matrix model integral for the expectation value of the 1/2 BPS loops. This is what we do in the rest of this section. We note that, in this thesis, we do not attempt to solve or indeed write down the explicit matrix model integral; we demonstrate the aforementioned cohomological equivalence and refer the reader to [29] for the form of the matrix model integral.

2.3.1 The straight line

We now demonstrate, closely following [27], that both the ψ_1 -loops and ψ_2 -loops are in the same cohomology class - that of a certain combination of 1/4 BPS straight loops. We note that this is a classical calculation; quantum corrections were considered in [40,41], where it was found that only a single linear combination of these loops will be exactly equivalent to the 1/4 BPS loop.

Considering the quiver segment of Section 2.1.1, the appropriate 1/4 BPS operator

has a purely bosonic connection from two adjacent nodes

$$\mathcal{L}^{1/4} = \begin{pmatrix} A_{(1)1} - \frac{2i}{k} ((\tilde{\mu}_{(0)})^1_{\underline{1}} + (\mu_{(1)})^{\bar{1}}_{\bar{1}}) & 0 \\ 0 & A_{(2)1} - \frac{2i}{k} ((\tilde{\mu}_{(1)})^{\bar{1}}_{\bar{1}} + (\mu_{(2)})^1_{\underline{1}}) \end{pmatrix}. \quad (2.78)$$

We note that this shares its four supersymmetries with our 1/2 BPS loops. As our localizing supercharge, we consider the linear combination

$$Q := Q_{-}^{\bar{1}2} + Q_{+}^{\bar{2}1}. \quad (2.79)$$

ψ_1 -loop

The difference between the ψ_1 -loop (2.18) found in Section 2.1.1 and the 1/4 BPS loop is given by

$$\mathcal{W}^{\psi_1} - \mathcal{W}^{1/4} = \text{tr } \vec{\mathcal{P}} \left[e^{-i \int ds \mathcal{L}^{\psi_1}} - e^{-i \int ds \mathcal{L}^{1/4}} \right]. \quad (2.80)$$

We wish to demonstrate that this is a Q -exact quantity by finding a V such that $\mathcal{W}^{\psi_1} - \mathcal{W}^{1/4} = QV$. Following [27], we take

$$V^{\psi_1} = -i \text{tr } \vec{\mathcal{P}} \left[\int_{-\infty}^{\infty} ds e^{-i \int_{-\infty}^s ds_1 \mathcal{L}^{1/4}} \Lambda^{\psi_1}(s) e^{-i \int_s^{\infty} ds_2 \mathcal{L}^{\psi_1}} \right], \quad (2.81)$$

where Λ^{ψ_1} is a $U(N_1|N_2)$ supermatrix. We fix Λ^{ψ_1} by imposing that $Q\Lambda^{\psi_1} = \mathcal{L}_F^{\psi_1}$ and $\mathcal{L}_F^{\psi_1}$ is the connection with fermionic entries equal to that of $\mathcal{L}^{\psi_1}$ and zeros everywhere else. It is easily seen that

$$\Lambda^{\psi_1} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & \bar{c} q_{(1)}^{\bar{2}} \\ -c \bar{q}_{(1)\bar{2}} & 0 \end{pmatrix}. \quad (2.82)$$

Using the $Q\Lambda^{\psi_1} = \mathcal{L}_F^{\psi_1}$ and the Leibniz rule, we see that acting with Q on V^{ψ_1} gives

$$QV^{\psi_1} = -i \text{tr } \vec{\mathcal{P}} \left[\int_{-\infty}^{\infty} ds e^{-i \int_{-\infty}^s ds_1 \mathcal{L}^{1/4}} \left(\mathcal{L}_F^{\psi_1}(s) + \Lambda^{\psi_1} Q \right) e^{-i \int_s^{\infty} ds_2 \mathcal{L}^{\psi_1}} \right]. \quad (2.83)$$

We pick up a boundary term from acting with the supercharges on the exponent. Noting that

$$Q\psi_{(1)\underline{1}}^+ = \sqrt{2} \mathcal{D}_s q_{(1)}^{\bar{2}}, \quad Q\bar{\psi}_{(1)+}^1 = -\sqrt{2} \mathcal{D}_s \bar{q}_{(1)\bar{2}}, \quad (2.84)$$

we find

$$\begin{aligned} QV^{\psi_1} &= -i \operatorname{tr} \vec{\mathcal{P}} \left[\int_{-\infty}^{\infty} ds e^{-i \int_{-\infty}^s ds_1 \mathcal{L}^{1/4}} \left(\mathcal{L}_F^{\psi_1}(s) + \tilde{\mathcal{L}}_B^{\psi_1} \right) e^{-i \int_s^{\infty} ds_2 \mathcal{L}^{\psi_1}} \right], \\ &= -i \operatorname{tr} \vec{\mathcal{P}} \left[\int_{-\infty}^{\infty} ds e^{-i \int_{-\infty}^s ds_1 \mathcal{L}^{1/4}} \tilde{\mathcal{L}}^{\psi_1}(s) e^{-i \int_s^{\infty} ds_2 \mathcal{L}^{\psi_1}} \right]. \end{aligned} \quad (2.85)$$

As in [27], upon Taylor expansion, we see that this is equal to the expansion of

$$\mathcal{W}^{\psi_1} - \mathcal{W}^{1/4} = \operatorname{tr} \vec{\mathcal{P}} \left[e^{-i \int_{-\infty}^{\infty} \mathcal{L}^{1/4}} \sum_{p=1}^{\infty} (-i)^p \int ds_1 \cdots ds_p \tilde{\mathcal{L}}^{\psi_1}(s_1) \cdots \tilde{\mathcal{L}}^{\psi_1}(s_p) \right], \quad (2.86)$$

demonstrating the cohomological equivalence of the ψ_1 -loop and 1/4 BPS loop.

ψ_2 -loop

The ψ_2 -loop (2.19) of Section 2.1.1 is also cohomologically equivalent to the 1/4 BPS loop. The result follows almost identically to that of the ψ_1 -loop, with

$$V^{\psi_2} = -i \operatorname{tr} \vec{\mathcal{P}} \left[\int_{-\infty}^{\infty} ds e^{-i \int_{-\infty}^s ds_1 \mathcal{L}^{1/4}} \Lambda^{\psi_2}(s) e^{-i \int_s^{\infty} ds_2 \mathcal{L}^{\psi_1}} \right], \quad (2.87)$$

where

$$\Lambda^{\psi_2} = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & \bar{c} q_{(1)}^{\bar{1}} \\ c \bar{q}_{(1)\bar{1}} & 0 \end{pmatrix}, \quad (\Lambda^{\psi_1})^2 = \frac{1}{2} \tilde{\mathcal{L}}_B^{\psi_2} \equiv \frac{1}{2} \left(\mathcal{L}_B^{\psi_2} - \mathcal{L}_B^{1/4} \right). \quad (2.88)$$

Noting also that

$$Q\psi_{(1)\underline{2}}^- = -\sqrt{2} \mathcal{D}_s q_{(1)}^{\bar{1}}, \quad Q\bar{\psi}_{(1)-}^2 = -\sqrt{2} \mathcal{D}_s \bar{q}_{(1)\bar{1}}, \quad (2.89)$$

it is straightforward to see that both loops are cohomologically equivalent to each other.

Vanishing CS levels

In the case of the quivers with vanishing CS levels discussed in Section 2.1.3, we have not demonstrated the cohomological equivalence to 1/4 BPS loops for the loops based on 2×2 blocks. Below we do it for the 4×4 superconnection (2.38). For the 1/2 BPS loop in Section 2.1.4 it is in fact identical to the 1/4 BPS loop.

We found the $U(N_1 + N_3 | 2N_2)$ valued superconnection (2.38) in Section 2.1.3. We now apply the arguments of the previous section to this loop. It is useful to

rearrange the superconnection to make manifest its $U(N_1 + N_3|2N_2)$ structure:

$$\mathcal{L} = \begin{pmatrix} A_{(1)1} & \bar{d}\epsilon_{\bar{A}\bar{B}}q_{(1)}^{\bar{A}}q_{(2)}^{\bar{B}} & \bar{c}_{(1)}^1\psi_{(1)1}^+ & \bar{c}_{(1)}^2\psi_{(1)2}^- \\ d\epsilon^{\bar{A}\bar{B}}\bar{q}_{(2)\bar{A}}q_{(1)\bar{B}} & A_{(3)1} & c_{(2)2}\bar{\psi}_{(2)-}^2 & c_{(2)1}\bar{\psi}_{(2)+}^1 \\ c_{(1)1}\bar{\psi}_{(1)+}^1 & \bar{c}_{(2)}^2\psi_{(2)2}^- & \mathcal{A}_{(2)} & 0 \\ c_{(1)2}\bar{\psi}_{(1)-}^2 & \bar{c}_{(2)}^1\psi_{(2)1}^+ & 0 & \mathcal{A}'_{(2)} \end{pmatrix}. \quad (2.90)$$

As in (2.81) we define

$$V^{4\times 4} = -i \operatorname{tr} \vec{\mathcal{P}} \left[\int_{-\infty}^{\infty} ds e^{-i \int_{-\infty}^s ds_1 \mathcal{L}^{1/4}} \Lambda^{4\times 4}(s) e^{-i \int_s^{\infty} ds_2 \mathcal{L}^{1/4}} \right], \quad (2.91)$$

where $\mathcal{L}^{1/4}$ is the combination of 1/4 BPS connections

$$\begin{aligned} \mathcal{L}^{1/4} = \operatorname{diag} \left\{ A_{(1)1} - \frac{2i}{k}(\mu_{(1)})^{\bar{1}}_{\bar{1}}, A_{(3)1} - \frac{2i}{k}(\tilde{\mu}_{(2)})^{\bar{1}}_{\bar{1}}, \right. \\ A_{(2)1} + i(\varphi_{(2)})^{\bar{1}}_{\bar{1}} - \frac{i}{k} \left((\tilde{\mu}_{(1)})^{\bar{1}}_{\bar{1}} + (\mu_{(2)})^{\bar{1}}_{\bar{1}} \right), \\ \left. A_{(2)1} + i(\varphi_{(2)})^{\bar{1}}_{\bar{1}} - \frac{i}{k} \left((\tilde{\mu}_{(1)})^{\bar{1}}_{\bar{1}} + (\mu_{(2)})^{\bar{1}}_{\bar{1}} \right) \right\}. \end{aligned} \quad (2.92)$$

We determine $\Lambda^{4\times 4}$ by the requirement that $Q\Lambda^{4\times 4} = \mathcal{L}_F$, with $\mathcal{L}_F$ the fermionic part of $\mathcal{L}$.

Given the same supercharge as for the alternating levels case, we may build $\Lambda^{4\times 4}$ from (2.82), (2.88)

$$\Lambda^{4\times 4} = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -\bar{c}_{(1)}^1 q_{(1)}^{\bar{2}} & \bar{c}_{(1)}^2 q_{(1)}^{\bar{2}} \\ 0 & 0 & c_{(2)2} \bar{q}_{(2)\bar{1}} & c_{(2)1} \bar{q}_{(2)\bar{2}} \\ c_{(1)1} \bar{q}_{(1)\bar{2}} & \bar{c}_{(1)}^2 q_{(2)}^{\bar{1}} & 0 & 0 \\ c_{(1)2} \bar{q}_{(1)\bar{1}} & -\bar{c}_{(1)}^1 q_{(2)}^{\bar{2}} & 0 & 0 \end{pmatrix}. \quad (2.93)$$

A short computation (making use of the constraints (2.41)) shows that

$$\begin{aligned} (\Lambda^{4\times 4})^2 &= \frac{1}{2} \tilde{\mathcal{L}}_B \\ &- \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{(1)1} \bar{c}_{(1)}^2 ((\mu_{(2)})^{\bar{1}}_{\bar{2}} - (\tilde{\mu}_{(1)})^{\bar{1}}_{\bar{2}}) \\ 0 & 0 & c_{(1)2} \bar{c}_{(1)}^1 ((\mu_{(2)})^{\bar{2}}_{\bar{1}} - (\tilde{\mu}_{(1)})^{\bar{2}}_{\bar{1}}) & 0 \end{pmatrix}. \end{aligned} \quad (2.94)$$

As before, $\tilde{\mathcal{L}}_B$ is the bosonic part of the difference $\mathcal{L} - \mathcal{L}^{1/4}$. We note that both of the non-zero entries are proportional to $(\mu_{(2)})^{\bar{A}}_{\bar{B}} - (\tilde{\mu}_{(1)})^{\bar{A}}_{\bar{B}}$, for fixed $\bar{A}, \bar{B}$. For

notational convenience, we define $(\Lambda^{4\times 4})^2 := \frac{1}{2}\tilde{\mathcal{L}}_B + \Delta\mu$.

Continuing as before, using the action of the supercharges on the fermions given in (2.84) and (2.89), we see that

$$Q\mathcal{L}_F = -2i\mathfrak{D}_s\Lambda, \quad (2.95)$$

where $\mathfrak{D}_s$ is the supercovariant derivative with respect to $\mathcal{L}$.

Acting with the supercharge on V thus gives

$$\begin{aligned} QV &= -i \operatorname{tr} \vec{\mathcal{P}} \left[\int_{-\infty}^{\infty} ds e^{-i \int_{-\infty}^s ds_1 \mathcal{L}^{1/4}} (\mathcal{L}_F(s) + \Lambda Q) e^{-i \int_s^{\infty} ds_2 \mathcal{L}} \right] \\ &= -i \operatorname{tr} \vec{\mathcal{P}} \left[\int_{-\infty}^{\infty} ds e^{-i \int_{-\infty}^s ds_1 \mathcal{L}^{1/4}} \left(\mathcal{L}_F(s) + \tilde{\mathcal{L}}_B + \Delta\mu \right) e^{-i \int_s^{\infty} ds_2 \mathcal{L}^{\psi_1}} \right] \\ &= -i \operatorname{tr} \vec{\mathcal{P}} \left[\int_{-\infty}^{\infty} ds e^{-i \int_{-\infty}^s ds_1 \mathcal{L}^{1/4}} \left(\tilde{\mathcal{L}}(s) + \Delta\mu \right) e^{-i \int_s^{\infty} ds_2 \mathcal{L}} \right]. \end{aligned} \quad (2.96)$$

As in the previous cases, the first term in brackets matches the expansion of $\mathcal{W} - \mathcal{W}^{1/4}$ exactly, where $\mathcal{W}$ is the loop defined with superconnection $\mathcal{L}$. The second term is however new.

In Section 2.1.3, we argued that the variation of the loop (2.38) vanishes inside correlation functions. Indeed there too we found terms proportional to $\mu_{(2)} - \tilde{\mu}_{(1)}$, which vanish once the auxiliary field is integrated out. This is enough to conclude that the expectation value of $\mathcal{W}$ and $\mathcal{W}^{1/4}$ are equal.

2.3.2 The circle

We now consider the circle. This will be cohomologically equivalent to the circular analogs of the 1/4 BPS loops above, which have a finite expectation value, which can be calculated by localization [29]. We take the localizing supercharge to be

$$Q = (Q_-^{\bar{1}2} - S_+^{\bar{1}2}) + (Q_-^{\bar{2}1} + S_+^{\bar{2}1}). \quad (2.97)$$

ψ_1 -loop

We begin by considering the difference $\mathcal{W}^{\psi_1} - \mathcal{W}^{1/4}$ between the ψ_1 -loop (2.51) found in Section 2.2.1 and the 1/4 BPS circle, similar to (2.78). We again look for V^{ψ_1} as in (2.81), this time perturbatively in a power series. It will be useful to consider the insertion

$$\Lambda^{\psi_1} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & \bar{c}(s)q_{(1)}^{\bar{2}} \\ -c(s)\bar{q}_{(1)\bar{2}} & 0 \end{pmatrix}, \quad (2.98)$$

with $c(s)$ and $\bar{c}(s)$ as in (2.62), (2.57)). It satisfies

$$Q\Lambda^{\psi_1} = \mathcal{L}_F^{\psi_1}, \quad 4 \cos s (\Lambda^{\psi_1})^2 = \tilde{\mathcal{L}}_B^{\psi_1}. \quad (2.99)$$

Furthermore, the action of the supercharges on the fermionic components of the ψ_1 -loop is given by

$$Q\mathcal{L}_F^{\psi_1} = 2\sqrt{2}\mathcal{D}_s \left[\cos s \begin{pmatrix} 0 & \bar{c}(s)q_{(1)}^{\bar{2}} \\ -c(s)\bar{q}_{(1)\bar{2}} & 0 \end{pmatrix} \right] = -4i\mathcal{D}_s [\cos s \Lambda^{\psi_1}]. \quad (2.100)$$

Order-by-order, we define

$$\begin{aligned} V_1^{\psi_1} &= -i \operatorname{tr} \vec{\mathcal{P}} \left\{ e^{-i \int_0^{2\pi} ds_1 \mathcal{L}^{1/4}(s_1)} \int_0^{2\pi} ds_2 \Lambda^{\psi_1} \right\}, \\ V_2^{\psi_1} &= -\frac{1}{2} \operatorname{tr} \vec{\mathcal{P}} \left\{ e^{-i \int_0^{2\pi} ds_1 \mathcal{L}^{1/4}(s_1)} \right. \\ &\quad \times \left. \int_0^{2\pi} ds_2 \int_{s_2}^{2\pi} ds_3 \left(\Lambda^{\psi_1}(s_2) \mathcal{L}_F^{\psi_1}(s_3) - \mathcal{L}_F^{\psi_1}(s_2) \Lambda^{\psi_1}(s_3) \right) \right\} \\ V_3^{\psi_1} &= \operatorname{tr} \vec{\mathcal{P}} \left\{ e^{-i \int_0^{2\pi} ds_1 \mathcal{L}^{1/4}(s_1)} \right. \\ &\quad \times \left(- \int_0^{2\pi} ds_2 \int_{s_2}^{2\pi} ds_3 \left(\tilde{\mathcal{L}}_B^{\psi_1}(s_2) \Lambda^{\psi_1}(s_3) + \Lambda^{\psi_1}(s_2) \tilde{\mathcal{L}}_B^{\psi_1}(s_3) \right) \right. \\ &\quad + i \int_{s_2 < s_3 < s_4} ds_2 ds_3 ds_4 \left(\Lambda^{\psi_1}(s_2) \mathcal{L}_F^{\psi_1}(s_3) \mathcal{L}_F^{\psi_1}(s_4) + \mathcal{L}_F^{\psi_1}(s_2) \Lambda^{\psi_1}(s_3) \mathcal{L}_F^{\psi_1}(s_4) \right. \\ &\quad \left. \left. + \mathcal{L}_F^{\psi_1}(s_2) \mathcal{L}_F^{\psi_1}(s_3) \Lambda^{\psi_1}(s_4) \right) \right) \left. \right\}. \end{aligned} \quad (2.101)$$

The action of the supercharges on these operators is given by

$$\begin{aligned} QV_1^{\psi_1} &= -i \operatorname{tr} \vec{\mathcal{P}} \left\{ e^{-i \int_0^{2\pi} ds_1 \mathcal{L}^{1/4}(s_1)} \int_0^{2\pi} ds_2 \mathcal{L}_F^{\psi_1} \right\}, \\ QV_2^{\psi_1} &= \\ &= - \operatorname{tr} \vec{\mathcal{P}} \left\{ e^{-i \int_0^{2\pi} ds_1 \mathcal{L}^{1/4}(s_1)} \int_0^{2\pi} ds_2 \left(\int_{s_2}^{2\pi} ds_3 \mathcal{L}_F^{\psi_1}(s_2) \mathcal{L}_F^{\psi_1}(s_3) + i \tilde{\mathcal{L}}_B^{\psi_1}(s_2) \right) \right\}, \\ QV_3^{\psi_1} &= \operatorname{tr} \vec{\mathcal{P}} \left\{ e^{-i \int_0^{2\pi} ds_1 \mathcal{L}^{1/4}(s_1)} \right. \\ &\quad \times \left(- \int_{s_2 < s_3} ds_2 ds_3 \left(\tilde{\mathcal{L}}_B^{\psi_1}(s_2) \mathcal{L}_F^{\psi_1}(s_3) + \mathcal{L}_F^{\psi_1}(s_2) \tilde{\mathcal{L}}_B^{\psi_1}(s_3) \right) \right. \\ &\quad \left. + i \int_{s_2 < s_3 < s_4} ds_2 ds_3 ds_4 \mathcal{L}_F^{\psi_1}(s_2) \mathcal{L}_F^{\psi_1}(s_3) \mathcal{L}_F^{\psi_1}(s_4) \right) \left. \right\}. \end{aligned} \quad (2.102)$$

These indeed match the terms in the expansion of $\mathcal{W}^{\psi_1} - \mathcal{W}^{1/4}$, demonstrating that to third order in the expansion the ψ_1 -loop is cohomologically equivalent to the 1/4 BPS loop. However, as in the case of ABJM, we expect this relationship to hold to all orders, and the corresponding localization calculation to reduce to the known matrix model calculation of the 1/4 BPS loops [29].

ψ_2 -loop

The calculation for the ψ_2 -loop, (2.67) found in Section 2.2.1, proceeds very similarly. Defining

$$\Lambda^{\psi_2} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & \bar{c}(s)q_{(1)}^{\bar{1}} \\ -c(s)\bar{q}_{(1)\bar{1}} & 0 \end{pmatrix}, \quad (2.103)$$

we see that it satisfies

$$Q\Lambda^{\psi_2} = \mathcal{L}_F^{\psi_2}, \quad 4 \cos s (\Lambda^{\psi_2})^2 = \tilde{\mathcal{L}}_B^{\psi_2}, \quad (2.104)$$

and

$$Q\mathcal{L}_F^{\psi_2} = 2\sqrt{2}\mathcal{D}_s \left[\cos s \begin{pmatrix} 0 & \bar{c}(s)q_{(1)}^{\bar{1}} \\ -c(s)\bar{q}_{(1)\bar{1}} & 0 \end{pmatrix} \right] = -4i\mathcal{D}_s [\cos s \Lambda^{\psi_2}]. \quad (2.105)$$

Equations (2.104) and (2.105) are the corresponding identities for the ψ_2 -loop as (2.99) and (2.100) for the ψ_1 -loop. As these are identical, it immediately follows that the action of the supercharge on

$$\begin{aligned} V_1^{\psi_2} &= -i \operatorname{tr} \vec{\mathcal{P}} \left\{ e^{-i \int_0^{2\pi} ds_1 \mathcal{L}^{1/4}(s_1)} \int_0^{2\pi} ds_2 \Lambda^{\psi_1} \right\}, \\ V_2^{\psi_2} &= -\frac{1}{2} \operatorname{tr} \vec{\mathcal{P}} \left\{ e^{-i \int_0^{2\pi} ds_1 \mathcal{L}^{1/4}(s_1)} \right. \\ &\quad \times \left. \int_0^{2\pi} ds_2 \int_{s_2}^{2\pi} ds_3 \left(\Lambda^{\psi_1}(s_2) \mathcal{L}_F^{\psi_1}(s_3) - \mathcal{L}_F^{\psi_1}(s_2) \Lambda^{\psi_1}(s_3) \right) \right\} \\ V_3^{\psi_2} &= \operatorname{tr} \vec{\mathcal{P}} \left\{ e^{-i \int_0^{2\pi} ds_1 \mathcal{L}^{1/4}(s_1)} \right. \\ &\quad \times \left(- \int_0^{2\pi} ds_2 \int_{s_2}^{2\pi} ds_3 \left(\tilde{\mathcal{L}}_B^{\psi_1}(s_2) \Lambda^{\psi_1}(s_3) + \Lambda^{\psi_1}(s_2) \tilde{\mathcal{L}}_B^{\psi_1}(s_3) \right) \right. \\ &\quad \left. \left. + i \int_{s_2 < s_3 < s_4} ds_2 ds_3 ds_4 \left(\Lambda^{\psi_1}(s_2) \mathcal{L}_F^{\psi_1}(s_3) \mathcal{L}_F^{\psi_1}(s_4) + \mathcal{L}_F^{\psi_1}(s_2) \Lambda^{\psi_1}(s_3) \mathcal{L}_F^{\psi_1}(s_4) \right. \right. \right. \\ &\quad \left. \left. \left. + \mathcal{L}_F^{\psi_1}(s_2) \mathcal{L}_F^{\psi_1}(s_3) \Lambda^{\psi_1}(s_4) \right) \right) \right\}, \end{aligned} \quad (2.106)$$

will exactly match the expansion of $\mathcal{W}^{\psi_2} - \mathcal{W}^{1/4}$, thus demonstrating the equivalence of the two loops with respect to the localizing supercharge.

2.4 Holographic description

The goal of this section is to find the M2-brane configurations which preserve half of the supersymmetries of the vacuum and are the holographic duals to the gauge theory operators discussed above. We identify the fixed point of the dual geometry with respect to the $SU(2)_A$ R-symmetry group (preserved by the 1/2 BPS loops of Section 2.1). We then demonstrate that M2-brane embeddings at this fixed point

of the geometry preserve half of the supersymmetry of the background geometry. ABJ(M) theory is a special case of the backgrounds we consider; in Section 2.4.2, we use this, along with known results for this theory, to provide an argument for the strong coupling expectation values of the dual Wilson loops.

We limit our attention to the case of circular quivers whose corresponding space-time metric is $AdS_4 \times M_{p,q,k}$ [66], where the compact manifold is an orbifold of the seven-sphere

$$M_{p,q,k} = (S^7 / (\mathbb{Z}_p \oplus \mathbb{Z}_q)) / \mathbb{Z}_k, \quad (2.107)$$

with radius $R^6 = 2^5 \pi^2 N k p q$ (in units of $\ell_s = 1$). The gauge group rank N corresponds here to the number of M2-branes located at the orbifold point. p and q are the numbers of untwisted and twisted hypermultiplets in the dual quiver, respectively. As discussed in Section 1.3.2, the geometry is insensitive to arrangement of these hypermultiplets in the dual quiver and only depends on their number. To distinguish between different arrangements, one should consider the IIB holographic duals of [67–70].

For the AdS_4 factor of these geometries, we consider a metric with an AdS_2 foliation

$$ds_{AdS_4}^2 = du^2 + \cosh^2 u ds_{AdS_2}^2 + \sinh^2 u d\phi^2, \quad (2.108)$$

where for a time-like (straight) Wilson loop we take a Lorentzian AdS_2

$$ds_{AdS_2}^2 = d\rho^2 - \cosh^2 \rho dt^2, \quad (2.109)$$

and for the space-like circle we consider the Euclidean metric

$$ds_{AdS_2}^2 = d\rho^2 + \sinh^2 \rho d\psi^2. \quad (2.110)$$

The M2-brane world-volume will be along these AdS_2 coordinates in AdS_4 , while on the internal space it will wrap an S^1 (*i.e.*, the M-theory circle) and sit at a fixed point in the remaining coordinates, which is determined by supersymmetry, as we shall see.

The orbifold action giving rise to the space $M_{p,q,k}$ is most easily described in terms of complex embedding coordinates z_i ($i = 1, \dots, 4$) subject to the constraint

$$\sum_{i=1}^4 |z_i|^2 = 1, \quad (2.111)$$

and to the identifications [37]

$$(z_1, z_2, z_3, z_4) \sim (\omega_{kp}^m z_1, \omega_{kp}^m z_2, \omega_{kq}^{-m} z_3, \omega_{kq}^{-m} z_4), \quad \omega_r = e^{2\pi i/r}, \quad m \in \mathbb{Z}. \quad (2.112)$$

This last line alone without the constraint (2.111) would define the manifold

$$\mathcal{M}_{p,q,k} = (\mathbb{C}^2/\mathbb{Z}_p \otimes \mathbb{C}^2/\mathbb{Z}_q) / \mathbb{Z}_k. \quad (2.113)$$

The $SU(2)_A$ subgroup of the R -symmetry acts on $\mathbb{C}^2/\mathbb{Z}_p$ and $SU(2)_B$ acts on $\mathbb{C}^2/\mathbb{Z}_q$.

The supersymmetries preserved by the Wilson loops (2.6) are doublets of the R -symmetry subgroup $SU(2)_A$, which is left unbroken. We therefore expect that the M2-brane embedding will be at a fixed point of the $SU(2)_B$ action. This is the tip of the $\mathbb{C}/\mathbb{Z}_p$ cone in $\mathcal{M}_{p,q,k}$. A similar story applies of course if we were to consider loops invariant under $SU(2)_B$ with the two cones exchanged.

2.4.1 Parameterizations of the 7-sphere

We now review two known parameterizations of the compact space $\mathcal{M}_{p,q,k}$ [25, 37]. These parameterizations, and the Killing spinor in these coordinate systems, are required for the following supersymmetry analysis of the M2-brane embeddings. In Appendix B, we derive the Killing spinor for one of these coordinate systems. For the other coordinate system, the Killing spinor is known [25].

Firstly, we may parametrize the S^7 by the coordinates [25]

$$\begin{aligned} z_1 &= \cos \alpha \cos \frac{\theta_1}{2} e^{i\xi_1}, & z_2 &= \cos \alpha \sin \frac{\theta_1}{2} e^{i\xi_2}, \\ z_3 &= \sin \alpha \cos \frac{\theta_2}{2} e^{i\xi_3}, & z_4 &= \sin \alpha \sin \frac{\theta_2}{2} e^{i\xi_4}. \end{aligned} \quad (2.114)$$

This choice induces the following metric

$$\begin{aligned} ds^2 &= \frac{1}{4} \left[4d\alpha^2 + \cos^2 \alpha \left(d\theta_1^2 + 4 \cos^2 \frac{\theta_1}{2} d\xi_1^2 + 4 \sin^2 \frac{\theta_1}{2} d\xi_2^2 \right) \right. \\ &\quad \left. + \sin^2 \alpha \left(d\theta_2^2 + 4 \cos^2 \frac{\theta_2}{2} d\xi_3^2 + 4 \sin^2 \frac{\theta_2}{2} d\xi_4^2 \right) \right], \end{aligned} \quad (2.115)$$

with appropriate ranges for the coordinates to cover the entire sphere once, namely $0 \leq \alpha \leq \pi/2$, $0 \leq \theta_{1,2} \leq \pi$ and $0 \leq \xi_i \leq 2\pi$.

To make the action of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ more transparent, we can relabel the ξ_i as

$$\xi_1 = \frac{\phi_1}{2} + x, \quad \xi_2 = -\frac{\phi_1}{2} + x, \quad \xi_3 = \frac{\phi_2}{2} + v, \quad \xi_4 = -\frac{\phi_2}{2} + v, \quad (2.116)$$

which results in

$$ds_{S^7/(\mathbb{Z}_p \oplus \mathbb{Z}_q)}^2 = \frac{1}{4} \left[4d\alpha^2 + \cos^2 \alpha ds_{S^3/\mathbb{Z}_p}^2 + \sin^2 \alpha ds_{S^3/\mathbb{Z}_q}^2 \right], \quad (2.117)$$

with

$$\begin{aligned} ds_{S^3/\mathbb{Z}_p}^2 &= d\theta_1^2 + d\phi_1^2 + 4dx^2 + 4\cos\theta_1 dx d\phi_1, \\ ds_{S^3/\mathbb{Z}_q}^2 &= d\theta_2^2 + d\phi_2^2 + 4dv^2 + 4\cos\theta_2 dv d\phi_2. \end{aligned} \quad (2.118)$$

We are implicitly assuming that $k = 1$, so that $\mathbb{Z}_k$ is trivial. The action of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ then identify [37, 70]

$$(z_1, z_2, z_3, z_4) \sim (\omega_p^m z_1, \omega_p^m z_2, \omega_q^n z_3, \omega_q^n z_4), \quad m, n \in \mathbb{Z}, \quad (2.119)$$

and therefore impose the following periodicities on the angular coordinate x and v

$$x \sim x + \frac{2\pi m}{p}, \quad v \sim v + \frac{2\pi n}{q}. \quad (2.120)$$

With these coordinates the action of $SU(2)_A$ and $SU(2)_B$ is manifest on the two S^3 orbifolds in (2.118).

The full metric of the space (before the $\mathbb{Z}_k$ orbifold) is given by [70]

$$ds^2 = R^2 \left(ds_{AdS_4}^2 + 4ds_{S^7/(\mathbb{Z}_p \oplus \mathbb{Z}_q)}^2 \right). \quad (2.121)$$

The Killing spinor for this 11-dimensional geometry is derived in Appendix B and is given by

$$\eta = \Psi \Xi \eta_0, \quad (2.122)$$

with

$$\begin{aligned} \Psi &= e^{-\frac{\alpha-\pi/2}{2}\Gamma_4\Gamma_\star} e^{\frac{\theta_1-\pi/2}{4}(\Gamma_{45}+\Gamma_{67})} e^{\frac{\theta_2-\pi/2}{4}(\Gamma_{9\#}-\Gamma_8\Gamma_\star)} e^{\frac{x}{2}(\Gamma_{46}+\Gamma_{57})} \\ &\quad \times e^{\frac{v}{2}(\Gamma_{8\#}-\Gamma_9\Gamma_\star)} e^{-\frac{\phi_1}{4}(\Gamma_{47}+\Gamma_{56})} e^{-\frac{\phi_2}{4}(\Gamma_{89}-\Gamma_\#\Gamma_\star)}, \quad (2.123) \\ \Xi &= e^{\frac{u}{2}\Gamma_2\Gamma_\star} e^{\frac{\rho}{2}\Gamma_1\Gamma_\star} e^{\frac{t}{2}\Gamma_0\Gamma_\star} e^{\frac{\phi}{2}\Gamma_{23}}, \end{aligned}$$

where η_0 is a 32-component constant spinor and $\Gamma_\star \equiv \Gamma_{0123}$. The gamma matrices live in the tangent space and the indices $0, 1, \dots, 9, \#$ are flat indices.

A second parameterization of the angles is useful to study the general case of $k \geq 2$ and is obtained by setting

$$\begin{aligned} \xi_1 &= \frac{1}{4}(-2\phi_1 + \chi + \zeta), & \xi_2 &= \frac{1}{4}(2\phi_1 + \chi + \zeta), \\ \xi_3 &= \frac{1}{4}(-2\phi_2 - \chi + \zeta), & \xi_4 &= \frac{1}{4}(2\phi_2 - \chi + \zeta), \end{aligned} \quad (2.124)$$

in the metric above. This results in [25]

$$\begin{aligned} ds_{M_{p,q,k}}^2 &= \frac{1}{4} \left[4d\alpha^2 + \sin^2\alpha (d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \sin^2\alpha (d\theta_2^2 + \cos^2\theta_2 d\phi_2^2) \right. \\ &\quad \left. + \cos^2\alpha \sin^2\alpha (d\chi + \cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2)^2 + \frac{1}{4}(d\zeta + A)^2 \right], \end{aligned} \quad (2.125)$$

where

$$A = \cos 2\alpha d\chi + 2\cos^2 \alpha \cos \theta_1 d\phi_1 + 2\sin^2 \alpha \cos \theta_2 d\phi_2. \quad (2.126)$$

In these coordinates, the action of the $\mathbb{Z}_k$ orbifold is along the ζ direction, *i.e.*

$$\zeta \sim \zeta + \frac{2\pi m}{k}, \quad m \in \mathbb{Z}. \quad (2.127)$$

The Killing spinor for these coordinates was obtained in [25] and is as in (2.122), with Ψ replaced by

$$\tilde{\Psi} = e^{\frac{\alpha}{2}(\Gamma_4\Gamma_\star - \Gamma_{7\#})} e^{\frac{\theta_1}{4}(\Gamma_5\Gamma_\star - \Gamma_{8\#})} e^{\frac{\theta_2}{4}(\Gamma_{79} + \Gamma_{46})} e^{-\frac{\xi_1}{2}\Gamma_{\#}\Gamma_\star} e^{-\frac{\xi_2}{2}\Gamma_{58}} e^{-\frac{\xi_3}{2}\Gamma_{47}} e^{-\frac{\xi_4}{2}\Gamma_{69}}, \quad (2.128)$$

where the ξ_i are given by (2.124).

2.4.2 Supersymmetry analysis

We now check that the internal space $M_{p,q,k}$ preserves sixteen supersymmetries. To address this, we consider the first parameterization, equations (2.117) and (2.123) above, and use the procedure employed in [25] for the ABJM case. This begins by choosing a basis for the gamma matrices such that

$$\Gamma_{0123456789\#} = 1, \quad (2.129)$$

and by writing the constant spinor η_0 in a basis which block-diagonalizes as follows

$$\Gamma_{46}\eta_0 = s_1\eta_0, \quad \Gamma_{57}\eta_0 = s_2\eta_0, \quad \Gamma_{8\#}\eta_0 = s_3\eta_0, \quad \Gamma_9\Gamma_\star\eta_0 = s_4\eta_0. \quad (2.130)$$

All s_i eigenvalues take values ± 1 , see [25]. A translation in x corresponds to an action of the $\mathbb{Z}_p$ orbifold and rescales the Killing spinor by

$$\Psi \Xi \eta_0 \rightarrow e^{i(s_1+s_2)\delta/2} \Psi \Xi \eta_0, \quad (2.131)$$

as can be readily seen from (2.123). In order for this to be a symmetry of the Killing spinor (for a generic translation parameter δ), we are restricted to

$$(s_1, s_2, s_3, s_4) \in \{(+, -, \cdot, \cdot), (-, +, \cdot, \cdot)\}, \quad (2.132)$$

where $\pm$ indicates ± 1 and $\cdot$ indicates an unrestricted value. The action of the $\mathbb{Z}_q$ orbifold (*i.e.*, a translation in v) similarly gives

$$(s_1, s_2, s_3, s_4) \in \{(\cdot, \cdot, +, +), (\cdot, \cdot, -, -)\}. \quad (2.133)$$

Now we notice that $\zeta \propto x + v$, so that the action of the $\mathbb{Z}_k$ orbifold rescales the Killing spinor as

$$\Psi \Xi \eta_0 \rightarrow e^{i(s_1+s_2+s_3-s_4)\delta/2} \Psi \Xi \eta_0. \quad (2.134)$$

This restricts to the following choices

$$(s_1, s_2, s_3, s_4) \in \{(+, -, -, -), (-, +, -, -), (-, -, +, -), \\ (-, +, +, +), (+, -, +, +), (+, +, -, +)\}. \quad (2.135)$$

Note that each set of s_i eigenvalues represents four supercharges. Thus, the $\mathbb{Z}_k$ orbifold by itself restricts us to 24 supercharges, *i.e.*, the ABJM theory, as expected. Taking the intersection of the three sets of allowed eigenvalues, we are left with

$$(s_1, s_2, s_3, s_4) \in \{(+, -, -, -), (-, +, -, -), (-, +, +, +), (+, -, +, +)\}, \quad (2.136)$$

namely 16 supercharges, as required.

2.4.3 The M2-brane embedding

We now proceed to finding the M2-brane embeddings in the above geometries, which preserve half of these supersymmetries. The supersymmetry analysis of the M2-brane embedding is most easily carried out using the second parametrization, equations (2.125) and (2.128) above. As anticipated earlier, the M2-brane is taken to be oriented along the AdS_2 directions in AdS_4 (while sitting at $u = 0$) and to wrap the S^1 parametrized by ζ . The world-volume coordinates are then given by (t, ρ, ζ) , with the remaining coordinates on the sphere assuming constant values. The induced metric is thus simply given by

$$\frac{ds_{\text{ind}}^2}{R^2} = d\rho^2 - \cosh^2 \rho dt^2 + \frac{1}{4} d\zeta^2, \quad (2.137)$$

where $R^6 = 2^5 \pi^2 N k p q$, as seen above [37]. The amount of supersymmetry preserved by this embedding is obtained by considering the projection equation [98]

$$\Gamma \eta = \eta, \quad (2.138)$$

for the Killing spinor (2.128), with the projector being given by

$$\Gamma = \frac{1}{\sqrt{-\det g}} \partial_t x^\mu \partial_\rho x^\nu \partial_\zeta x^\sigma \gamma_\mu \gamma_\nu \gamma_\sigma = \Gamma_{01\#}. \quad (2.139)$$

This is formally identical to the ABJM case considered in [25], so that we can quote the results from that case. Setting therefore $\alpha = 0$ and $\theta_1 = 0$ the projection equation for ϵ reduces to an equation for ϵ_0 and the brane breaks half of the supercharges in (2.136). Selecting $\alpha = \pi/2$ and $\theta_2 = 0$ also gives 1/2 BPS embeddings, which are

the loops invariant under $SU(2)_A$.

Unlike the ABJM case, in the present case there are also the $\mathbb{Z}_p$ and $\mathbb{Z}_q$ orbifolds. A crucial fact is that $\alpha = \pi/2$, where our M2-brane is located, is a fixed point of the $\mathbb{Z}_p$ orbifold. In fact, we expect p different states at the singular point and find holographic duals for p different fundamental Wilson loops. This mirrors (part of) the degeneracy of $1/2$ BPS loop operators we have uncovered in the gauge theory side. We expect there to be a single fundamental Wilson loop for every one of the p hypermultiplets. It would be advantageous to use the type IIB description [67–70], where this singularity is resolved, and which allows also to study linear quivers. See more in the discussion below.

We may also exploit the connection with ABJ(M) theory to provide an argument for the computation of the renormalized on-shell action of the M2-branes: the brane action (after renormalization) will be proportional to $R^3 \int d\zeta$. Using the known result for the on-shell action for the case of ABJ(M) theory [25] (*i.e.* the $p = q = 1$ case for our geometries), the explicit expression of the radius written above and that the range of ζ goes up to $1/kq$, the expectation value would then seem to be given by

$$\langle W \rangle \simeq e^{\pi \sqrt{\frac{2Np}{kq}}} . \quad (2.140)$$

As all of the M2-branes/Wilson loops preserving the $SU(2)_A$ R-symmetry group factor sit at the tip of the orbifold, it is actually unlikely that this expression be valid for all the p different Wilson loops, as the answer should depend on the details of the resolution of the orbifold singularity (and fluxes, and fractional branes). This expression does in fact hold at least in the simplest case where the Wilson loops are all equivalent, so theories with $p = q$ and alternating $\pm k$ CS levels. Those in fact are orbifolds of the ABJM theory.

As discussed in Section 1.3.2, to answer the question of the expectation values more concretely one should study the type IIB duals of these theories [67–70]. The orbifold singularity in the M-theory description is resolved in these theories and also allow us to study linear quivers. Moreover, this would allow us to study the degeneracy observed in the field theory side at strong coupling; in contrast to studying the M-theory dual described above.

2.5 Summary

In this chapter, we construct the $1/2$ BPS Wilson loops for a class of $\mathcal{N} = 4$ Chern-Simons matter theories. These theories, a subset of those constructed in [35–37], are quiver gauge theories. Each node in the quiver corresponds to a $U(N_i)$ vector multiplet (for the i th node) and Chern-Simons level k , $-k$ or 0 . The nodes are connected by links corresponding to hypermultiplets transforming in the bifundamental representation of the adjacent gauge group factors. There are two types of hyper-

multiplets: twisted and untwisted. The matter of the two types of hypermultiplets form different representations of the $SO(4)$ R-symmetry group.

We find that the $1/2$ BPS loops are associated with segments of the quiver (rather than the entire quiver). For two adjacent nodes, each with non-vanishing CS levels, there are two $1/2$ BPS Wilson loops preserving the same supersymmetries (with the geometry of either a straight line or a circle). Such a pairwise degeneracy has not previously been seen in the context of Wilson loops. This degeneracy has since been found to be lifted at the quantum level by other authors in [41]. There it was found that only the average of the two $1/2$ BPS loops which we find remains $1/2$ BPS at the quantum level.

For segments of the quivers with vanishing CS levels, the story is somewhat different. For such segments, we find *weakly* $1/2$ BPS Wilson loops. Such loops are not themselves invariant under the supersymmetry transformations; rather their expectation values are. For a segment of a quiver comprising two nodes with non-vanishing CS level connected by nodes of vanishing CS level and repeated untwisted hypermultiplets, we find two weakly $1/2$ BPS loops of a given geometry associated with any adjacent pair of nodes (and the link joining them). We also find a single weakly $1/2$ BPS loop associated with the entire segment. Preserving a different set of supersymmetries to those preserved by the loops described above, we also find $1/2$ BPS loops associated to each of the $k = 0$ nodes in the quiver (these are BPS in the strong sense).

In [29], it was demonstrated that the computation of the expectation values of the $1/4$ BPS loops of the theories we consider may be reduced, via localization, to a matrix model integral. The resulting matrix model has not been solved, however. We demonstrate that the $1/2$ BPS loops we find are cohomologically equivalent to a combination of $1/4$ BPS loops, with respect to a supercharge preserved by both the $1/2$ and $1/4$ BPS loops. This supercharge may be used to perform the localization procedure of [29]. The cohomological equivalence of the two loops implies that their expectation values are given by the same matrix model integral. We may thus use the results of [29] to write the expectation value of the $1/2$ BPS loops as a matrix model integral.

Lastly, we study the M-theory holographic duals of the $1/2$ BPS Wilson loops. These M-theory duals describe the subset of quivers which are circular, with all non-vanishing CS levels and vector multiplets of equal rank. The holographic duals of the $1/2$ BPS Wilson loops are M2-branes. We find the M2-brane embeddings in these holographic duals which preserve the required supersymmetry. Using the fact that ABJ(M) theory is a special case of the theories which we consider, we provide an argument for the leading order strong coupling expectation value of the $1/2$ BPS loops.

Chapter 3

Operator insertions and wavy Wilson lines

In this chapter we study operator insertions into the $1/2$ BPS straight Wilson line in planar $\mathcal{N} = 4$ super Yang-Mills theory. By performing a functional Taylor expansion of a Wilson loop, one may demonstrate an equivalence between the Wilson loop with a set of operator insertions along the loop and small deformations of the Wilson loop (with no operator insertions). These operator insertions may be associated with orders of the functional Taylor expansion about the $1/2$ BPS line. Previous work [52] has studied aspects of operator insertions up to, and including, the second order in this expansion. We study the operator insertions up to, and including, up to the fourth order in this expansion.

Via the aforementioned equivalence, the expectation value of the deformed loop is given by a sum of n -point functions of the operator insertions, integrated along the loop. By focusing on one- and two-loop order at weak coupling, this expectation value is given by sum of two- and three-point functions, as well as factorized four point functions. The functional form of these correlation functions is fixed, up to a set of constants, by the residual conformal symmetry preserved by the $1/2$ BPS line. These constants are the two- and three-point functions coefficients and one-loop anomalous dimensions of the operator insertions. We refer to these constants as the CFT data.

It is this CFT data which we compute for the operator insertions, up to the fourth order in the deformation. At one- and two-loop order at weak coupling, there are known results for the expectation values of smooth Wilson loops. We manipulate these expressions for the case a small deformation of the $1/2$ BPS straight line. We massage them into the form of a sum of two-, three- and four-point functions of operator insertions. From the resulting expressions, we may read off the aforementioned CFT data.

Our procedure necessitates the introduction of a regularization scheme. We study two different regularization schemes and find scheme-dependence in the results. We

discuss this scheme-dependence and comment on current avenues of investigation to resolve this.

3.1 Operator insertions in Wilson loops

We begin by recapping a couple of definitions from Section 1.2. We also review the constraints imposed by conformal symmetry on the correlation functions of operator insertions (this has previously been discussed in [56, 58]). We define a Wilson loop with insertions as

$$W [\mathcal{O}^{(1)}(x_1) \dots \mathcal{O}^{(n)}(x_n)] := \frac{1}{N} \text{tr} \overleftarrow{\mathcal{P}} \left[\mathcal{O}^{(1)}(x_1) \dots \mathcal{O}^{(n)}(x_n) e^{\int i(\dot{x}^\mu A_\mu(x(s)) - \sqrt{\dot{x}^2} \Phi^1(x(s))) ds} \right], \quad (3.1)$$

where $\mathcal{O}^{(n)}$ are generic local operators transforming in the adjoint of the $SU(N)$ gauge group, $x(s)^\mu$ describes a path in $\mathbb{R}^{1,3}$ (with metric $\eta_{\mu\nu}$) and $\dot{x}^2 = \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$. We define the n -point functions of operator insertions as

$$\langle\langle \mathcal{O}^{(1)}(x_1) \dots \mathcal{O}^{(n)}(x_n) \rangle\rangle := \langle W [\mathcal{O}^{(1)}(x_1) \dots \mathcal{O}^{(n)}(x_n)] \rangle / \langle W \rangle. \quad (3.2)$$

As discussed in Section 1.2, due to the residual $SO(1, 2)_c$ conformal group preserved by the line, these correlation functions satisfy the axioms of a CFT.

The two-point function of a scalar primary operator, $\mathcal{O}$, satisfies

$$\langle\langle \mathcal{O}(s) \mathcal{O}(0) \rangle\rangle = \frac{a(\lambda)}{s^{2\Delta(\lambda)}}, \quad (3.3)$$

where $a(\lambda)$ is the two-point function coefficient and $\Delta(\lambda)$ is the scaling dimension of $\mathcal{O}$. In a similar way, the three-point function of three scalar primary operators satisfies

$$\begin{aligned} \langle\langle \mathcal{O}^{(1)}(s_1) \mathcal{O}^{(2)}(s_2) \mathcal{O}^{(3)}(s_3) \rangle\rangle \\ = \frac{c^{(123)}(\lambda)}{|s_{12}|^{\Delta^{(1)} + \Delta^{(2)} - \Delta^{(3)}} |s_{13}|^{\Delta^{(1)} - \Delta^{(2)} + \Delta^{(3)}} |s_{23}|^{-\Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)}}}, \end{aligned} \quad (3.4)$$

where $s_{ij} = s_i - s_j$ and $c^{(123)}(\lambda)$ is the structure constant.

The four-point function is

$$\begin{aligned} \langle\langle \mathcal{O}^{(1)}(s_1) \mathcal{O}^{(2)}(s_2) \mathcal{O}^{(3)}(s_3) \mathcal{O}^{(4)}(s_4) \rangle\rangle \\ = \prod_{1 \leq i \leq j \leq 4} |s_{ij}|^{\frac{1}{3} \sum_{k=0}^3 \Delta_0^{(k)} - \Delta_0^{(i)} - \Delta_0^{(j)}} G(u, v), \end{aligned} \quad (3.5)$$

where G is a function of the cross-ratios u and v . It can in principle be determined from the structure constants via the operator product expansion, but we shall not

explore that relation.

Above we have considered scalar primary operators. All of the above constraints generalize to descendant operators and to operators with tensor structure. The correlation function of a descendant operator is obtained from the correlation function of the primary operator of the multiplet by the application of the lowering operator of the residual conformal algebra. For the 1/2 BPS line along the ‘3’ direction, this lowering operator is P_3 .

The correlation functions of tensor operators is constructed from the inversion tensor

$$I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}, \quad Y^\mu(x_1, x_2, x_3) = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}. \quad (3.6)$$

For insertions along the line however, the latter vanishes and the former reduces in the transverse directions to $I_{ij} = \eta_{ij}$. Let us consider a vector primary operator $\mathcal{O}_i$ as an example. Its two point function is given by

$$\langle\langle \mathcal{O}_i(s) \mathcal{O}_j(0) \rangle\rangle = \frac{a(\lambda)\eta_{ij}}{s^{2\Delta}}, \quad (3.7)$$

and its three-point function is constrained to vanish. This generalizes straightforwardly to higher rank tensor operators. In this way, as for a CFT, the correlation functions of the insertions are determined up to a set of coefficients.

We now discuss in detail the equivalence between deformations of Wilson loops and the insertion of certain operator insertions into the undeformed loop. We shall later exploit this to extract two- and three-point function coefficients and anomalous dimensions in Section 3.4.

3.2 Equivalence of operator insertions and deformations

In this section we review the equivalence between small deformations of a Wilson loop and operator insertions along the loop. This has previously been discussed, to quadratic order in the deformation parameter, in [52, 56–58]. We extend this discussion up to, and including, the fourth order in the deformation parameter. The procedure is general and holds for deformations of any smooth loop, however we shall later restrict ourselves to deformations of the 1/2 BPS straight line.

3.2.1 The operator expansion

Let us consider a Wilson loop (or line) along a contour $\mathcal{C}_0$, defined by $x^\mu(s)$ in $\mathbb{R}^{1,3}$. We consider a small deformation of the contour $\mathcal{C}_0$ to define a new contour $\mathcal{C}$, by adding a small deformation to $x^\mu(s)$ denoted by $\delta x^\mu(s)$. The Wilson loop $W[\mathcal{C}]$ is

then given formally by the functional Taylor expansion

$$W[\mathcal{C}] = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n W[\mathcal{C}_0], \quad (3.8)$$

where δ corresponds to taking $x^\mu(s) \rightarrow x^\mu(s) + \delta x^\mu(s)$. We do not consider deformations of the scalar couplings, beyond the change of the magnitude of the scalar coupling, which is $|\dot{x}|$.

It is straight-forward to take the functional variation of the Wilson loop. Taking the variation $\delta W[\mathcal{C}_0]$ is equivalent to the insertion of a (collection of) operator(s) $\mathcal{S}^{(1)}(s_1)$ into $W[\mathcal{C}_0]$:

$$\delta W[\mathcal{C}_0] = \text{tr} \overleftarrow{\mathcal{P}} \oint ds_1 \mathcal{S}^{(1)}(s_1) e^{i \oint ds \mathcal{A}_0(s)}, \quad (3.9)$$

where $\mathcal{A}_0(s)$ is the connection of the undeformed loop $W[\mathcal{C}_0]$ and $\oint ds_1$ denotes integration along the contour $\mathcal{C}_0$.

The second order variation is equivalent to two insertions of $\mathcal{S}^{(1)}$ in $W[\mathcal{C}_0]$ as well as the insertion of a new operator $\mathcal{S}^{(2)}(s)$ into the loop, *i.e.*

$$\begin{aligned} \delta^2 W[\mathcal{C}_0] \\ = \text{tr} \overleftarrow{\mathcal{P}} \oint ds_1 \mathcal{S}^{(2)}(s_1) e^{i \oint ds \mathcal{A}_0(s)} + \text{tr} \overleftarrow{\mathcal{P}} \oint ds_1 ds_2 \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(1)}(s_2) e^{i \oint ds \mathcal{A}_0(s)}. \end{aligned} \quad (3.10)$$

The $\mathcal{S}^{(n)}$ are related by the recursion relation

$$\mathcal{S}^{(n+1)} = \delta \mathcal{S}^{(n)} - i \delta x^\mu [A_\mu, \mathcal{S}^{(n)}] = \left(\delta x^\mu D_\mu + \delta \dot{x}^\mu \frac{\partial}{\partial \dot{x}^\mu} \right) \mathcal{S}^{(n)}. \quad (3.11)$$

We may use (3.9) to relate the expectation values $\langle W[\mathcal{C}_0] \rangle$ and $\langle W[\mathcal{C}] \rangle$. Using the double bracket notation (3.2) we have

$$\frac{\langle \delta W[\mathcal{C}_0] \rangle}{\langle W[\mathcal{C}_0] \rangle} = \oint ds_1 \langle\langle \mathcal{S}^{(1)}(s_1) \rangle\rangle, \quad (3.12)$$

$$\frac{\langle \delta^2 W[\mathcal{C}_0] \rangle}{\langle W[\mathcal{C}_0] \rangle} = \oint ds_1 \left(\langle\langle \mathcal{S}^{(2)}(s_1) \rangle\rangle + \oint ds_2 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(1)}(s_2) \rangle\rangle \right). \quad (3.13)$$

At higher orders we find

$$\begin{aligned} \frac{\langle \delta^3 W[\mathcal{C}_0] \rangle}{\langle W[\mathcal{C}_0] \rangle} &= \oint ds_1 \langle\langle \mathcal{S}^{(3)}(s_1) \rangle\rangle + \oint ds_1 ds_2 3 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(2)}(s_2) \rangle\rangle \\ &\quad + \oint ds_1 ds_2 ds_3 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(1)}(s_2) \mathcal{S}^{(1)}(s_3) \rangle\rangle \end{aligned} \quad (3.14)$$

$$\frac{\langle \delta^4 W[\mathcal{C}_0] \rangle}{\langle W[\mathcal{C}_0] \rangle} = \oint ds_1 \langle\langle \mathcal{S}^{(4)}(s_1) \rangle\rangle \quad (3.15)$$

$$\begin{aligned} &+ \oint ds_1 ds_2 (4 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(3)}(s_2) \rangle\rangle + 3 \langle\langle \mathcal{S}^{(2)}(s_1) \mathcal{S}^{(2)}(s_2) \rangle\rangle) \\ &+ \oint ds_1 ds_2 ds_3 6 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(1)}(s_2) \mathcal{S}^{(2)}(s_3) \rangle\rangle \\ &+ \oint ds_1 ds_2 ds_3 ds_4 \langle\langle \mathcal{S}^{(1)}(s_1) \mathcal{S}^{(1)}(s_2) \mathcal{S}^{(1)}(s_3) \mathcal{S}^{(1)}(s_4) \rangle\rangle, \end{aligned} \quad (3.16)$$

etc. We restrict our study to fourth order in the deformation.

3.2.2 Deformations of the straight line and explicit form of the operators

The operators $\mathcal{S}^{(i)}$ introduced in the previous section are linear combinations of basic insertions. In this section we first explain the explicit form of these sums of operators for general deformations of the 1/2 BPS Wilson line in $\mathbb{R}^{1,3}$ (signature $(+ - - -)$)¹. The 1/2 BPS line with spacetime contour $x^\mu(s) = (0, 0, 0, s)$ is given by

$$W_{\text{BPS}} = \text{tr} \overleftarrow{\mathcal{P}} \exp \left(- \oint ds [iA_3 - \Phi^1] \right). \quad (3.17)$$

The first combination of operators which appears in the expansion about the line is (3.12)

$$\mathcal{S}^{(1)} = -i\delta x^\mu \left(\dot{x}^\nu F_{\mu\nu} - \sqrt{\dot{x}^2} D_\mu \Phi^1 \right) + \frac{i\delta \dot{x} \cdot \dot{x}}{\sqrt{\dot{x}^2}} \Phi^1, \quad (3.18)$$

where $a \cdot b := \eta_{\mu\nu} a^\mu b^\nu$. The rest of the operators are given by the relation (3.11). By reparametrization of the loop, we can assume that the deformations take the form $\delta x^\mu(s) = (\epsilon^i(s), 0)$ where $i = 0, 1, 2$. $\epsilon = |\epsilon| = \sqrt{-\eta_{ij} \epsilon^i \epsilon^j} \ll 1$ serves as our expansion parameter, and we assume that $\frac{d^n}{ds^n} \epsilon(s) \sim \epsilon(s)$. To fourth order we then have

$$\begin{aligned} \mathcal{S}^{(1)} &= i\mathbb{F}_{i3} \epsilon^i, \\ \mathcal{S}^{(2)} &= iD_i \mathbb{F}_{j3} \epsilon^i \epsilon^j + iF_{ij} \epsilon^i \dot{\epsilon}^j - \Phi^1 |\dot{\epsilon}|^2, \\ \mathcal{S}^{(3)} &= iD_i D_j \mathbb{F}_{k3} \epsilon^i \epsilon^j \epsilon^k + 2iD_i F_{jk} \epsilon^i \epsilon^j \dot{\epsilon}^k - 3D_i \Phi^1 |\dot{\epsilon}|^2 \epsilon^i, \\ \mathcal{S}^{(4)} &= iD_i D_j D_k \mathbb{F}_{m3} \epsilon^i \epsilon^j \epsilon^k \epsilon^m + 3iD_i D_j F_{km} \epsilon^i \epsilon^j \epsilon^k \dot{\epsilon}^m \\ &\quad - 6D_i D_j \Phi^1 |\dot{\epsilon}|^2 \epsilon^i \epsilon^j + 3\Phi^1 |\dot{\epsilon}|^4. \end{aligned} \quad (3.19)$$

¹The internal space has completely negative signature. That is $\eta_{IJ} \Phi^I \Phi^J = -(\Phi^1)^2 - \dots - (\Phi^6)^2$ where $I, J = 1, 2, \dots, 6$.

Here we have introduced the notation $\mathbb{F}_{i3} = F_{i3} + iD_i\Phi^1$. Generally, the operators one encounters in the expansion are

$$\begin{aligned}\mathcal{S}_{\mathbb{F}}^{(n)} &= D_{i_1} \cdots D_{i_n} \mathbb{F}_{i_{n+1}3}, \\ \mathcal{S}_F^{(n)} &= D_{i_1} \cdots D_{i_n} F_{i_{n+1}j}, \\ \mathcal{S}_{\Phi}^{(n)} &= D_{i_1} \cdots D_{i_n} \Phi^1,\end{aligned}\tag{3.20}$$

with $n \geq 0$. Here all the indices, except for the j index in $\mathcal{S}_F^{(n)}$ are symmetrized, since these operators always appear contracted to ϵ^i (and ϵ^j). The last operator can appear with arbitrary powers of $|\epsilon|^2$.

At strong coupling there is an algorithm to study the minimal surface associated to small deformations of the straight Wilson line, as long as the deformations leave the contour $\mathcal{C}$ in $\mathbb{R}^2$ [99]. In that case the operator F_{ij} and its derivatives do not appear in the expansion. This is a direction of future work.

3.3 Operators representation and multiplets

The introduction of the 1/2 BPS line breaks the $PSU(2,2|4)$ symmetry of $\mathcal{N} = 4$ SYM to a $OSp(2,2|4)$ residual superconformal group. In this section we shall decompose the operator insertions found in the previous section into irreducible representations of this residual symmetry group and classify them in terms of their multiplets and supermultiplets. In previous works discussing the operator insertions into the straight line operator, this multiplet structure of the insertions has not been studied.

The 1/2 BPS line breaks the $SO(1,3)$ Lorentz group of SYM to $SO(1,2)_l$. We adopt the conventions of [53–55] for $\mathcal{N} = 4$ SYM. We now detail the conventions in terms of the residual symmetry group. We take the chiral form of the gamma matrices, *i.e.*

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \bar{\sigma}^{\mu,\dot{\alpha}\beta} & 0 \end{pmatrix},\tag{3.21}$$

where we choose $\sigma^\mu = (1, \tau^1, \tau^3, \tau^2)$ and τ^a are the Pauli matrices.² These split according to

$$\sigma^{\mu,\dot{\alpha}\beta} \rightarrow \{\sigma^{i,\alpha\beta}, -i\epsilon^{\alpha\beta},\} , \quad \bar{\sigma}^\mu_{\alpha\dot{\beta}} \rightarrow \{\sigma^i_{\alpha\beta}, i\epsilon_{\alpha\beta}\} .\tag{3.22}$$

Since we now have only one $SU(2)$ (more precisely $SO(1,2)_l$, there is no way to distinguish dotted and undotted indices and we can raise and lower them using $\epsilon^{\alpha\beta}$, arising from σ^3 .

The $SO(6)$ R-symmetry also breaks to $SO(5)$. Taking the $SO(6)$ gamma matrices

²And $\bar{\sigma}^\mu = (1, -\tau^1, -\tau^3, -\tau^2)$.

to also be in chiral form, in terms of $\rho^{A,ab}$ and $\bar{\rho}_{ab}^A$, which split as

$$\rho^{I,ab} \rightarrow \{\omega^{ab}, \rho^{A,ab}\}, \quad \bar{\rho}_{ab}^I \rightarrow \{\omega_{ab}, \rho_{ab}^A\}. \quad (3.23)$$

Again we can use the distinguished $\rho^1 = \omega$ to raise and lower indices according to $s^a = \omega^{ab}s_b$, $s_a = s^b\omega_{ba}$ and consequently $\omega^{ab}\omega_{bc} = -\delta^a_c$. Similarly, we raise and lower $SU(2)$ spinor indices as $s^\alpha = \epsilon^{\alpha\beta}s_\beta$ and $s_\alpha = s^\beta\epsilon_{\beta\alpha}$ with $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^\alpha_\gamma$.

The $OSP(2, 2|4)$ Poincaré supercharges are

$$\mathfrak{Q}_{a,\alpha}^+ = q_{a,\alpha} + \epsilon_{\alpha\beta}\omega_{ab}\bar{q}^{b,\beta} = q_{a,\alpha} + \bar{q}_{a,\alpha}, \quad (3.24)$$

where $q_{a,\alpha}$ and $\bar{q}^{a,\dot{\alpha}}$ are the usual $\mathcal{N} = 4$ SYM Poincaré supercharges. Given a superconformal primary operator, we find their superconformal descendants by acting with $\mathfrak{Q}^+$.

The scalar fields Φ^I (with $I = 1, \dots, 6$), having the lowest dimension, are primary operators in $\mathcal{N} = 4$ SYM. In terms of the residual symmetry group, we have Φ^A (with $A = 2, \dots, 6$) in the $\mathbf{5}$ of $SO(5)$ and Φ^1 in the $\mathbf{1}$. It is known that Φ^A is protected operator while Φ^1 is not [56, 61]. In the expansion (3.19) the only operator of classical scaling dimension one is the operator Φ^1 . For higher scaling dimension we have multiple operators in various representations and various multiplets - we now classify these.

3.3.1 Operators of classical dimension one

The only operator of classical dimension one in (3.19) is Φ^1 . This is an unprotected operator, and we thus expect to find its anomalous dimension in our two-loop analysis of the deformed Wilson loop.

The other operators in the theory of classical dimension one are Φ^A , in the $(\mathbf{1}, \mathbf{5})$ of $SO(1, 2)_I \times SO(5)$. They are protected and do not mix with Φ^1 . It does not arise in our expansion, but as we show below, some of its superconformal descendants do.

3.3.2 Operators of classical dimension two

In (3.19) the operators

$$\{i\mathbb{F}_{i3}, D_i\Phi^1, iF_{ij}\}, \quad (3.25)$$

are of classical scaling dimension two. The former is in the protected Φ^A supermultiplet, which is easily seen, as follows: acting on the primary Φ^A with the supercharges (3.24) we find

$$\mathfrak{Q}_{a,\alpha}^+ \Phi^A = i\bar{\rho}_{ab}^A \lambda_{\alpha}^{-,b}, \quad (3.26)$$

where $\lambda_{\alpha}^{\pm,a} = \lambda_{\alpha}^a \pm \bar{\lambda}_{\alpha}^a$ are the linear combinations of SYM fermions λ_{α}^a and $\bar{\lambda}_{\alpha}^{\dot{a}}$. Acting again with $\mathfrak{Q}^+$ gives

$$\mathfrak{Q}_{a,\alpha}^+ \lambda_{\beta}^{-,b} = 2i\delta_a^b \bar{\sigma}_{\alpha\beta}^i \mathbb{F}_{i3} + 2i\epsilon_{\alpha\beta} \omega^{bc} \bar{\rho}_{ca}^A \mathbb{D}_3 \Phi^A, \quad (3.27)$$

where

$$i\mathbb{A}_3 = iA_3 - \Phi^1, \quad (3.28)$$

and the covariant derivative $\mathbb{D}_3$ is defined with respect to this modified connection. We thus see that $i\mathbb{F}_{i3}$ is a superconformal descendant of Φ^A (and a conformal primary). The conformal descendant of Φ^A of scaling dimension two is given by the second operator in (3.27), $\mathbb{D}_3 \Phi^A$, which does not appear in (3.25).

The second operator in (3.25) is $D_i \Phi^1$. Recalling that $i\mathbb{F}_{i3} = iF_{i3} - D_i \Phi^1$ we see that $D_i \Phi^1$ is a linear combination of $i\mathbb{F}_{i3}$ and its orthogonal³ operator $i\tilde{\mathbb{F}}_{i3} = iF_{i3} + 2D_i \Phi^1$. Thus

$$D_i \Phi^1 = \frac{1}{3} \left(i\tilde{\mathbb{F}}_{i3} - i\mathbb{F}_{i3} \right). \quad (3.29)$$

$i\tilde{\mathbb{F}}_{i3}$ is in neither the Φ^A nor the Φ^1 multiplet.

Finally, we have iF_{ij} , also in the $\mathbf{3}$ of $SO(1,2)_l$. This is in the Φ^1 supermultiplet. Acting with $\mathfrak{Q}^+$ on Φ^1 gives

$$\mathfrak{Q}_{a,\alpha}^+ \Phi^1 = i\lambda_{a,\alpha}^+, \quad (3.30)$$

and acting again with $\mathfrak{Q}^+$ gives

$$\begin{aligned} \mathfrak{Q}_{a,\alpha}^+ \lambda_{\beta}^{+,b} &= 2i\epsilon_{\alpha\beta} \delta_a^b \mathbb{D}_3 \Phi^1 - 2\omega^{bc} \bar{\rho}_{ca}^A \bar{\sigma}_{\alpha\beta}^i D_i \Phi^A \\ &\quad + (\sigma^{ij})_{\beta}^{\gamma} F_{ij} \epsilon_{\alpha\gamma} \delta_a^b + i\epsilon_{\alpha\beta} \rho^{A,bc} \bar{\rho}_{ca}^B [\Phi^A, \Phi^B]. \end{aligned} \quad (3.31)$$

Projecting onto the $(\mathbf{3}, \mathbf{1})$ of $SO(1,2)_l \times SO(5)$ gives F_{ij} .

3.3.3 Operators of classical dimension three

The operators of classical scaling dimension three in (3.19) are

$$\{iD_i \mathbb{F}_{j3}, iD_i F_{jk}\}, \quad (3.32)$$

Both of the operators above are in reducible representations of $SO(1,2)_l \times SO(5)$. The former reduces to the trace $(\mathbf{1}, \mathbf{1})$ and the traceless symmetric $(\mathbf{5}, \mathbf{1})$. The latter reduces to the $(\mathbf{3}, \mathbf{1})$ and the $(\mathbf{5}, \mathbf{1})$. Recall that we only have symmetric representations, as operators are contracted with $\epsilon^{i_1} \epsilon^{i_2} \dots \epsilon^{i_n}$, with the exception of iF_{ij} (and its derivatives) which is contracted with $\epsilon^{i_1} \dots \epsilon^{i_{n-1}} \epsilon^{i_n}$.

³Here orthogonality refers to the two-point functions.

Both $iD^i\mathbb{F}_{i3}$ and iD^iF_{ij} are related to descendants by the equations of motion:

$$\begin{aligned} iD^i\mathbb{F}_{i3} &= D^3D_3\Phi^1 + [D_3\Phi^1, \Phi^1] + [D_3\Phi^A, \Phi^A] \\ &\quad + [[\Phi^1, \Phi^A], \Phi^A] + \frac{1}{2} [\lambda_{a,\alpha}^+, \lambda^{-a,\alpha}] \\ &= -\mathbb{D}_3\mathbb{D}_3\Phi^1 + [\mathbb{D}_3\Phi^A, \Phi^A] + \frac{1}{2} [\lambda_{a,\alpha}^+, \lambda^{-a,\alpha}] . \end{aligned} \quad (3.33)$$

As can be seen in this equation, the operators of dimension three can mix with operators made of fermions, which does not happen for operators of lower dimension. We note that the tree-level two-point functions of the latter two operators contribute to the expectation value $\langle W[C] \rangle$ at two-loop order. Similarly, for iD^iF_{ij} , we see from the equations of motion that

$$iD^iF_{ij} = -\frac{1}{3}\mathbb{D}_3 \left(2i\mathbb{F}_{i3} + i\tilde{\mathbb{F}}_{i3} \right) + (\text{bilinears}) , \quad (3.34)$$

where we have suppressed bilinear composite operators which do not contribute to our computation.

$iD_{\{i}\mathbb{F}_{j\}3}$ is also in the Φ^1 supermultiplet (but not the Φ^1 multiplet). Projecting onto the $(\mathbf{5}, \mathbf{1})$ of $SO(1,2)_l \times SO(5)$, we have

$$\epsilon^{abcd} \bar{\sigma}_{\{i}^{\alpha\beta} \bar{\sigma}_{j\}^{\gamma\delta} \mathfrak{Q}_{a,\alpha}^+ \mathfrak{Q}_{b,\beta}^+ \mathfrak{Q}_{c,\gamma}^+ \mathfrak{Q}_{d,\delta}^+ \Phi^1 = -1024 D_{\{i}\mathbb{F}_{j\}3} . \quad (3.35)$$

3.3.4 Operators of classical dimension four

The operator of classical scaling dimension four in (3.19) is

$$\{iD_iD_j\mathbb{F}_{k3}\} . \quad (3.36)$$

From the equations of motion, one can see that the irreducible representation $iD_iD^j\mathbb{F}_{j3}$ is related to the descendant $i\mathbb{D}_3\mathbb{D}_3\mathbb{F}_{i3}$, via

$$iD^iF_{ij} = -\frac{1}{3}\mathbb{D}_3 \left(2i\mathbb{F}_{j3} + i\tilde{\mathbb{F}}_{j3} \right) + [\Phi^1, i\mathbb{F}_{j3}] + [D_j\Phi^A, \Phi^A] + \frac{1}{2}\sigma_j^{\alpha\beta} [\lambda_{a,\alpha}^+, \lambda_{\beta}^{a,-}] . \quad (3.37)$$

The $iD^iD_{(i}\mathbb{F}_{j3)}$ operator is similarly related by the equations of motion to a conformal descendant of $\mathbb{F}_{i3}$.

We summarize the results of this section in the Table 3.1.

3.4 Extracting the CFT data

We now use the equivalence between the operator insertions and small deformations of the 1/2 BPS Wilson line to extract the anomalous dimensions, as well as two- and three-point functions coefficients of the relevant operator insertions at weak

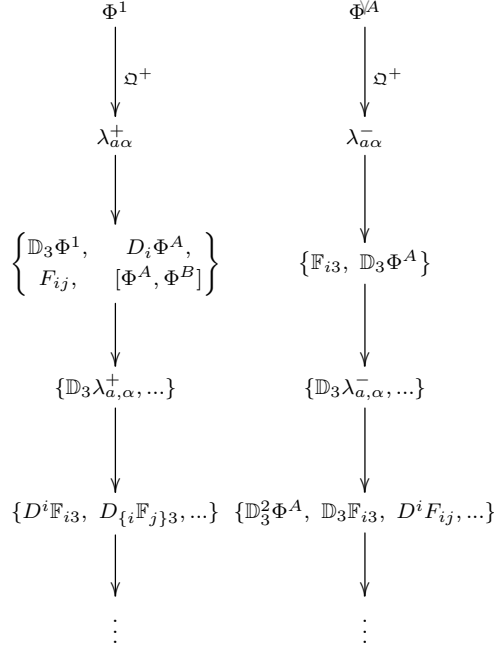


Figure 3.1: **Supermultiplet structure of the operator insertions:** The diagram shows how the super-multiplets are constructed, starting with the primary fields Φ^1 and Φ^A and acting with the unbroken supercharges $\Omega_{\alpha,a}^+$ to generate the super-descendants. We present only a small portion of the diagram which is relevant to our discussion. The ellipsis represent more fields which we do not write explicitly. We write the fields in their irreducible representations under the residual symmetry group.

coupling. The idea is to use the known perturbative expressions for a general Wilson loop at one and two-loops, then expand these expressions for a small deformation and rewrite it in the form of a sum of n -point functions. Exploiting the equivalence between deformations of Wilson loops and operator insertions in Wilson loops, we then extract the data.

At one-loop order the Wilson loop is sensitive only to two point functions. In particular, the one-loop expansion of the Wilson loop is sensitive only to the *classical* scaling dimension of the operator insertions and their tree-level two-point function coefficients⁴. At two-loops things get more interesting, with the Wilson loop sensitive to two-, three- and four-point functions. The two-loop expansion of the Wilson loop will supply the *one-loop* two-point functions coefficient and anomalous dimension, as well as the *classical* three-point function structure constant. The four-point function at this loop order is factorized and supplies no new information, just the classical dimensions already found before.

There are no expressions for generic Wilson loops at higher loop order, so any higher loop expressions cannot be found at the moment using this method. As noted in Section 1.2, there exist methods to compute the expectation values of deformations (in $\mathbb{R}^2$) of the straight line to leading order in strong coupling [99]. As

⁴We note that this is not true for composite operators, which we discuss later.

such, it may be possible to extend this analysis to that regime. It should also be possible using our present technology to go to higher orders in ϵ .

3.4.1 The one- and two-loop expectation values for general contours

At one-loop order, the expectation value of a general smooth contour is given by the well-known expression

$$\begin{aligned}\langle W[\mathcal{C}] \rangle_{1\text{-loop}} &= -\frac{\lambda}{16\pi^2} \oint ds_1 ds_2 \frac{\dot{x}_1 \cdot \dot{x}_2 - \sqrt{\dot{x}_1^2} \sqrt{\dot{x}_2^2}}{x_{12}^2} \\ &=: -\frac{\lambda}{16\pi^2} \oint ds_1 ds_2 I(s_1, s_2).\end{aligned}\tag{3.38}$$

At two-loop order, there is an analogous expression for a general smooth contour with a constant scalar coupling. The sum of all two-loop Feynman diagrams in the planar approximation can be combined to the elegant expression [60]

$$\begin{aligned}\langle W[\mathcal{C}] \rangle_{2\text{-loop}} &= -\frac{\lambda^2}{128\pi^4} \oint ds_1 ds_2 ds_3 \epsilon(s_1, s_2, s_3) I(s_1, s_3) \frac{x_{32} \cdot \dot{x}_2}{x_{32}^2} \log \frac{x_{21}^2}{x_{31}^2} \\ &\quad + \frac{\lambda^2}{2} \left(\frac{1}{16\pi^2} \oint ds_1 ds_2 I(s_1, s_2) \right)^2 \\ &\quad - \frac{\lambda^2}{64\pi^4} \int_{s_1 > s_2 > s_3 > s_4} ds_1 ds_2 ds_3 ds_4 I(s_1, s_3) I(s_2, s_4).\end{aligned}\tag{3.39}$$

where $\epsilon(s_1, s_2, s_3)$ is completely antisymmetric and takes value 1 for $s_1 > s_2 > s_3$.

Divergences and regularization

The one- and two-loop integrals above are finite. However, if we write the integrand as a sum of terms and integrate each term separately, then each integral is separately divergent. For example, in the case of the one-loop integrand

$$\begin{aligned}& -\frac{\lambda}{8\pi^2} \oint_{s_1 > s_2} ds_1 ds_2 \frac{\dot{x}_1 \cdot \dot{x}_2 - \sqrt{\dot{x}_1^2} \sqrt{\dot{x}_2^2}}{x_{12}^2} \\ &= \frac{\lambda}{8\pi^2} \left(\oint_{s_1 > s_2} ds_1 ds_2 \frac{\sqrt{\dot{x}_1^2} \sqrt{\dot{x}_2^2}}{x_{12}^2} - \oint_{s_1 > s_2} ds_1 ds_2 \frac{\dot{x}_1 \cdot \dot{x}_2}{x_{12}^2} \right),\end{aligned}\tag{3.40}$$

where each term on the right hand side is divergent. If, for example, we wish to integrate the second term by parts then we must introduce a regulator to keep track of the divergences. This is an example of the type of manipulation we must perform in order to express the one- and two-loop results as the sum of two-, three- and (factorized) four-point functions of operator insertions.

We consider two types of regulation: a cutoff and ‘mass’ terms. We firstly illus-

trate these regulation schemes for the second order in the deformation at one- and two-loops. We subsequently present the results for the third and fourth orders in the deformation for the cutoff regularization and comment on results for the ‘mass’ regularization for deformations in $\mathbb{R}^2$.

Before proceeding to present the results, we note that while there our results match a number of previously known results, there are certain inconsistencies in what we find. We highlight these as we present our findings. We now proceed by presenting the results for the second order in the deformation.

3.4.2 Order ϵ^2 : cutoff

At first order in the deformation, equation (3.12) includes only the one-point function of $i\mathbb{F}_{i3}$. This vanishes, as discussed earlier, and we therefore start our discussion at order ϵ^2 , also known as the wavy line approximation, which is well studied and somewhat special [52]. We now re-derive the known results to illustrate our approach.

Denoting by $\langle W[\mathcal{C}] \rangle|_{\epsilon^2}$ the contributions of order $\mathcal{O}(\epsilon^2)$ to the expectation value $\langle W[\mathcal{C}] \rangle$, the relevant terms from the expansion (3.12), (3.13) with the explicit operators (3.19) is

$$\langle W[\mathcal{C}] \rangle|_{\epsilon^2} = \frac{1}{2!} \oint ds_1 ds_2 \epsilon_1^i \epsilon_2^j \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) \rangle\rangle. \quad (3.41)$$

The two-point functions are constrained (since $i\mathbb{F}_{i3}$ is a conformal primary) to take the form

$$\langle W[\mathcal{C}] \rangle|_{\epsilon^2} = \oint_{s_1 > s_2} ds_1 ds_2 \frac{a_F(\lambda)}{s_{12}^{2\Delta_F(\lambda)}} \eta_{ij} \epsilon_1^i \epsilon_2^j, \quad (3.42)$$

where $a_F(\lambda)$ is the two-point coefficient of $i\mathbb{F}_{i3}$ and $\Delta_F(\lambda)$ is its scaling dimension. (3.42) is true for all values of the coupling.

1-loop

To match with the perturbative expansion of the Wilson loop, we should expand the all-coupling expression (3.42) at weak coupling. The $\mathcal{O}(\lambda)$ term is

$$\langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = \lambda \oint_{s_1 > s_2} ds_1 ds_2 \frac{a_F^0}{s_{12}^{2\Delta_F^0}} \eta_{ij} \epsilon_1^i \epsilon_2^j, \quad (3.43)$$

where a_F^0 is the tree-level two-point function coefficient and Δ_F^0 is the classical scaling dimension. In this case clearly $\Delta_F^0 = 2$, and it thus remains to determine a_F^0 .

To do this, we compare the above expression with (3.38) for the contour $\mathcal{C}$ defined

by $x^\mu = (\epsilon^i(s), s)$ and expand to order $\mathcal{O}(\epsilon^2)$

$$\langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = \frac{\lambda}{16\pi^2} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_2 \frac{|\dot{\epsilon}_1 - \dot{\epsilon}_2|^2}{s_{12}^2}. \quad (3.44)$$

where $|a| = \sqrt{-\eta_{ij}a^ia^j}$. We now split this integral into a sum of integrals, and introduce a cutoff parameter, μ :

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = & \frac{\lambda}{16\pi^2} \left(\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{|\dot{\epsilon}_1|^2}{s_{12}^2} \right. \\ & \left. + 2 \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{\dot{\epsilon}_1 \cdot \dot{\epsilon}_2}{s_{12}^2} + \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{|\dot{\epsilon}_2|^2}{s_{12}^2} \right). \end{aligned} \quad (3.45)$$

We may perform the integration in s_2 and s_1 for the first and third terms, respectively, without knowledge of the precise form of $\epsilon^i(s)$. These both lead to purely divergent terms, which we do not consider henceforth. We integrate the second term by parts to find:

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = & -\frac{3\lambda}{4\pi^2} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{\epsilon_1 \cdot \epsilon_2}{s_{12}^4} \\ & + \frac{\lambda}{8\pi^2} \int_{-\infty}^{\infty} ds_1 \frac{\dot{\epsilon}(s_1) \cdot \epsilon(s_1 - \mu)}{\mu^2} + \frac{\lambda}{4\pi^2} \int_{-\infty}^{\infty} ds_1 \frac{\epsilon(s_1 + \mu) \cdot \epsilon(s_1)}{\mu^3}. \end{aligned} \quad (3.46)$$

We take $\epsilon^i(s)$ to be analytic in s and may thus perform a Taylor expansion, in the cutoff μ , of the final two integrands. Neglecting terms of order μ and higher, as well as pure divergences and total derivatives, we find

$$\langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = -\frac{3\lambda}{4\pi^2} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{\epsilon_1 \cdot \epsilon_2}{s_{12}^4}. \quad (3.47)$$

We compare this with the corresponding terms in (3.43). The first term is the tree-level contribution of the two-point of $i\mathbb{F}_{i3}$, from which we may extract the tree-level two-point function coefficient:

$$a_F^0 = -\frac{3}{4\pi^2}. \quad (3.48)$$

This matches the result of [52], up to a difference in sign due to our spacetime signature.

2-loop

At order $\mathcal{O}(\lambda^2)$, equation (3.42) is

$$\langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = \lambda^2 \oint_{s_1 > s_2} ds_1 ds_2 \frac{a_F^1 - 2a_F^0 \gamma_F \log \frac{s_{12}}{\xi}}{s_{12}^{2\Delta_F^0}} \eta_{ij} \epsilon_1^i \epsilon_2^j, \quad (3.49)$$

where a_F^1 is the one-loop two-point function coefficient of $i\mathbb{F}_{i3}$, γ_F is the one-loop anomalous dimension of $i\mathbb{F}_{i3}$ and ξ is a scale parameter. Expanding (3.39) to second order in ϵ gives

$$\langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = -\frac{\lambda^2}{256\pi^4} \oint ds_1 ds_2 ds_3 \epsilon(s_1, s_2, s_3) \frac{|\dot{\epsilon}_1 - \dot{\epsilon}_3|^2 \log\left(\frac{s_{12}^2}{s_{13}^2}\right)}{s_{13}^2 s_{23}}. \quad (3.50)$$

We restrict the domain of integration to $s_1 > s_2 > s_3$ and symmetrize the integrand accordingly. We also introduce two independent cutoffs μ_1 and μ_2

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = & \frac{\lambda^2}{256\pi^4} \left(- \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1} ds_2 \int_{-\infty}^{s_2-\mu_2} ds_3 \frac{|\dot{\epsilon}_1|^2 \log\left(\frac{s_{12}^2}{s_{13}^2}\right)}{s_{13}^2 s_{23}} \right. \\ & - \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1} ds_2 \int_{-\infty}^{s_2-\mu_2} ds_3 \frac{|\dot{\epsilon}_3|^2 \log\left(\frac{s_{12}^2}{s_{13}^2}\right)}{s_{13}^2 s_{23}} \\ & - 2 \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1} ds_2 \int_{-\infty}^{s_2-\mu_2} ds_3 \frac{\dot{\epsilon}_1 \cdot \dot{\epsilon}_3 \log\left(\frac{s_{12}^2}{s_{13}^2}\right)}{s_{13}^2 s_{23}} \Bigg) \\ & + (\text{symmetrization}). \end{aligned} \quad (3.51)$$

In the rest of this section, unless otherwise indicated, we explicitly write the steps of the calculation only for the terms explicitly written above. Analogous steps are implicit for the symmetrized terms, and the results of which denoted by “(symmetrization)”. For the third term above, we may perform the s_2 integral and we have (dropping the divergences for presentational purposes)

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} &= \frac{\lambda^2}{128\pi^4} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1-\mu_2} ds_3 \frac{\dot{\epsilon}_1 \cdot \dot{\epsilon}_3}{s_{13}^2} (-2 \log \mu_1 \log(s_{13} - \mu_1) \\ &+ 2 \log s_{13} [\log(s_{13} - \mu_1) + \log \mu_1]) \\ &+ \frac{\lambda^2}{128\pi^4} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1-\mu_2} ds_3 \frac{\dot{\epsilon}_1 \cdot \dot{\epsilon}_3}{s_{13}^2} (2 \log s_{13} [-\log(s_{13} - \mu_2) - \log \mu_2] \\ &+ 2 \log \mu_2 \log(s_{13} - \mu_2) - 2 \text{Li}_2\left(\frac{\mu_2}{s_{13}}\right) + 2 \text{Li}_2\left(1 - \frac{\mu_2}{s_{13}}\right)) \\ &+ (\text{symmetrization}), \\ &=: \lambda^2 \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1-\mu_2} ds_3 (f(s_1, s_2; \mu_1) \dot{\epsilon}_1 \cdot \dot{\epsilon}_3 + g(s_1, s_2; \mu_2) \dot{\epsilon}_1 \cdot \dot{\epsilon}_3). \end{aligned} \quad (3.52)$$

We observe that f and g above depend on μ_1 and μ_2 , respectively, but not on both. If a term depends on μ_1 then we shift s_1 such that the domain of integration is independent of μ_1 , and similarly for terms which depend on μ_2 . The result of these

shifts is

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = & \lambda^2 \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_2} ds_3 f(s_1 + \mu_1, s_3; \mu_1) \dot{\epsilon}(s_1 + \mu_1) \cdot \dot{\epsilon}_3 \\ & + \lambda^2 \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1} ds_3 g(s_1 + \mu_2, s_3; \mu_2) \dot{\epsilon}(s_1 + \mu_2) \cdot \dot{\epsilon}_3. \end{aligned} \quad (3.53)$$

At this point, we re-emphasize the goal of this section: we wish to manipulate (3.53) into the form of a sum of two-, three- and (factorized) four-point functions of operator insertions (plus purely divergent terms). We do so via integration-by-parts. Performing this integration by parts, we find

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = & -\lambda^2 \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_2} ds_3 \epsilon(s_1 + \mu_1) \cdot \epsilon_3 \left(\frac{d^2}{dt_1 ds_3} f(t_1, s_3; \mu_1) \right) \Big|_{t_1=s_1+\mu_1} \\ & - \lambda^2 \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1} ds_3 \epsilon(s_1 + \mu_2) \cdot \epsilon_3 \left(\frac{d^2}{dt_1 ds_3} g(t_1, s_3; \mu_2) \right) \Big|_{t_1=s_1+\mu_2}. \end{aligned} \quad (3.54)$$

We have neglected the boundary terms here. Boundary terms give rise to divergences or total derivatives. For example, performing the integration by parts in s_3 in (3.52) the boundary terms cancel. Performing the integration by parts in s_1 of the terms explicitly written in (3.52) gives (after Taylor expanding in μ_1 and μ_2)

$$\begin{aligned} & -\lambda^2 \log \frac{\mu_1}{\mu_2} \oint ds_1 \left(-\frac{|\epsilon_1|^2}{64\pi^4 \mu_2^3} + \frac{\epsilon_1 \cdot \dot{\epsilon}_1}{64\pi^4 \mu_2^2} - \frac{\epsilon_1 \cdot \ddot{\epsilon}_1}{128\pi^4 \mu_2} + \frac{\epsilon_1 \cdot \ddot{\ddot{\epsilon}}_1}{384\pi^4} \right) \\ & + (\text{symmetrization}). \end{aligned} \quad (3.55)$$

The first and third terms give divergences, while the second and fourth are total derivatives. We see that we thus have no finite single integrals, in keeping with the operator insertion picture for which we have no one-point functions.

We return to the double integrals in (3.54), and Taylor expand the integrands in μ_1 and μ_2 . We neglect terms of order μ_1, μ_2 or higher to find

$$\begin{aligned} \langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = & -\frac{\lambda^2}{32\pi^4} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_2} ds_3 \frac{\epsilon_1^i \cdot \epsilon_2^j}{s_{13}^4} [1 - 5 \log s_{13} + 3(\log s_{13})^2] \\ & - \frac{\lambda^2}{64\pi^4} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu_1} ds_3 \frac{\epsilon_1^i \cdot \epsilon_2^j}{s_{13}^4} [-2 + \pi^2 + 10 \log s_{13} - 6(\log s_{13})^2] \\ & + (\text{symmetrization}). \end{aligned} \quad (3.56)$$

Finally, identifying $\mu_1 = \mu_2 =: \mu$ in these integrals, we find

$$\langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = \frac{\lambda^2}{32\pi^2} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{\epsilon_1^i \cdot \epsilon_2^j}{s_{12}^4}, \quad (3.57)$$

Where we have included all terms appearing in (3.51). Comparison with (3.49) gives

$$a_F^1 = \frac{1}{32\pi^2}, \quad \gamma_F = 0. \quad (3.58)$$

This two-point function coefficient and anomalous dimension agrees with the results of [52] and the fact that $i\mathbb{F}_{i3}$ is a protected operator.

3.4.3 Order ϵ^2 : ‘masses’

An alternative regularization scheme is the introduction of a small imaginary ‘mass’ piece into the propagator-like terms $1/s_{ij}$. At one- and two-loops, the expectation value of the Wilson loop at order ϵ^n is given by a sum of double, triple and quartic integrals. The integrands are functions of the form $b(s_{ij}; \epsilon_k, \dot{\epsilon}_m)$. That is, the s_i dependence of the integrands is completely encoded in the deformation ϵ_i and its derivative, as well as the differences s_{ij} . We replace $s_{ij} \rightarrow s_{ij} + i\zeta_{ij}$ and use these ζ_{ij} ’s to regularize the integrals.

One-loop

In this regularization scheme (3.44) becomes

$$\langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = \frac{\lambda}{16\pi^2} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_2 \frac{|\dot{\epsilon}_1 - \dot{\epsilon}_2|^2}{(s_{12} + i\zeta_{12})^2}. \quad (3.59)$$

As for the cutoff regularization scheme, we split the integral into a sum of three integrals. We evaluate the integrals which are possible to perform without specific knowledge of the deformation and perform integration by parts on the remaining terms to bring it to the form of an operator insertion two-point function. This yields

$$\begin{aligned} & \langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} \\ &= \frac{i\lambda}{16\pi^2} \oint ds_1 \left(\frac{\dot{\epsilon}_1^2}{\zeta_{12}} + \frac{4\epsilon_1^2}{\zeta_{12}^3} \right) - \frac{3\lambda}{4\pi^2} \oint_{s_1 > s_2} ds_1 ds_2 \frac{\epsilon_1 \cdot \epsilon_2}{(s_{12} + i\zeta_{12})^4}, \end{aligned} \quad (3.60)$$

where we have thrown away a total derivative. From the last term, we read off the tree-level normalization of the two-point function (3.48). This is from comparison with (3.43) with mass regularization, *i.e.*

$$\langle W[\mathcal{C}] \rangle_{1\text{-loop}}|_{\epsilon^2} = \lambda \oint_{s_1 > s_2} ds_1 ds_2 \frac{a_F^0}{(s_{12} + i\zeta_{12})^{2\Delta_F^0}} \eta_{ij} \epsilon_1^i \epsilon_2^j. \quad (3.61)$$

2-loops

At two-loops we have

$$\begin{aligned}
& \langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} \\
&= \frac{\lambda^2}{256\pi^4} \left(- \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_2 \int_{-\infty}^{s_2} ds_3 \frac{|\dot{\epsilon}_1|^2 \log \left(\frac{(s_{12}+i\zeta_{12})^2}{(s_{13}+i\zeta_{13})^2} \right)}{(s_{13}+i\zeta_{13})^2 (s_{23}+i\zeta_{23})} \right. \\
&\quad - \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_2 \int_{-\infty}^{s_2} ds_3 \frac{|\dot{\epsilon}_3|^2 \log \left(\frac{(s_{12}+i\zeta_{12})^2}{(s_{13}+i\zeta_{13})^2} \right)}{(s_{13}+i\zeta_{13})^2 (s_{23}+i\zeta_{23})} \\
&\quad \left. - 2 \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_2 \int_{-\infty}^{s_2} ds_3 \frac{\dot{\epsilon}_1 \cdot \dot{\epsilon}_3 \log \left(\frac{(s_{12}+i\zeta_{12})^2}{(s_{13}+i\zeta_{13})^2} \right)}{(s_{13}+i\zeta_{13})^2 (s_{23}+i\zeta_{23})} \right) \\
&\quad + (\text{symmetrization}).
\end{aligned} \tag{3.62}$$

Here we understand that under symmetrization we preserve the combination $s_{ij} + i\zeta_{ij}$.

The first and second terms are divergent. We may perform the integration with respect to s_2 in the third term and subsequently integrate by parts in s_1 and s_3 . Having performed the aforementioned integration by parts in s_1 and s_3 , we Taylor expand the first term in ζ_{12} and the second in ζ_{23} ; these are the masses which regulate the divergences in the respective integrals. We keep the finite terms in both of these expansions. As per the cutoff method, the expression we wish to compare the result to is

$$\langle W[\mathcal{C}] \rangle_{2\text{-loop}}|_{\epsilon^2} = \lambda^2 \oint_{s_1 > s_2} ds_1 ds_2 \frac{a_F^1 - 2a_F^0 \gamma_F \log \frac{s_{12}}{\xi}}{(s_{12} + i\zeta_{12})^{2\Delta_F^0}} \eta_{ij} \epsilon_1^i \epsilon_2^j. \tag{3.63}$$

We identified $\mu_1 = \mu_2 =: \mu$ in the cutoff method (see (3.57)) when we came to the two point functions. The analog here is identifying $\zeta_{13} = \zeta_{23} =: \zeta$ in the first term in (3.62) and $\zeta_{12} = \zeta_{13} \equiv \zeta$ in the second. Doing this, we obtain

$$a_F^1 = \frac{1}{32\pi^2}, \quad \gamma_F = 0, \tag{3.64}$$

as per the cutoff method.

3.4.4 Order ϵ^3

At $\mathcal{O}(\epsilon^3)$, the expectation value of the deformed loop is given in terms of the expectation values of operator insertion two- and three-point functions by substituting (3.19) into (3.14). The resulting expression involves the following correlation functions:

$$\begin{aligned}
& \langle\langle i\mathbb{F}_{i3}(s_1) iD^j \mathbb{F}_{j3}(s_2) \rangle\rangle, \quad \langle\langle i\mathbb{F}_{i3}(s_1) iD_{\{j} \mathbb{F}_{k\}3}(s_2) \rangle\rangle, \quad \langle\langle i\mathbb{F}_{i3}(s_1) iF_{jk}(s_2) \rangle\rangle, \\
& \langle\langle i\mathbb{F}_{i3}(s_1) \Phi^1(s_2) \rangle\rangle, \quad \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) i\mathbb{F}_{k3}(s_3) \rangle\rangle.
\end{aligned} \tag{3.65}$$

From the analysis in Section 3.3, we see that all two-point functions involve pairs of operators from different supermultiplets, and we would therefore expect them to vanish. The last expression, which is the three-point function of $\mathbb{F}_{i3}$ also vanishes, because the only invariant three-tensor is antisymmetric.

Indeed the perturbative expressions at one-loop (3.38) and two-loop (3.39) do not contain any terms at $\mathcal{O}(\epsilon^3)$.

3.4.5 Order ϵ^4

At $\mathcal{O}(\epsilon^4)$ there are contributions from two-, three- and four-point functions. The operator expansion, written in terms of irreducible representations, *i.e.* (3.16), is quite long and messy at this order. As such we deal with the four-, three- and two-point functions separately. We further deal with the two-point functions of the different supermultiplets separately. The results given were obtained in the cutoff regularization scheme. For the case of a deformation in $\mathbb{R}^2$, we have also performed the computation with the ‘mass’ regularization, which we comment also discuss.

The four-point functions

The sole four-point function appearing in the operator expansion at $\mathcal{O}(\epsilon^4)$ is

$$\oint_{s_1 > s_2 > s_3 > s_4} ds_1 ds_2 ds_3 ds_4 \ll i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) i\mathbb{F}_{k3}(s_3) i\mathbb{F}_{m3}(s_4) \gg \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_4^m, \quad (3.66)$$

where the coefficient is exactly as it appears in the operator expansion. At two-loops (tree-level in terms of the operator insertions) this four-point function factorizes:

$$\begin{aligned} & \oint_{s_1 > s_2 > s_3 > s_4} ds_1 ds_2 ds_3 ds_4 \ll i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) i\mathbb{F}_{k3}(s_3) i\mathbb{F}_{m3}(s_4) \gg \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_4^m = \\ & - \oint_{s_1 > s_2 > s_3 > s_4} ds_1 ds_2 ds_3 ds_4 \ll i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{k3}(s_3) \gg \ll i\mathbb{F}_{j3}(s_2) i\mathbb{F}_{m3}(s_4) \gg \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_4^m \\ & + \frac{1}{2} \left(\oint ds_1 ds_2 \ll i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) \gg \epsilon_1^i \epsilon_2^j \right)^2. \end{aligned} \quad (3.67)$$

We have already extracted the two-point function coefficient of $i\mathbb{F}_{i3}$ from the results for the second order in the expansion (see (3.48)). We therefore obtain no new information about the operator insertions from these terms in the operator expansion. We can however, extract the tree-level two-point function coefficient here and compare it to (3.48). Doing this, we find agreement between the two methods (and also between the two different regularization schemes).

The three-point functions

Using the results of Section 3.2 we find that the contribution of the three-point functions in the operator expansion is

$$\begin{aligned}
& \oint_{s_1 > s_2 > s_3} ds_1 ds_2 ds_3 \left(\frac{1}{2} \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) iD_{\{k} \mathbb{F}_{m\}3} \rangle\rangle \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_3^m \right. \\
& + \frac{1}{2} \langle\langle i\mathbb{F}_{i3}(s_1) iD_{\{j} \mathbb{F}_{k\}3}(s_2) i\mathbb{F}_{m3} \rangle\rangle \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_3^m + \frac{1}{2} \langle\langle iD_{\{i} \mathbb{F}_{j\}3}(s_1) i\mathbb{F}_{k3}(s_2) i\mathbb{F}_{m3} \rangle\rangle \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_3^m \\
& + \frac{1}{6} \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) iD^n \mathbb{F}_{n3} \rangle\rangle \eta_{km} \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_3^m + \frac{1}{6} \langle\langle i\mathbb{F}_{i3}(s_1) iD^n \mathbb{F}_{n3}(s_2) i\mathbb{F}_{m3} \rangle\rangle \eta_{jk} \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_3^m \\
& + \frac{1}{6} \langle\langle iD^n \mathbb{F}_{n3}(s_1) i\mathbb{F}_{k3}(s_2) i\mathbb{F}_{m3} \rangle\rangle \eta_{ij} \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_3^m + \frac{1}{2} \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) iF_{km} \rangle\rangle \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_3^m \\
& + \frac{1}{2} \langle\langle i\mathbb{F}_{i3}(s_1) iF_{jk}(s_2) i\mathbb{F}_{m3} \rangle\rangle \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_3^m + \frac{1}{2} \langle\langle iF_{ij}(s_1) i\mathbb{F}_{k3}(s_2) i\mathbb{F}_{m3} \rangle\rangle \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_3^m \\
& + \frac{1}{2} \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) \Phi^1(s_3) \rangle\rangle \eta_{km} \epsilon_1^i \epsilon_2^j \epsilon_3^k \epsilon_3^m + \frac{1}{2} \langle\langle i\mathbb{F}_{i3}(s_1) \Phi^1(s_2) i\mathbb{F}_{m3}(s_3) \rangle\rangle \eta_{jk} \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_3^m \\
& \left. + \frac{1}{2} \langle\langle \Phi^1(s_1) i\mathbb{F}_{k3}(s_2) i\mathbb{F}_{m3}(s_3) \rangle\rangle \eta_{ij} \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_3^m \right). \tag{3.68}
\end{aligned}$$

Up to symmetrization, we have four different three point functions to consider. As discussed in Section 3.3, both $i\mathbb{F}_{i3}$ and Φ^1 are conformal primaries. Applying the constraints of the residual conformal symmetry, we see that

$$\langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) \Phi^1(s_3) \rangle\rangle = \frac{\lambda^2 c_{FF\Phi}^0 \eta_{ij}}{|s_{12}|^3 |s_{13}| |s_{23}|} + \mathcal{O}(\lambda^3). \tag{3.69}$$

We emphasize again that the tree-level three-point function appears at two-loop order in the expectation value of the deformed Wilson loop. The other primaries above iF_{ij} and $iD_{\{i} \mathbb{F}_{j\}3}$. The three-point functions involving these primaries are given by

$$\begin{aligned}
\langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) iF_{km}(s_3) \rangle\rangle &= \frac{c_{FF\hat{F}}^0 (\eta_{ik} \eta_{jm} - \eta_{im} \eta_{jk})}{|s_{12}|^2 |s_{13}|^2 |s_{23}|^2}, \\
\langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) iD_{\{k} \mathbb{F}_{m\}3}(s_3) \rangle\rangle &= \frac{c_{FFDF,ts}^0 (\eta_{ik} \eta_{jm} + \eta_{im} \eta_{jk} - \frac{2}{3} \eta_{ij} \eta_{km})}{|s_{12}|^2 |s_{13}|^2 |s_{23}|^2}, \tag{3.70}
\end{aligned}$$

where we have suppressed the λ dependence. We have seen in Section 3.2 that $iD^i \mathbb{F}_{i3}$ is a second level descendant of Φ^1 . Thus we have the descent condition

$$\langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) iD^k \mathbb{F}_{k3}(s_3) \rangle\rangle = -\frac{d^2}{ds_3^2} \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) \Phi^1(s_3) \rangle\rangle. \tag{3.71}$$

Performing an analogous computation as that described in Section 3.4.2 for the second order in the deformation, we find

$$c_{FF\Phi}^0 = -\frac{1}{32\pi^4}, \quad c_{FF\hat{F}}^0 = \frac{1}{16\pi^4}, \quad c_{FFDF,ts}^0 = \frac{5}{16\pi^4}, \tag{3.72}$$

and that the descent condition (3.71) is satisfied. We find that the results for the two different regularization schemes are identical.⁵

The two-point functions of the Φ^A supermultiplet

We now proceed to discuss the two-point functions of the Φ^A multiplet. The relevant terms in the operator expansion are

$$\begin{aligned}
& \oint_{s_1 > s_2} ds_1 ds_2 \left(\frac{1}{30} \left(\langle\langle i\mathbb{F}_{i3}(s_1) iD^n D_{(n}\mathbb{F}_{m)3}(s_2) \rangle\rangle \eta_{jk} \right. \right. \\
& + \langle\langle i\mathbb{F}_{i3}(s_1) iD^n D_{(n}\mathbb{F}_{k)3}(s_2) \rangle\rangle \eta_{jm} - \frac{2}{3} \langle\langle i\mathbb{F}_{i3}(s_1) iD^n D_{(n}\mathbb{F}_{j)3}(s_2) \rangle\rangle \eta_{km} \left. \right) \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_2^m \\
& + \frac{1}{18} \langle\langle i\mathbb{F}_{i3}(s_1) iD_j D^n \mathbb{F}_{n3}(s_2) \rangle\rangle \eta_{km} \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_2^m + \frac{1}{30} \left(\langle\langle iD^n D_{(n}\mathbb{F}_{k)3}(s_1) i\mathbb{F}_{m3}(s_2) \rangle\rangle \eta_{ij} \right. \\
& + \langle\langle iD^n D_{(n}\mathbb{F}_{k)3}(s_1) i\mathbb{F}_{m3}(s_2) \rangle\rangle \eta_{ik} - \frac{2}{3} \langle\langle iD^n D_{(n}\mathbb{F}_{i)3}(s_1) i\mathbb{F}_{m3}(s_2) \rangle\rangle \eta_{jk} \left. \right) \epsilon_1^i \epsilon_1^j \epsilon_1^k \epsilon_2^m \\
& + \frac{1}{18} \langle\langle iD_i D^n \mathbb{F}_{n3}(s_1) i\mathbb{F}_{m3}(s_2) \rangle\rangle \eta_{jk} \epsilon_1^i \epsilon_1^j \epsilon_1^k \epsilon_2^m - \frac{1}{6} \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) \rangle\rangle \eta_{km} \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_2^m \\
& - \frac{1}{6} \langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{m3}(s_2) \rangle\rangle \eta_{jk} \epsilon_1^i \epsilon_1^j \epsilon_1^k \epsilon_2^m + \frac{1}{6} \langle\langle i\mathbb{F}_{i3}(s_1) i\tilde{\mathbb{F}}_{j3}(s_2) \rangle\rangle \eta_{km} \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_2^m \\
& + \frac{1}{6} \langle\langle i\tilde{\mathbb{F}}_{i3}(s_1) i\mathbb{F}_{m3}(s_2) \rangle\rangle \eta_{jk} \epsilon_1^i \epsilon_1^j \epsilon_1^k \epsilon_2^m + \frac{1}{9} \langle\langle i\mathbb{F}_{i3}(s_1) iD^n F_{nm}(s_2) \rangle\rangle \eta_{jk} \epsilon_1^i \epsilon_2^j \epsilon_2^k \epsilon_2^m \\
& + \frac{1}{9} \langle\langle iD^n F_{ni}(s_1) i\mathbb{F}_{m3}(s_2) \rangle\rangle \eta_{jk} \epsilon_1^i \epsilon_1^j \epsilon_1^k \epsilon_2^m \left. \right). \tag{3.73}
\end{aligned}$$

All of the physical information in the correlation functions above is determined by descent relations, and we thus do not extract any new information from here.

The two-point functions of the Φ^1 supermultiplet

The relevant terms in the operator expansion are

$$\begin{aligned}
& \oint_{s_1 > s_2} ds_1 ds_2 \left(\frac{1}{36} \langle\langle iD^n \mathbb{F}_{n3}(s_1) iD^p \mathbb{F}_{p3}(s_2) \rangle\rangle \eta_{ij} \eta_{km} \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_2^m + \right. \\
& \frac{1}{4} \langle\langle \Phi^1(s_1) \Phi^1(s_2) \rangle\rangle \eta_{ij} \eta_{km} \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_2^m + \frac{1}{12} \langle\langle \Phi^1(s_1) iD^n \mathbb{F}_{n3}(s_2) \rangle\rangle \eta_{ij} \eta_{km} \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_2^m \tag{3.74} \\
& \left. + \frac{1}{12} \langle\langle iD^n \mathbb{F}_{n3}(s_1) \Phi^1(s_2) \rangle\rangle \eta_{ij} \eta_{km} \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_2^m \right).
\end{aligned}$$

Via our procedure, we may extract a_Φ^0 , a_Φ^1 and γ_Φ . That is, the tree-level and one-loop two-point function coefficient of Φ^1 and its anomalous dimension. We note that the anomalous dimension was computed using Feynman diagrams in [61]. We find

$$a_\Phi^0 = \frac{1}{8\pi^2}, \quad a_\Phi^1 = -\frac{5}{64\pi^4} - \frac{1}{192\pi^2}, \quad \gamma_\Phi = \frac{1}{4\pi^2}, \tag{3.75}$$

⁵If we consider deformations in $\mathbb{R}^2$, the three-point function $\langle\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) iF_{km}(s_3) \rangle\rangle$ does not appear in the operator expansion. This is the case for our study using the ‘mass’ regularization scheme.

where a_{Φ}^0 , a_{Φ}^1 and γ_{Φ} are the tree-level and one-loop two-point function coefficients and anomalous dimension of Φ^1 , respectively. For the mass regularization scheme, we find the same tree-level and anomalous dimensions, but a different one-loop two point function coefficient a_{Φ}^1 .⁶ The anomalous dimension matches that found in [61]. We note that in the cutoff regularization scheme, the scale parameter ξ in the anomalous dimension term of the two-point function (see (3.63)) is played by the cutoff, μ . In the ‘mass’ regularization scheme, this role is played by the mass ζ .

From the equations of motion (3.33), we also find, in the cutoff regularization, scheme

$$a_{\Phi\Phi}^0 + \frac{1}{4}a_{\lambda\lambda}^0 = \frac{81}{16\pi^4}, \quad (3.76)$$

where $a_{\Phi\Phi}^0$ and $a_{\lambda\lambda}^0$ are the tree-level two-point function coefficients of $[\mathbb{D}_3\Phi^A, \Phi^A]$ and $[\lambda_{a,\alpha}^+, \lambda^{-a,\alpha}]$, respectively. We emphasize that the *tree-level* two-point functions of these operators contribute to the two-loop expectation value of the Wilson loop.

In this supermultiplet we also have the iF_{ij} multiplet

$$\oint_{s_1 > s_2} ds_1 ds_2 \langle\langle iF_{ij}(s_1) iF_{km}(s_2) \rangle\rangle \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_2^m. \quad (3.77)$$

$$a_{\hat{F}}^0 = -\frac{1}{4\pi^2}, \quad a_{\hat{F}}^1 = \frac{7}{64\pi^4} + \frac{1}{96\pi^2}, \quad \gamma_{\hat{F}} = \frac{1}{4\pi^2}, \quad (3.78)$$

where $a_{\hat{F}}^0$, $a_{\hat{F}}^1$ and $\gamma_{\hat{F}}$ are the tree-level and one-loop two-point function coefficients and anomalous dimension of F_{ij} , respectively. The iF_{ij} operator insertion does not appear for deformations in $\mathbb{R}^2$ and we therefore do not obtain results for this operator in the mass regularization scheme. Similarly, we have

$$\oint_{s_1 > s_2} ds_1 ds_2 \frac{1}{4} \langle\langle iD_{\{i}\mathbb{F}_{j\}3}(s_1) iD_{\{k}\mathbb{F}_{m\}3}(s_2) \rangle\rangle \epsilon_1^i \epsilon_1^j \epsilon_2^k \epsilon_2^m. \quad (3.79)$$

$$a_{DF,ts}^0 = \frac{5}{\pi^2}, \quad a_{DF,ts}^1 = \frac{19}{16\pi^4} - \frac{5}{24\pi^2}, \quad \gamma_{DF,ts} = \frac{1}{4\pi^2}, \quad (3.80)$$

where $a_{DF,ts}^0$, $a_{DF,ts}^1$ and $\gamma_{DF,ts}$ are the tree-level and one-loop two-point function coefficients and anomalous dimension of $iD_{\{i}\mathbb{F}_{j\}3}$, respectively. As was the case for the Φ^1 operator, we find agreement between the cutoff and mass regularization schemes for the tree-level two-point coefficient and anomalous dimension of $iD_{\{i}\mathbb{F}_{j\}3}$. For deformations in $\mathbb{R}^2$ the one-loop two-point coefficient mixes with the tree-level two-point coefficients in (3.76) and we thus can not extract an answer for the mass regularization scheme.

⁶Specifically, we find $a_{\Phi}^1 = \frac{1}{32\pi^4} - \frac{1}{192\pi^2}$.

3.4.6 Order ϵ^4 : the results and scheme dependence

Above, we have listed results for order ϵ^4 . In this section, we summarize and discuss the results given above.

The three- and four-point functions

The three- and four-point coefficients above are consistent with all descent relations. Furthermore, they are consistent with results obtained via ‘mass’ regularization. This agreement for the three- and four-point functions is to be expected as none of the manipulation we perform to obtain these results actually requires a regularization procedure.

The two-point functions

The tree-level two-point functions we extract are consistent with all descent relations and between both regularization schemes. This is also to be expected for the same reason as given above for the agreement between the regularization schemes for the three- and four-point function coefficients.

The anomalous dimensions are also consistent between the regularization schemes, satisfy the descent relations and are consistent with previously known results. However, the one-loop two-point function coefficients are not consistent between the two regularization schemes. Moreover, they do not satisfy all of the descent relations.

For the cutoff regularization, we also obtain the terms

$$\begin{aligned}
& -\frac{1}{16\pi^4} \frac{(\dot{\epsilon}_1 \cdot \dot{\epsilon}_1)(\dot{\epsilon}_1 \cdot \epsilon_2)}{s_{12}^3} + \frac{1}{16\pi^4} \frac{(\epsilon_1 \cdot \dot{\epsilon}_2)(\dot{\epsilon}_2 \cdot \dot{\epsilon}_2)}{s_{12}^3} + \frac{1}{64\pi^4} \frac{(\epsilon_1 \cdot \dot{\epsilon}_1)(\ddot{\epsilon}_1 \cdot \epsilon_2)}{s_{12}^3} \\
& - \frac{1}{64\pi^4} \frac{(\epsilon_1 \cdot \ddot{\epsilon}_2)(\epsilon_2 \cdot \dot{\epsilon}_2)}{s_{12}^3}.
\end{aligned} \tag{3.81}$$

which do not have an interpretation as an operator expansion, as well as finite single integrals.

In our procedure, we have discounted operator mixing. This may occur for operators of classical dimension two and higher in the operator expansion. To account for this, we should replace the bare operators considered above with mixtures of renormalized operators (which must carry the same quantum numbers as the bare operator). This may resolve the issue of the two-point functions and the descent relations, but not the extra terms in (3.81).

The project this chapter is based on is still ongoing and it remains to understand these extra terms in the cutoff regularization, and to allow for operator mixing which we hope may resolve the issues in the two-point function coefficients.

3.5 Summary

In this chapter, we exploit an equivalence between operator insertions into a Wilson loop and small deformations of that Wilson loop (without operator insertions), in order to extract physical information about the operator insertions. Due to the aforementioned equivalence, the expectation value of the Wilson loop with insertions and the deformed loop are the same. The former is given by the sum of n -point functions of the operator insertions. The latter, for one- and two-loop order at weak coupling, has a known integral form [60].

We focus on the case of operator insertions into the 1/2 BPS straight line in $\mathcal{N} = 4$ SYM, which preserves an $OSp(2, 2|4)$ subgroup of the $PSU(2, 2|4)$ symmetry of the theory. In particular, the $SO(1, 2)_c$ group of conformal transformations along the line is preserved. The form of the n -point functions of the operator insertions into the Wilson loop are heavily constrained by this conformal symmetry. In particular, it constrains one-, two- and three-point functions up to a set of constants. These constants are the two- and three-point function coefficients and the anomalous dimensions of the operator insertions.

By focusing on one- and two-loop order at weak coupling, we need only consider one-, two- and three-point functions (as well as factorized four-point functions) of operator insertions. We may compute the expectation value of a small deformation of the 1/2 BPS straight line using the results of [60] and we compute the expectation value of the 1/2 BPS straight line with operator insertions up to the set of aforementioned coefficients. Using the equivalence between the two expectation values, we extract the two- and three-point coefficients and anomalous dimensions of the operator insertions. We do so by manipulating the expression for the expectation value of the deformed loop such that it matches the operator insertion expression.

This manipulation requires a regularization scheme. We find some discrepancies between the two regularization schemes. Firstly, in the case of the cutoff regularization scheme, we find “extra” terms, which do not have an interpretation as a correlation function of the operator insertions - see (3.81).

Secondly, the one-loop two-point function coefficients differ between regularization schemes for all the operator insertions. Furthermore, they are not consistent with the multiplet structure of the operator insertions. For example, the multiplet structure of the operator insertions imposes

$$\langle\langle \Phi^1(s_1) iD^i \mathbb{F}_{i3}(s_2) \rangle\rangle = -\frac{d^2}{ds_2^2} \langle\langle \Phi^1(s_1) \Phi^1(s_2) \rangle\rangle. \quad (3.82)$$

The form of $\langle\langle \Phi^1(s_1) \Phi^1(s_2) \rangle\rangle$ at two-loop order is fixed by conformal symmetry to be

$$\langle\langle \Phi^1(s_1) \Phi^1(s_2) \rangle\rangle = \frac{a_\Phi^1 - 2a_\Phi^0 \gamma_\Phi \log \frac{s_{12}}{\xi}}{s_{12}^{\frac{2\Delta_\Phi^0}{2}}}. \quad (3.83)$$

The values which we extract from our calculation for a_Φ^1 from $\langle\langle \Phi^1(s_1)\Phi^1(s_2) \rangle\rangle$ and $\langle\langle \Phi^1(s_1)iD^i\mathbb{F}_{i3} \rangle\rangle$ are inconsistent with the descent relation (3.83).

We observe no inconsistencies between regularization schemes or with descent relations for the tree-level two- and three-point function coefficients or anomalous dimensions of the operators. Furthermore, at second order in the deformation, we match the results of [52] for the $i\mathbb{F}_{i3}$ operator insertion. At fourth order in the deformation, we also match the result for the anomalous dimension of the Φ^1 operator insertion found in [61].

The research for this project is ongoing, and the outstanding issues listed above are the subject of this ongoing research.

Chapter 4

The algebraic curve of minimal surfaces in $\mathbb{H}_3$

In this chapter, we derive the Lax operator and corresponding algebraic curve for a very large family of known minimal surfaces in $\mathbb{H}_3$ (constructed by Ishizeki *et al.* in [1]). We begin the chapter by reviewing the $\mathbb{H}_3$ sigma model in Section 4.1. The minimal surfaces of [1] are described in terms of Riemann theta functions, whose analytic properties we review in Section 4.2. In this section we also present the minimal surfaces of [1].

Our construction of the Lax operator for these minimal surfaces is based on a procedure outlined in [2]. In Section 4.3 we review this procedure. Having reviewed the above background material, exploiting the analytic properties of theta functions, we proceed to the central result of the chapter. That is, we derive the Lax operator and algebraic curve of the minimal surfaces of [1] in Section 4.3. For the surfaces corresponding to a genus-one algebraic curve, we are able to be much more explicit than for the general genus case; in Section 4.4 we study this genus-one case in more detail.

4.1 The $\mathbb{H}_3$ sigma-model

We now review the $\mathbb{H}_3$ sigma model. The integrability of the $\mathbb{H}_3$ subsector of the $AdS_5 \times S^5$ sigma model is most manifest when considered as the coset manifold $SL(2; \mathbb{C})/SU(2)$. It may be parametrized by the group elements

$$g = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix} \in SL(2; \mathbb{C})/SU(2), \quad (4.1)$$

where X_i are the embedding coordinates of $\mathbb{H}_3$ in $\mathbb{R}^{1,3}$. The $SU(2)$ factor relates to the hermiticity of g . The sigma-model action is written in terms of $\mathfrak{sl}(2; \mathbb{C})/\mathfrak{su}(2)$

Maurer-Cartan forms as

$$S = \frac{1}{4\pi} \int d^2w \operatorname{tr}(\mathcal{J} \bar{\mathcal{J}}), \quad (4.2)$$

where w and $\bar{w}$ are complex world-sheet coordinates (with ∂ and $\bar{\partial}$ their respective derivatives) and the Maurer-Cartan forms are $\mathcal{J} = g^{-1}\partial g$ and $\bar{\mathcal{J}} = g^{-1}\bar{\partial}g$. The sigma-model equations of motion and Virasoro constraints may then be written in terms of these currents.

Instead of working with these currents, we choose, following [1], to employ the Pohlmeyer reduction [100] of this model. Due to the hermiticity of g , we may perform the decomposition

$$g = hh^\dagger, \quad h \in SL(2; \mathbb{C}). \quad (4.3)$$

Two $\mathfrak{sl}(2; \mathbb{C})$ currents may naturally be associated with h via $j := h^{-1}\partial h$ and $\bar{j} := h^{-1}\bar{\partial}h$. As a consequence of their definition, the Virasoro constraints and the equations of motion, these currents satisfy

$$\operatorname{tr} j = \operatorname{tr} \bar{j} = 0, \quad \bar{\partial}j - \partial\bar{j} + [\bar{j}, j] = 0, \quad (4.4a)$$

$$\det(\bar{j} + j^\dagger) = 0, \quad (4.4b)$$

$$\partial(\bar{j} + j^\dagger) + \frac{1}{2} [j - \bar{j}^\dagger, \bar{j} + j^\dagger] = 0. \quad (4.4c)$$

A *spectral parameter*, $x \in \mathbb{C}$ may be introduced, as in [2,83], to define new currents

$$J(x) := \frac{1}{1-x} (j + x\bar{j}^\dagger), \quad \bar{J}(x) := \frac{1}{1+x} (\bar{j} - xj^\dagger), \quad (4.5)$$

such that the equations for j and $\bar{j}$, (4.4a) and (4.4c), are equivalent to the flatness condition for J and $\bar{J}$

$$\bar{\partial}J - \partial\bar{J} + [\bar{J}, J] = 0. \quad (4.6)$$

While it is clear that (4.4a) is satisfied by $J(x)$ and $\bar{J}(x)$ for all x , we note that for $\operatorname{Re}(x) = 0$ they also solve the equations of motion, (4.4c).

An explicit parameterization of the currents j and $\bar{j}$ was given in [1]

$$j = \begin{pmatrix} -\frac{1}{2}\partial\alpha & e^{-\alpha} \\ e^\alpha & \frac{1}{2}\partial\alpha \end{pmatrix}, \quad \bar{j} = \begin{pmatrix} \frac{1}{2}\bar{\partial}\alpha & e^\alpha \\ -e^{-\alpha} & -\frac{1}{2}\bar{\partial}\alpha \end{pmatrix}, \quad (4.7)$$

where $\alpha(w, \bar{w})$ is the induced world-sheet metric, satisfying the cosh-Gordon equation

$$\partial\bar{\partial}\alpha = 2 \cosh(2\alpha). \quad (4.8)$$

Substituting (4.7) into (4.5), we have

$$J = \begin{pmatrix} -\frac{1}{2}\partial\alpha & e^{-\alpha} \\ \frac{1+x}{1-x}e^{\alpha} & \frac{1}{2}\partial\alpha \end{pmatrix}, \quad \bar{J} = \begin{pmatrix} \frac{1}{2}\bar{\partial}\alpha & \frac{1-x}{1+x}e^{\alpha} \\ -e^{-\alpha} & -\frac{1}{2}\bar{\partial}\alpha \end{pmatrix}. \quad (4.9)$$

Indeed it was already noted in [1] that, for $\lambda \equiv \frac{1+x}{1-x}$ on the unit circle, this describes real solutions, which corresponds to imaginary x . x serves thus in a dual role: shifts along the imaginary axis lead to different solutions, while expanding the flatness condition around any such point leads to the usual equations of motion.

The cosh-Gordon equation may be solved in terms of hyperelliptic theta functions. Given such a solution and a choice of x , we have the currents J and $\bar{J}$ and can reconstruct the full solution by reversing the procedure above: we firstly identify the flatness condition in (4.6) as the compatibility condition for the auxiliary linear problem

$$\partial\psi = \psi J, \quad \bar{\partial}\psi = \psi \bar{J}. \quad (4.10)$$

For imaginary x this connection also satisfies the equations of motion (4.4c) and we may then identify $h(x)$ in (4.3) with the matrix consisting of the two linearly independent solutions to the auxiliary problem, *i.e.*

$$h(x) = (\psi \quad \tilde{\psi}), \quad (4.11)$$

where ψ and $\tilde{\psi}$ are the two independent solutions of the auxiliary linear problem (4.10). Secondly, from $h(x)$, we can then construct the group element $g(x)$ (4.3), giving the solution to the $\mathbb{H}_3$ sigma-model. In this way, changing the spectral parameter enables us to construct from J and $\bar{J}$ a solution given by the group element $g(x)$ *at any point along the imaginary-axis*.

In [1], a family of solutions, $h(x)$, corresponding to minimal surfaces in $\mathbb{H}_3$ were constructed by Ishizeki *et al.* The minimal surfaces are described in terms of Riemann theta functions. In order to construct the Lax operator and extract the algebraic curve for these minimal surfaces, we shall exploit analytic properties of theta functions. Thus, before presenting the minimal surfaces of [1] and constructing the corresponding Lax operator, we review Riemann theta functions and their analytic properties.

4.2 Riemann theta functions

The procedure outlined above gives solutions to the $\mathbb{H}_3$ sigma-model from solutions of the cosh-Gordon equation. This equation may be solved by theta functions associated with hyperelliptic Riemann surfaces of odd genus [101], allowing one to find a very large family of solutions to the sigma-model [1]. We provide a quick review

of theta functions, focusing on properties relevant for our purposes. Much more can be found in the abundant literature on the subject, *e.g.* [102–106].

The surface associated with the theta functions is a hyperelliptic Riemann surface defined by a function $f(\lambda)$ of the form¹

$$f(\lambda) = \sqrt{\lambda} \prod_{i=1}^{2g} \sqrt{\lambda - \lambda_i}, \quad (4.12)$$

where $\lambda_i \in \mathbb{C}$, together with $\lambda = 0$ and $\lambda = \infty$, are the branch points of f . It is apparent that such surfaces are two-sheeted, corresponding to the two branches of the square root. To ensure real solutions to the sigma-model, we require the surface to be equipped with an *anti-holomorphic involution*, $s : \lambda \mapsto -1/\bar{\lambda}$, and the set of $2g + 2$ branch points $\{\lambda_i\}$ to be invariant under this involution.

For any a Riemann surface Σ there exists a canonical basis of homology cycles $\{a_i, b_i\}$ with $i = 1, \dots, g$, satisfying $a_i \circ a_j = b_i \circ b_j = 0$, and, $a_i \circ b_j = \delta_{ij}$. A set of cohomology one-forms $\{\omega_i\}$ dual to $\{a_i\}$, *i.e.*, satisfying $\oint_{a_i} \omega_j = \delta_{ij}$, may be defined. With this one constructs the *period matrix*

$$\Pi_{ij} = \oint_{b_i} \omega_j. \quad (4.13)$$

This is a symmetric $g \times g$ matrix with positive definite imaginary part.

Riemann theta functions are defined as

$$\theta(\Pi; z) := \sum_{n \in \mathbb{Z}^g} \exp \left(2\pi i \left(\frac{1}{2} n^\top \Pi n + n^\top z \right) \right), \quad (4.14)$$

where $z \in \mathbb{C}^g$ and Π is a $g \times g$ symmetric matrix with positive definite imaginary part. We consider Riemann theta function associated with Σ by taking Π to be the period matrix (4.13). We keep throughout a fixed Riemann surface and therefore write $\theta(z) := \theta(\Pi; z)$.

The $g \times 2g$ matrix (I, Π) (where I is the $g \times g$ identity matrix) generates a lattice, denoted by $\mathcal{L}(\Sigma) = (I, \Pi) \times \mathbb{Z}^{2g}$. The *Jacobian variety* of the surface Σ is the quotient $\mathfrak{J}(\Sigma) = \mathbb{C}^g / \mathcal{L}(\Sigma)$. One may then define the *Abel-Jacobi mapping*

$$\phi : \Sigma \rightarrow \mathfrak{J}(\Sigma), \quad \phi(\lambda) = \int_{\lambda_0}^{\lambda} \omega, \quad (4.15)$$

where λ_0 is the *base point* of the mapping, which we choose to be at $\lambda_0 = 0$. We often represent the Abel-Jacobi mapping by the integral $\int_0^\lambda := \phi(\lambda)$.

Strictly, we should distinguish between points $p \in \Sigma$ and their projection λ_p onto the complex plane. We may represent the uplift to the Riemann surface as

¹We will also refer to the function itself, and not only its solutions, as the curve.

$p = (\lambda, f(\lambda))$, and denote the other uplift $p' = (\lambda, -f(\lambda))$ (at branch points they are of course degenerate). The endpoint of integration in (4.15) is really not λ itself, but one of its uplifts. We will often use this notation and label corresponding points on the two sheets as λ and λ' .

It is useful to define theta functions with *characteristics*. Given $n, m \in \mathbb{C}^g$, the theta function with these characteristics is given by

$$\theta \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] (z) = e^{\pi i \left(\frac{1}{4} n^\top \Pi n + n^\top z + \frac{1}{2} n^\top m \right)} \theta \left(z + \frac{1}{2} m + \frac{1}{2} \Pi n \right). \quad (4.16)$$

We will concern ourselves only with integer characteristics $n, m \in \mathbb{Z}^g$. From (4.16) and (for the second equality) the definition (4.14) we find

$$\theta \left[\begin{smallmatrix} 2n \\ 2m \end{smallmatrix} \right] (z) = e^{\pi i (n^\top \Pi n + 2n^\top z + 2n^\top m)} \theta(z + m + \Pi n) = \theta(z). \quad (4.17)$$

The theta functions are therefore quasi-periodic so they are defined, up to monodromies, on the Jacobian. An integer characteristic is said to be *odd* or *even* if the scalar product, $n^\top m$, is odd or even. We note that theta functions with odd characteristics are odd functions of $z \in \mathbb{C}^g$ and theta functions with even characteristics are even.

Let us consider now the Abel-Jacobi map more explicitly. The differentials in (4.13) are given, up to a normalization, by [106]

$$\omega_i(\lambda) = \frac{\lambda^{i-1} d\lambda}{f(\lambda)}, \quad (4.18)$$

with $f(\lambda)$ given by (4.12). Since $f(\lambda') = -f(\lambda)$, then also $\omega(\lambda') = -\omega(\lambda)$ and consequently, $\int_0^{\lambda'} = -\int_0^\lambda$.

Despite the fact that the Abel-Jacobi mapping is not single-valued (it depends on the contour of integration), the composite map $\theta \circ \phi : \Sigma \rightarrow \mathbb{C}$ has a well-defined set of zeros. In particular, it has g zeros and no poles [106]. The set of zeros is similarly well defined for theta functions with characteristics.

There are $2g+2$ branch points of the algebraic curve defining the Riemann surface. To each of these branch points we may naturally associate a characteristic. The integral $\int_0^{\lambda_i}$ is half of the period integral between 0 to λ_i and back to 0 on the second sheet. Therefore it can be expressed as $\int_0^{\lambda_i} = \frac{1}{2} m I + \frac{1}{2} \Pi n$, for some $m, n \in \mathbb{Z}$. Now, applying (4.16)

$$\theta(\int_0^{\lambda_i}) = e^{\pi i \left(-\frac{1}{4} n^\top \Pi n - \frac{1}{2} n^\top m \right)} \theta \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] (0), \quad (4.19)$$

If $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$ is an odd characteristic, then $\theta \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] (z)$ is an odd function and thus $\theta(\int_0^{\lambda_i})$ vanishes. We refer to branch points for which the corresponding characteristics are odd, as *odd branch points* (and correspondingly for even characteristics). We denote by λ_i^- the odd branch points and by λ_i^+ the even branch points. There are

precisely g odd branch points of a hyperelliptic surface, corresponding to the g zeros of $\theta \circ \phi$ [106]. We fixed our curve to have branch points at 0 and at ∞ and a special role will be played by the theta functions with the corresponding characteristics. For zero, this is the original theta function. For infinity, this is defined as $\hat{\theta}(z) = \theta[\begin{smallmatrix} k \\ l \end{smallmatrix}](z)$, where $2 \int_0^\infty = l + \Pi k$ where k and l are some integers.

For λ_i a branch point, $2 \int_0^{\lambda_i} = Im + \Pi n$ ($m, n \in \mathbb{Z}$) is a full period. Combining this with the fact that $\hat{\theta}(0) = 0$ implies $\hat{\theta}(2 \int_0^{\lambda_i}) = 0$. In fact for any integer p , a theta functions of the form $\theta[\begin{smallmatrix} n \\ m \end{smallmatrix}](p \int_0^\lambda)$, has $p^2 g$ zeros (accounting for multiplicities) [106]. As we show below, for the case of $\hat{\theta}(2 \int_0^\lambda)$ these are just the $2g + 2$ branch points, of which $g - 1$ (the finite odd ones) are triple zeros and the other $g + 3$ branch points are simple zeros on the Riemann surface.²

Let us mention one more necessary tool: *directional derivatives* with respect to a point λ on the Riemann surface are defined as

$$D_\lambda \theta(z) \equiv \omega(\lambda) \cdot \frac{\partial}{\partial z} \theta(z). \quad (4.20)$$

4.2.1 The zeros of $\hat{\theta}(2 \int_0^\lambda)$

We now show that $\hat{\theta}(2 \int_0^\lambda)$ has triple zeros at the *odd finite branch points*, $\{\lambda_i^-\}$. As demonstrated in Appendix C, we may Taylor expand $\hat{\theta}(2 \int_0^\lambda)$ about a branch point λ_i . Defining $y := -i\sqrt{\lambda - \lambda_i}$ we expand about $y = 0$

$$\hat{\theta}(2 \int_0^\lambda) = 2y D_{\lambda_i} \hat{\theta}(2 \int_0^{\lambda_i}) + \mathcal{O}(y^3), \quad (4.21)$$

This is a simple zero if $D_{\lambda_i^-} \hat{\theta}(2 \int_0^{\lambda_i^-})$ is finite and at least a triple zero (since the function is odd) if it vanishes. As discussed above, the path $2 \int_0^{\lambda_i}$ defines a closed cycle and we may write $2 \int_0^{\lambda_i} := m_i + \Pi n_i$ for $m_i, n_i \in \mathbb{Z}$. Applying the identities (4.17) and (4.16), and recalling that $2 \int_0^\infty = l + \Pi k$, we may write

$$\hat{\theta}(z + 2 \int_0^{\lambda_i}) = C \exp \{ \pi i (k^\top - 2n^\top) z \} \theta(z + \int_0^\infty), \quad (4.22)$$

where C is a nonzero constant. It is thus apparent that

$$\begin{aligned} D_{\lambda_i} \hat{\theta}(2 \int_0^{\lambda_i}) &= C \exp \{ \pi i (l^\top - 2n^\top) z \} D_{\lambda_i} \theta(z + \int_0^\infty) \Big|_{z=0} \\ &= C D_{\lambda_i} \theta(\int_0^\infty). \end{aligned} \quad (4.23)$$

In particular, $D_{\lambda_i} \theta(\int_0^\infty) = 0$ is equivalent to $D_{\lambda_i} \hat{\theta}(2 \int_0^{\lambda_i}) = 0$. Therefore, in order to demonstrate that the odd finite branch points are triple zeros of $\hat{\theta}(2 \int_0^\lambda)$, it suffices to show that the directional derivative, $D_{\lambda_i^-} \theta(\int_0^\infty)$, vanishes. To do so, we shall

²The order of the zeros are given as a function on the Riemann surface Σ , as opposed to as a function of the projection onto the complex plane λ_p .

employ a corollary of *Fay's trisecant identity*, see e.g., [103]. This corollary states that

$$\begin{aligned} D_\lambda \ln \frac{\theta(z)}{\theta(z + \int_\eta^\rho)} \\ = -D_\lambda \ln \frac{\theta(a + \int_\eta^\lambda)}{\theta(a + \int_\rho^\lambda)} - \frac{D_\lambda \theta(a) \theta(a + \int_\rho^\eta)}{\theta(a + \int_\rho^\lambda) \theta(a + \int_\lambda^\eta)} \frac{\theta(z + \int_\eta^\lambda) \theta(z + \int_\lambda^\rho)}{\theta(z) \theta(z + \int_\eta^\rho)}, \end{aligned} \quad (4.24)$$

where a is a non-singular zero of theta and ρ, η, λ are arbitrary points on the Riemann surface. Let us choose $\rho = 0$, $a = \int_0^\infty$ and $z = \int_0^\infty + \int_0^\lambda$:

$$\begin{aligned} D_\lambda \ln \frac{\theta(\int_0^\infty + \int_0^\lambda)}{\theta(\int_0^\infty + \int_\eta^\lambda)} \\ = -D_\lambda \ln \frac{\theta(\int_0^\infty + \int_\eta^\lambda)}{\theta(\int_0^\infty + \int_0^\lambda)} - \frac{D_\lambda \theta(\int_0^\infty) \theta(\int_0^\infty + \int_0^\eta)}{\theta(\int_0^\infty + \int_0^\lambda) \theta(\int_0^\infty + \int_\lambda^\eta)} \frac{\theta(\int_\eta^\infty + 2 \int_0^\lambda) \theta(\int_0^\infty)}{\theta(\int_0^\infty + \int_0^\lambda) \theta(\int_0^\infty + \int_\eta^\lambda)}. \end{aligned} \quad (4.25)$$

It is apparent that the left-hand side cancels with the first term of the right-hand side. Let us examine the analytic structure of the second term on the right-hand side. Taking λ and η to be regular points on the surface we have a simple zero due to the $\theta(\int_0^\infty)$ factor in the numerator. The identity is thus satisfied.

The situation is different when λ is an odd finite branch point, since $\theta(\int_0^\infty + \int_0^{\lambda_i})$ in the denominator vanishes. This may be seen by noting that $\theta(z) = 0$ if and only if $z \in W_{g-1} + \kappa$, where W_{g-1} is the image of integral divisors of degree $g-1$ on Σ under the Abel-Jacobi mapping and κ is the *vector of Riemann constants* [106]. The vector of Riemann constants may be given by the image of the divisor of the odd branch point characteristics under the Abel-Jacobi mapping, i.e. $\kappa = \int_0^\infty + \sum_j^{g-1} \int_0^{\lambda_j^-}$. We may write

$$\int_0^\lambda + \int_0^\infty = \int_0^\lambda - \sum_j^{g-1} \int_0^{\lambda_j^-} + \kappa, \quad (4.26)$$

for an arbitrary point λ . Taking λ to be an odd finite branch point λ_i^- , it is apparent that $\int_0^{\lambda_i^-} - \sum_j^{g-1} \int_0^{\lambda_j^-} \in W_{g-1}$. From this we see that $\theta(\int_0^{\lambda_i^-} + \int_0^\infty) = 0$.

Thus, in order for the identity (4.25) to be satisfied for $\lambda = \lambda_i^-$, we find

$$D_{\lambda_i^-} \theta(\int_0^\infty) = 0, \quad (4.27)$$

and λ_i^- is at least a triple zero of $\hat{\theta}(2 \int_0^\lambda)$. We emphasize that the same procedure does not apply for $D_\infty \theta(\int_0^\infty)$, or indeed for the even branch points.

We therefore find that $\hat{\theta}(2 \int_0^\lambda)$ as a function on Σ has a zero of order (at least) three at the odd finite branch points. As there are $g-1$ finite odd branch points and $g+1$ other branch points which are also zeros, we find a total of at least $3(g-1) + (g+3) = 4g$, zeros. But as stated earlier, this is exactly the number of

zeros of $\hat{\theta}(2 \int_0^\lambda)$, and so odd finite branch points are triple zeros.

The solutions of Ishizeki *et al.* [1]

Having reviewed Riemann theta functions and their properties, we are now in a position to present the solutions to the $\mathbb{H}_3$ sigma model found in [1]. Solutions to this sigma model may be described in terms of solutions to the auxiliary linear problem (4.10) (specifically a 2×2 matrix of linearly independent solutions to this differential equation). From these, one may easily construct the group element g describing the string solution. In [1], a family of minimal surfaces was found. The corresponding solutions to the auxiliary problem are given by [1]

$$h(w, \bar{w}; \lambda) = \left(\frac{\theta^2(\int_0^\lambda)}{2\hat{\theta}(2 \int_0^\lambda) D_\infty \hat{\theta}(0)} \right)^{1/2} \times \begin{pmatrix} \sqrt{-\lambda} \frac{\hat{\theta}(z+\int_0^\lambda)}{\hat{\theta}(z)} \left(\frac{\hat{\theta}(z)}{\theta(z)} \right)^{1/2} e^{\mu(\lambda)w+\nu(\lambda)\bar{w}} & \frac{\theta(z+\int_0^\lambda)}{\theta(z)} \left(\frac{\theta(z)}{\hat{\theta}(z)} \right)^{1/2} e^{\mu(\lambda)w+\nu(\lambda)\bar{w}} \\ -\sqrt{-\lambda} \frac{\hat{\theta}(z-\int_0^\lambda)}{\hat{\theta}(z)} \left(\frac{\hat{\theta}(z)}{\theta(z)} \right)^{1/2} e^{-\mu(\lambda)w-\nu(\lambda)\bar{w}} & \frac{\theta(z-\int_0^\lambda)}{\theta(z)} \left(\frac{\theta(z)}{\hat{\theta}(z)} \right)^{1/2} e^{-\mu(\lambda)w-\nu(\lambda)\bar{w}} \end{pmatrix}, \quad (4.28)$$

where $z = z(w, \bar{w})$ is a function of the world-sheet coordinates. The precise form of $z(w, \bar{w})$, of $\mu(\lambda)$ and of $\nu(\lambda)$ are not important for our discussion. As outlined in Section 4.1, the group element/minimal surface may be readily reconstructed from (4.28).

4.3 The Lax operator and algebraic curve

We now review the procedure, introduced in [2], to associate a Lax operator to solutions of the $\mathbb{H}_3$ sigma model (and subsequently extract an algebraic curve). As discussed in Chapter 1, for infinite-dimensional systems such as the $\mathbb{H}_3$ sigma-model, we may treat the two world-sheet coordinates as independent ‘time’ parameters and study

$$dL = [L, M], \quad (4.29)$$

where d is the exterior derivative with respect to the world-sheet coordinates of the string, $d = dw \partial + d\bar{w} \bar{\partial}$. An algebraic curve is naturally associated to this Lax operator via the characteristic polynomial

$$\det(L - yI) = 0. \quad (4.30)$$

For $\mathbb{H}_3$, this simplifies to

$$y^2 = -\det L. \quad (4.31)$$

The Lax equations ensure that this curve is independent of the world-sheet position.

We now proceed to employ the method proposed in [2] to find the Lax operator

and from it the algebraic curve for the minimal surface solutions discussed above. It is natural that the result should be the algebraic curve (4.12) associated to the theta function, but it is rather opaque how this would come about. The prescription in [2] is to take the Lax operator as

$$L(w, \bar{w}; x) = \hat{\Psi}(w, \bar{w}; x)^{-1} \cdot A(x) \cdot \hat{\Psi}(w, \bar{w}; x), \quad (4.32)$$

where $\hat{\Psi}(w, \bar{w}; x)$ is the matrix whose rows are the linearly independent solutions of an auxiliary linear problem of the form,

$$\partial\psi = \psi\mathcal{J}(x), \quad \bar{\partial}\psi = \psi\bar{\mathcal{J}}(x). \quad (4.33)$$

and the matrix $A(x)$ is determined by the requirement that $L(w, \bar{w}; x)$ have polynomial dependence on the spectral parameter. Here

$$\mathcal{J}(x) = \frac{\mathcal{J}}{1-x}, \quad \bar{\mathcal{J}}(x) = \frac{\bar{\mathcal{J}}}{1+x}, \quad (4.34)$$

are defined in terms of the Maurer-Cartan forms $\mathcal{J}$ and $\bar{\mathcal{J}}$ in (4.2).

We wish to modify this prescription for our purposes. Firstly, we will construct a Lax operator with respect to the Pohlmeyer reduced connection j and $\bar{j}$, *i.e.*, for $M = J(\lambda)dw + \bar{J}(\lambda)d\bar{w}$, rather than $M = \mathcal{J}(x)dw + \bar{\mathcal{J}}(x)d\bar{w}$. The sigma-model connection and the Pohlmeyer reduced connection are related by a gauge transformation [107], and the resulting characteristic polynomials are the same. We demonstrate the equivalence of solving the respective Lax equations in Appendix D.1.

As we expect the curve to be hyperelliptic, we make the ansatz [108]

$$A(\lambda) = c(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4.35)$$

where $c(\lambda)$ is a to-be-determined function of λ .

4.3.1 The Lax operator for the minimal surfaces of Ishizeki *et al.* [1]

We now apply the procedure reviewed above to the minimal surfaces found in [1]. We recall, from (4.28), that the matrix of linearly independent solutions to the auxiliary linear problem, $\hat{\Psi}$, is known for these minimal surfaces. It remains to determine $c(\lambda)$ appearing in the expression (4.32) for the Lax operator. Substituting (4.28) into (4.32) gives

$$L(w, \bar{w}; \lambda) := \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \quad (4.36)$$

where

$$\begin{aligned} L_{11} &= -L_{22} \\ &= \frac{c(\lambda)\sqrt{-\lambda}\theta^2(\int_0^\lambda)}{2\hat{\theta}(2\int_0^\lambda)D_\infty\hat{\theta}(0)} \left(\frac{\hat{\theta}(z+\int_0^\lambda)\theta(z-\int_0^\lambda) - \theta(z+\int_0^\lambda)\hat{\theta}(z-\int_0^\lambda)}{\theta(z)\hat{\theta}(z)} \right), \end{aligned} \quad (4.37a)$$

$$L_{12} = \frac{c(\lambda)\theta^2(\int_0^\lambda)}{\hat{\theta}(2\int_0^\lambda)D_\infty\hat{\theta}(0)} \frac{\theta(z+\int_0^\lambda)\theta(z-\int_0^\lambda)}{\theta(z)\hat{\theta}(z)}, \quad (4.37b)$$

$$L_{21} = -\frac{\lambda c(\lambda)\theta^2(\int_0^\lambda)}{\hat{\theta}(2\int_0^\lambda)D_\infty\hat{\theta}(0)} \frac{\hat{\theta}(z+\int_0^\lambda)\hat{\theta}(z-\int_0^\lambda)}{\theta(z)\hat{\theta}(z)}. \quad (4.37c)$$

We now emphasize a point about this construction. It is apparent that $c(\lambda)$ must be the algebraic curve as, by the definition of L , $\det L = -c(\lambda)^2$. This observation advocates the converse procedure of that in [2], *i.e.*, we make an ansatz for $c(\lambda)$, and then verify that the resulting Lax matrix has polynomial entries. The ansatz which we make is that the algebraic curve/characteristic polynomial of the Lax operator is given by the Riemann surface associated with the theta functions, *i.e.* $c(\lambda) = f(\lambda)$. In order to investigate the spectral parameter dependence of the Lax operator, we require algebraic relations between $f(\lambda)$ and the theta functions.

As can be seen from (4.37), the dependence of the Lax operator on the spectral parameter is not manifestly polynomial. Thus, we shall need to use analytic properties of theta functions to verify this polynomial dependence. In Section 4.3.2, we define and discuss the analytic properties a function, defined in terms of Riemann theta functions, which will prove useful in verifying this dependence. For the genus one case, we may additionally use algebraic relations for the theta functions in order to make this polynomial dependence manifest (see Section 4.4).

4.3.2 A meromorphic function

Consider the function on the Riemann surface Σ , given by

$$G(p) = \frac{\hat{\theta}(2\int_0^{\lambda_p})}{\theta^4(\int_0^{\lambda_p})}. \quad (4.38)$$

Recall that $p = (\lambda_p, f(\lambda_p))$, where λ_p is the projection of p onto the complex plane. First note that it is single valued on Σ and in particular, it is independent of the contour of integration. To see that, consider the monodromy around a closed cycle $\int_0^{\lambda_p} \rightarrow \int_0^{\lambda_p} + m + \Pi n$, for $m, n \in \mathbb{Z}$. Applying (4.17) and (4.16) we find

$$G(p) \rightarrow \frac{\hat{\theta}(2\int_0^{\lambda_p} + 2m + 2\Pi n)}{\theta^4(\int_0^{\lambda_p} + m + \Pi n)} = \frac{\hat{\theta}(2\int_0^{\lambda_p})}{\theta^4(\int_0^{\lambda_p})}, \quad (4.39)$$

Since both θ and $\hat{\theta}$ are analytic, $G(p)$ is meromorphic on Σ .

If instead we consider G as a function on the complex λ -plane, it will have the same branch-cuts as $f(\lambda)$, and then $G^2(\lambda)$ is meromorphic on the complex λ -plane. One may be tempted to guess that $G(\lambda) = f(\lambda)$, but this is not the case. The zeros of the numerator and denominator have already been discussed in Section 4.2. Combining these results, we see that G^2 has³

- simple zeros at the $g + 2$ even branch points, $\{\lambda_i^+\}$,
- simple poles at the $g - 1$ odd finite branch points, $\{\lambda_i^-\}$,
- a third order pole at ∞ ,
- no other zeros or poles.

We may therefore identify G as

$$G(\lambda) = G_0 \frac{\prod_i^{g+2} \sqrt{\lambda - \lambda_i^+}}{\prod_i^{g-1} \sqrt{\lambda - \lambda_i^-}}. \quad (4.40)$$

Though it is not crucial for our purposes, we may determine the constant from the asymptotics of the theta functions

$$\lim_{\lambda \rightarrow \infty} \lambda^{-3/2} G(\lambda) = \frac{1}{4} \left(\frac{1}{D_\infty \hat{\theta}(0)} \right)^3. \quad (4.41)$$

It was shown in [1] that

$$\sqrt{-\lambda} = 2 \frac{D_\infty \hat{\theta}(0) \hat{\theta}(\int_0^\lambda)}{\theta(0) \theta(\int_0^\lambda)}, \quad (4.42)$$

so we find that

$$G_0 = \frac{D_0 \hat{\theta}(0)}{\left(\theta(0) D_\infty \hat{\theta}(0) \right)^2}. \quad (4.43)$$

Finally, splitting the product (4.12) into terms with the odd finite branch points $\{\lambda_i^-\}$ and with the even branch points $\{\lambda_i^+\}$ we have

$$G(\lambda) \prod_i^{g-1} (\lambda - \lambda_i^-) = \sqrt{\lambda} \prod_i^{g-1} \sqrt{\lambda - \lambda_i^-} \prod_j^{g+1} \sqrt{\lambda - \lambda_j^+} = f(\lambda) =: c(\lambda). \quad (4.44)$$

So while $G(\lambda)$ is not equal to $f(\lambda)$, their ratio is a polynomial.

We now use this representation of $c(\lambda)$ to prove that the entries of the Lax matrix (4.37) are indeed polynomials in λ . We also find the order of these polynomials.

³Since all the zero and poles are branch points of $G(\lambda)$, the same statements actually hold for G as a function on Σ .

4.3.3 L_{11}

Using the theta function representation of $c(\lambda)$ (4.44) and $\int_0^{\lambda'} = -\int_0^\lambda$, where the integrals are along image paths on the two sheets, we can rewrite (4.37a) as

$$L_{11} = \frac{\theta^2(0)D_\infty\hat{\theta}(0)}{2D_0\hat{\theta}(0)} \frac{\hat{\theta}(z + \int_0^\lambda)\theta(z - \int_0^\lambda) - \hat{\theta}(z + \int_0^{\lambda'})\theta(z - \int_0^{\lambda'})}{\theta(z)\hat{\theta}(z)} \times \sqrt{-\lambda} \frac{\prod_i^{g-1}(\lambda - \lambda_i^-)}{\theta^2(\int_0^\lambda)}, \quad (4.45)$$

- **No branch cuts:** Clearly the $\sqrt{-\lambda}$ term introduces a branch cut which changes sign between the two sheets $\lambda \rightarrow \lambda'$, but this is negated by the branch cut of the second fraction (see Appendix C). Therefore L_{11} is single valued.
- **Analyticity:** As discussed in Section 4.2, the denominator of the last fraction has zeros at the odd finite branch points and at infinity. The finite branch points are simple zeros which are cancelled by corresponding zeros of the polynomial in the numerator.
- **Order:** Near the origin the two terms with branch cuts in (4.45) mean that $L_{11}(0) = 0$, so L_{11} has no constant piece. At large λ , it is shown in Appendix C that the second fraction scales together like $\lambda^{-1/2}$, as does $\theta(\int_0^\lambda)$. We find that $L_{11} = O(\lambda^g)$ and is therefore a degree g polynomial in λ .

4.3.4 L_{12}

$$L_{12} = \frac{\theta^2(0)D_\infty\hat{\theta}(0)}{D_0\hat{\theta}(0)} \frac{\theta(z + \int_0^\lambda)\theta(z - \int_0^\lambda)}{\theta(z)\hat{\theta}(z)} \frac{\prod_i^{g-1}(\lambda - \lambda_i^-)}{\theta^2(\int_0^\lambda)}, \quad (4.46)$$

- **No branch cuts:** L_{12} is invariant under $\lambda \rightarrow \lambda'$.
- **Analyticity:** The only possible zeros of the denominator are at λ_i^- and are cancelled by those in the numerator.
- **Order:** The z -dependant fraction has a finite limit at large λ . The last one scales like λ^g , so L_{12} is a polynomial of degree g .

4.3.5 L_{21}

$$L_{21} = -\frac{\theta^2(0)D_\infty\hat{\theta}(0)}{D_0\hat{\theta}(0)} \frac{\hat{\theta}(z + \int_0^\lambda)\hat{\theta}(z - \int_0^\lambda)}{\theta(z)\hat{\theta}(z)} \frac{\lambda \prod_i^{g-1}(\lambda - \lambda_i^-)}{\theta^2(\int_0^\lambda)}, \quad (4.47)$$

- The same arguments as in the case of L_{12} show that L_{21} is a polynomial of degree $g + 1$ with no constant term.

We conclude that, with the theta function representation (4.44) of $c(\lambda)$, the Lax operator $L(w, \bar{w}; \lambda)$, given by (4.37), is indeed the Lax operator describing the family of solutions and its determinant gives the corresponding algebraic curve. Thus the procedure advocated in [2] applies to the open string solutions of [1] with the spectral parameter $\lambda = (1+x)/(1-x)$ and, as expected, the resulting curve is none other than the hyperelliptic curve defining the Riemann theta functions.

4.4 The genus-one case

We have constructed the Lax operator for a solution based on an arbitrary (odd) genus curve, shown that its entries are polynomial in λ and proven that the resulting algebraic curve is indeed $f(\lambda)$, the Riemann surface defining the theta functions. We have not presented, however, explicit expressions for the coefficients of the polynomials in λ in the different matrix elements of L . To do that would require to disentangle expressions like $\theta(z + \int_0^\lambda) \theta(z - \int_0^\lambda)$. In fact the addition theorem (see *e.g.*, [105]) may be employed to do just that. This allows one to write such products as sums over products of theta functions of z and theta functions of λ , thus splitting the spectral parameter and world-sheet dependence in (4.37). It does so at the expense of introducing a sum over all possible integer (mod 2) characteristics. We have not found the resulting expressions for arbitrary genus illuminating. However, in the case of genus-one, which we present here, they are very explicit and rather simple.

For genus-one the Riemann theta functions reduce to elliptic theta functions (see [109])

$$\theta(z) = \vartheta_3(\pi z; q), \quad \hat{\theta}(z) = -\vartheta_1(\pi z; q), \quad \text{with } q = e^{i\pi\Pi}. \quad (4.48)$$

The period matrix is given by $\Pi = i\mathbb{K}(k')/\mathbb{K}(k)$, where $\mathbb{K}$ is the complete elliptic integral of the first kind, and k and k' are the elliptic modulus and complementary elliptic modulus, respectively.

Employing the theta function representation of $f(\lambda)$ from Section 4.3, we have

$$f(\lambda) = \sqrt{\lambda} \sqrt{\lambda - \lambda_1} \sqrt{\lambda + 1/\bar{\lambda}_1} = -\vartheta_3^2(0) D_\infty \vartheta_1(0) \frac{\vartheta_1(2 \int_0^\lambda)}{\vartheta_3^4(\int_0^\lambda)}. \quad (4.49)$$

We have absorbed a factor of π in the definition of the elliptic functions and used that for genus-one $\omega(0) = -\omega(\infty)$. Note that the cross-ratio of the branch points $\{0, \infty, \lambda_1, -1/\bar{\lambda}_1\}$ is always real, so they are all along a line, which without loss of generality we take to be the real line. Furthermore the only odd branch point is at infinity. We also note that the elliptic modulus and complementary modulus may

be expressed in terms of the branch points via

$$k = \frac{\lambda_1}{\sqrt{1 + \lambda_1^2}}, \quad k' = \frac{1}{\sqrt{1 + \lambda_1^2}}, \quad (4.50)$$

where $\lambda_1 \in \mathbb{R}$.

The expression for the Lax operator, (4.37), becomes

$$\begin{aligned} L_{11} &= -\frac{\sqrt{-\lambda}}{2} \vartheta_3^2(0) \frac{\vartheta_1(z + \int_0^\lambda) \vartheta_3(z - \int_0^\lambda) - \vartheta_3(z + \int_0^\lambda) \vartheta_1(z - \int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3^2(\int_0^\lambda)}, \\ L_{12} &= \vartheta_3^2(0) \frac{\vartheta_3(z + \int_0^\lambda) \vartheta_3(z - \int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3^2(\int_0^\lambda)}, \\ L_{21} &= -\lambda \vartheta_3^2(0) \frac{\vartheta_1(z + \int_0^\lambda) \vartheta_1(z - \int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3^2(\int_0^\lambda)}. \end{aligned} \quad (4.51)$$

We would like to evaluate the coefficients of the polynomials, and to explicitly recover the spectral curve. For genus-one, the expression for the spectral parameter (4.42) simplifies to

$$\lambda = \frac{\vartheta_1^2(\int_0^\lambda)}{\vartheta_3^2(\int_0^\lambda)}. \quad (4.52)$$

The addition theorem at genus-one is particularly simple and may be employed to split the world-sheet and spectral parameter dependence

$$\begin{aligned} &\vartheta_1(z + \int_0^\lambda) \vartheta_3(z - \int_0^\lambda) - \vartheta_3(z + \int_0^\lambda) \vartheta_1(z - \int_0^\lambda) \\ &= 2 \frac{\vartheta_2(z) \vartheta_4(z) \vartheta_1(\int_0^\lambda) \vartheta_3(\int_0^\lambda)}{\vartheta_2(0) \vartheta_4(0)}, \\ &\vartheta_3(z + \int_0^\lambda) \vartheta_3(z - \int_0^\lambda) = \frac{\vartheta_1^2(z) \vartheta_1^2(\int_0^\lambda) + \vartheta_3^2(z) \vartheta_3^2(\int_0^\lambda)}{\vartheta_3^2(0)}, \\ &\vartheta_1(z + \int_0^\lambda) \vartheta_1(z - \int_0^\lambda) = \frac{\vartheta_1^2(z) \vartheta_3^2(\int_0^\lambda) - \vartheta_3^2(z) \vartheta_1^2(\int_0^\lambda)}{\vartheta_3^2(0)}. \end{aligned} \quad (4.53)$$

This gives

$$\begin{aligned} L_{11} &= -\sqrt{-\lambda} \frac{\vartheta_3^2(0)}{\vartheta_2(0) \vartheta_4(0)} \frac{\vartheta_2(z) \vartheta_4(z) \vartheta_1(\int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3(\int_0^\lambda)} = \mp i \frac{\lambda}{\sqrt{k k'}} \frac{\vartheta_2(z) \vartheta_4(z)}{\vartheta_3(z) \vartheta_1(z)}, \\ L_{12} &= \frac{\vartheta_1^2(z) \vartheta_1^2(\int_0^\lambda) + \vartheta_3^2(z) \vartheta_3^2(\int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3^2(\int_0^\lambda)} = \lambda \frac{\vartheta_1(z)}{\vartheta_3(z)} + \frac{\vartheta_3(z)}{\vartheta_1(z)}, \\ L_{21} &= -\lambda \frac{\vartheta_1^2(z) \vartheta_3^2(\int_0^\lambda) - \vartheta_3^2(z) \vartheta_1^2(\int_0^\lambda)}{\vartheta_3(z) \vartheta_1(z) \vartheta_3^2(\int_0^\lambda)} = \lambda^2 \frac{\vartheta_3(z)}{\vartheta_1(z)} - \lambda \frac{\vartheta_1(z)}{\vartheta_3(z)}, \end{aligned} \quad (4.54)$$

where we have used the identity $\vartheta_2(0) \vartheta_4(0) = \sqrt{k k'} \vartheta_3^2(0)$. We note that all the components of the Lax operator have polynomial dependence on the spectral parameter

and with the orders of the polynomials matching the results of Section 4.3.

Recall that, due to the tracelessness of L , we have that

$$\det(yI - L(w, \bar{w}; \lambda)) = 0 \quad \Leftrightarrow \quad y^2 = -\det L(w, \bar{w}; \lambda), \quad (4.55)$$

which is the equation of the algebraic curve. It is straightforward to read off the algebraic curve from (4.54)

$$y^2 = \lambda^3 + \lambda^2 \left\{ -\frac{1}{kk'} \left(\frac{\vartheta_2(z)\vartheta_4(z)}{\vartheta_3(z)\vartheta_1(z)} \right)^2 + \left(\frac{\vartheta_3(z)}{\vartheta_1(z)} \right)^2 - \left(\frac{\vartheta_1(z)}{\vartheta_3(z)} \right)^2 \right\} - \lambda. \quad (4.56)$$

We may express the elliptic theta functions of the second and fourth kinds here in terms of the first and third kinds via the identity

$$\vartheta_2^2(z)\vartheta_4^2(z) = kk' \left(\vartheta_3^4(z) - \vartheta_1^4(z) + \vartheta_3^2(z)\vartheta_1^2(z) \left[\frac{\vartheta_2^2(0)}{\vartheta_4^2(0)} - \frac{\vartheta_4^2(0)}{\vartheta_2^2(0)} \right] \right), \quad (4.57)$$

which is derived from the addition theorem. From this we find the nice expression for the curve

$$y^2 = \lambda \left(\lambda^2 + \lambda \left[\frac{\vartheta_4^2(0)}{\vartheta_2^2(0)} - \frac{\vartheta_2^2(0)}{\vartheta_4^2(0)} \right] - 1 \right). \quad (4.58)$$

A final identity giving the position of the cuts of the Riemann surface associated to the theta functions

$$\left(\frac{\vartheta_2(0)}{\vartheta_4(0)} \right)^2 = \lambda_1, \quad (4.59)$$

leads to

$$y^2 = \lambda (\lambda - \lambda_1) (\lambda + 1/\lambda_1). \quad (4.60)$$

We have thus recovered the genus-one algebraic curve explicitly from the Lax operator. We recover this curve via an alternative approach (that of [83]) in Appendix D.2.

4.5 Summary

In this chapter, we construct the Lax operator for the family of minimal surfaces in $\mathbb{H}_3$ found by Ishizeki *et al.* in [1]. These minimal surfaces are holographically dual to Wilson loops in $\mathcal{N} = 4$ SYM. From the Lax operator, we also extract an algebraic curve.

As reviewed in Section 1.4, for closed strings in $AdS_5 \times S^5$ and its subsectors there exists an elegant and powerful framework which captures the integrability of the system. This is the spectral curve approach. This approach involves the construction of a monodromy operator for closed string solutions to the sigma model. From this monodromy, an algebraic curve may be extracted. This algebraic curve characterizes

the closed string solutions to the sigma model.

As also discussed in Section 1.4, the construction of a monodromy operator for open strings is more involved. It has been achieved in certain cases, see [74]. However, in general, the construction of such a monodromy operator has remained elusive. In [2], an approach to extract an algebraic curve from solutions to the sigma model was proposed. This approach constructs a Lax operator rather than a monodromy operator, from which the algebraic curve is extracted. This approach lacks some of the power of the spectral curve approach; in particular, it is not clear how to extract the conserved charges from the Lax operator (in contrast to the monodromy operator). However, one may hope that the algebraic curve extracted from the Lax operator will coincide with that extracted from a monodromy operator in the future.

In [2], this approach was applied to two open string solutions of the $\mathbb{H}_3$ sigma model holographically dual to Wilson loops. Here, we apply this approach to the very large family of minimal surfaces of Ishizeki *et al.* [1]. This family are defined in terms of a Riemann surface Σ . We find that the algebraic curve extracted from the Lax operator is exactly this Riemann surface Σ for these minimal surfaces.

Appendix A

Notation and conventions

A.1 $\mathcal{N} = 4$ Chern-Simons matter theories

We consider the loop in $\mathbb{R}^3$. Spinor indices are raised and lowered as

$$\psi^a = \epsilon^{ab}\psi_b, \quad \psi_a = \epsilon_{ab}\psi^b, \quad \epsilon^{+-} = -\epsilon_{+-} = 1, \quad (\text{A.1})$$

and we employ the gamma matrix basis

$$(\gamma^\mu)^a{}_b = \{\sigma^3, \sigma^1, -\sigma^2\}, \quad (\text{A.2})$$

satisfying $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + i\epsilon^{\mu\nu\rho}\gamma_\rho$ (with $\epsilon^{123} = 1$).

As in [37], we use the epsilon symbol to raise and lower the indices of the supersymmetry parameters via $\xi^{\bar{A}\underline{B}} = \epsilon^{\bar{A}\bar{C}}\epsilon^{\underline{B}\underline{D}}\xi_{\bar{C}\underline{D}}$, with $\epsilon^{12} = \epsilon_{12} = 1$ for both overlined and underlined indices. In the main text, we shall always write the supersymmetry parameters with lowered indices.

In this gamma matrix basis, the supersymmetry transformations are given by [37]

$$\delta q_{(I)}^{\bar{A}} = i\sqrt{2}(\xi^{\bar{A}\underline{B}})^a(\psi_{(I)\underline{B}})_a, \quad \delta \bar{q}_{(I)\bar{A}} = i\sqrt{2}(\xi_{\bar{A}\underline{B}})^a(\bar{\psi}_{(I)}^{\underline{B}})_a, \quad (\text{A.3a})$$

$$\begin{aligned} \delta(\psi_{(I)\underline{A}})^a &= \sqrt{2}(\not{D})^a{}_b q_{(I)}^{\bar{B}}(\xi_{\bar{B}\underline{A}})^b - \frac{\sqrt{2}s_I}{k}(\xi_{\bar{B}\underline{A}})^{a\epsilon} \left[\nu_{(I)} q_{(I)}^{\bar{B}} - q_{(I)}^{\bar{B}} \tilde{\nu}_{(I)} \right] \\ &+ \left(\sqrt{2}(\xi_{\bar{C}\underline{B}})^a(\varphi_{(I)})^{\underline{B}}_{\underline{A}} q_{(I)}^{\bar{C}} \right)_{k_I=0} - \left(\frac{2\sqrt{2}s_I}{k}(\xi_{\bar{C}\underline{B}})^a(\tilde{\mu}_{(I-1)})^{\underline{B}}_{\underline{A}} q_{(I)}^{\bar{C}} \right)_{k_I \neq 0} \\ &- \left(\sqrt{2}(\xi_{\bar{C}\underline{B}})^a q_{(I)}^{\bar{C}}(\varphi_{(I+1)})^{\underline{B}}_{\underline{A}} \right)_{k_{I+1}=0} \\ &+ \left(\frac{2\sqrt{2}s_I}{k}(\xi_{\bar{C}\underline{B}})^a q_{(I)}^{\bar{C}}(\mu_{(I+1)})^{\underline{B}}_{\underline{A}} \right)_{k_{I+1} \neq 0}, \end{aligned} \quad (\text{A.3b})$$

$$\begin{aligned}
\delta(\bar{\psi}_{(I)}^{\underline{A}})_a &= \sqrt{2}(\not{D})_{ab}\bar{q}_{(I)}^{\bar{B}}(\xi^{\bar{B}\underline{A}})^b + \frac{\sqrt{2}s_I}{k}(\xi^{\bar{B}\underline{A}})_a [\tilde{\nu}_{(I)}\bar{q}_{(I)\bar{B}} - \bar{q}_{(I)\bar{B}}\nu_{(I)}] \\
&+ \left(\sqrt{2}(\xi^{\bar{C}\underline{B}})_a\bar{q}_{(I)\bar{C}}(\varphi_{(I)})^{\underline{A}}_{\underline{B}} \right)_{k_I=0} - \left(\frac{2\sqrt{2}s_I}{k}(\xi^{\bar{C}\underline{B}})_a\bar{q}_{(I)\bar{C}}(\tilde{\mu}_{(I-1)})^{\underline{A}}_{\underline{B}} \right)_{k_I \neq 0} \\
&- \left(\sqrt{2}(\xi^{\bar{C}\underline{B}})_a(\varphi_{(I+1)})^{\underline{A}}_{\underline{B}}\bar{q}_{(I)\bar{C}} \right)_{k_{I+1}=0} \\
&+ \left(\frac{2\sqrt{2}s_I}{k}(\xi^{\bar{C}\underline{B}})_a(\mu_{(I+1)})^{\underline{A}}_{\underline{B}}\bar{q}_{(I)\bar{C}} \right)_{k_{I+1} \neq 0}, \tag{A.3c}
\end{aligned}$$

$$\begin{aligned}
\delta A_{(I)\mu} &= - \left(\frac{s_I}{k}(\xi_{\bar{A}\underline{B}})^a(\gamma_\mu)_{ab} \left[(j_{(I)}^{\bar{A}\underline{B}})^b - (j_{(I-1)}^{\bar{A}\underline{B}})^b \right] \right)_{k_I \neq 0} \\
&- \left((\xi_{\bar{A}\underline{B}})^a(\gamma_\mu)_{ab} \left((\lambda_{(I)}^{\bar{A}\underline{B}})^b + \frac{s_I}{2k} \left[(j_{(I)}^{\bar{A}\underline{B}})^b + (j_{(I-1)}^{\bar{A}\underline{B}})^b \right] \right) \right)_{k_I=0}, \tag{A.3d}
\end{aligned}$$

$$\delta(\varphi_{(I)})^{\underline{A}}_{\underline{B}} = 2i(\xi_{\bar{C}\underline{B}})^a(\lambda_{(I)}^{\bar{C}\underline{A}})_a - \delta^{\underline{A}}_{\underline{B}}i(\xi_{\bar{C}\underline{D}})^a(\lambda_{(I)}^{\bar{C}\underline{D}})_a, \tag{A.3e}$$

where for the hypermultiplet fields the label (I) indicates the link between the I^{th} and $(I+1)^{\text{st}}$ node and $s_I = 1$ for a hypermultiplet. The bracketed terms vanish unless the condition in the subscript is satisfied. To obtain the transformations for the twisted multiplets, we replace the overlined $SU(2)_A$ indices for underlined $SU(2)_B$ indices and choose $s_I = -1$.

The superconformal transformations of the fields are found by replacing $\xi^a \rightarrow x^\mu(\gamma_\mu)^a_b \eta^b$, with the exception of the fermions. The variation of the fermions with respect to the superconformal supercharges are given by

$$\begin{aligned}
\delta(\psi_{(I)\underline{A}})^a &= \sqrt{2}(\not{D})^a_b q_{(I)}^{\bar{B}} x^\mu(\gamma_\mu)^b_c (\eta_{\bar{B}\underline{A}})^c + \sqrt{2}q_{(I)}^{\bar{B}} \eta_{\bar{B}\underline{A}}^a \\
&- \frac{\sqrt{2}s_I}{k} x^\mu(\gamma_\mu)^a_b (\eta_{\bar{B}\underline{A}})^b \left[(\nu_{(I)})^{\bar{C}} \bar{q}_{(I)}^{\bar{B}} - q_{(I)}^{\bar{B}} (\tilde{\nu}_{(I)})^{\bar{C}} \bar{C} \right] \\
&+ \left(\sqrt{2}x^\mu(\gamma_\mu)^a_b (\eta_{\bar{C}\underline{B}})^b (\varphi_{(I)})^{\underline{B}}_{\underline{A}} q_{(I)}^{\bar{C}} \right)_{k_I=0} \\
&- \left(\frac{2\sqrt{2}s_I}{k} x^\mu(\gamma_\mu)^a_b (\eta_{\bar{C}\underline{B}})^b (\tilde{\mu}_{(I-1)})^{\underline{B}}_{\underline{A}} q_{(I)}^{\bar{C}} \right)_{k_I \neq 0} \\
&- \left(\sqrt{2}x^\mu(\gamma_\mu)^a_b (\eta_{\bar{C}\underline{B}})^b q_{(I)}^{\bar{C}} (\varphi_{(I+1)})^{\underline{B}}_{\underline{A}} \right)_{k_{I+1}=0} \\
&+ \left(\frac{2\sqrt{2}s_I}{k} x^\mu(\gamma_\mu)^a_b (\eta_{\bar{C}\underline{B}})^b q_{(I)}^{\bar{C}} (\mu_{(I+1)})^{\underline{B}}_{\underline{A}} \right)_{k_{I+1} \neq 0}, \tag{A.4}
\end{aligned}$$

$$\begin{aligned}
\delta(\bar{\psi}_{(I)}^A)_a &= \sqrt{2}(\not{D})_{ab}\bar{q}_{(I)}^{\bar{B}}x^\mu(\gamma_\mu)^b{}_c\eta^{\bar{B}A})^c + \sqrt{2}\bar{q}_{(I)\bar{B}}\eta_a^{\bar{B}A} \\
&+ \frac{\sqrt{2}s_I}{k}x^\mu(\gamma_\mu)^a{}_b\eta^{\bar{B}A})^b \left[(\tilde{\nu}_{(I)})^{\bar{C}}{}_{\bar{C}}\bar{q}_{(I)\bar{B}} - \bar{q}_{(I)\bar{B}}(\nu_{(I)})^{\bar{C}}{}_{\bar{C}} \right] \\
&+ \left(\sqrt{2}x^\mu(\gamma_\mu)^a{}_b\eta^{\bar{C}B})^b\bar{q}_{(I)\bar{C}}(\varphi_{(I)})^A{}_{\underline{B}} \right)_{k_I=0} \\
&- \left(\frac{2\sqrt{2}s_I}{k}x^\mu(\gamma_\mu)^a{}_b\eta^{\bar{C}B})^b\bar{q}_{(I)\bar{C}}(\tilde{\mu}_{(I-1)})^A{}_{\underline{B}} \right)_{k_I \neq 0} \\
&- \left(\sqrt{2}x^\mu(\gamma_\mu)^a{}_b\eta^{\bar{C}B})^b(\varphi_{(I+1)})^A{}_{\underline{B}}\bar{q}_{(I)\bar{C}} \right)_{k_{I+1}=0} \\
&+ \left(\frac{2\sqrt{2}s_I}{k}x^\mu(\gamma_\mu)^a{}_b\eta^{\bar{C}B})^b(\mu_{(I+1)})^A{}_{\underline{B}}\bar{q}_{(I)\bar{C}} \right)_{k_{I+1} \neq 0}.
\end{aligned} \tag{A.5}$$

Appendix B

Derivation of the Killing spinor (2.123)

In this appendix we work out the explicit expression for the Killing spinor associated to the first parameterization of the space employed in the main text, namely equations (2.117) and (2.121).

The AdS part is standard. For example, for a time-like line the vielbeins can be chosen as follows

$$e^0 = R \cosh u \cosh \rho dt, \quad e^1 = R \cosh u d\rho, \quad e^2 = R du, \quad e^3 = R \sinh u d\phi, \quad (\text{B.1})$$

and the relevant non-vanishing components of the spin connection are given by

$$\omega^{01} = \sinh \rho dt, \quad \omega^{02} = \sinh u \cosh \rho dt, \quad \omega^{12} = \sinh u d\rho, \quad \omega^{32} = \cosh u d\phi, \quad (\text{B.2})$$

The indices $0, 1, \dots, 9, \#$ are on the tangent space and in the following we denote them by the letters from the beginning of the Latin alphabet $a, b, \dots$. We can take the background 4-form field-strength to be proportional to the volume form of AdS_4

$$H_{(4)} = \kappa e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \kappa R^4 \cosh^2 u \cosh \rho \sinh u dt \wedge d\rho \wedge du \wedge d\phi, \quad (\text{B.3})$$

with κ a constant to be determined. The Killing spinor equation is given by the variation of the gravitino and reads [110]

$$D_\mu \eta + \frac{1}{288} (\gamma_{\mu\nu\rho\sigma\tau} H^{\nu\rho\sigma\tau} - 8 H_{\mu\nu\rho\sigma} \gamma^{\nu\rho\sigma}) \eta = 0, \quad (\text{B.4})$$

where $D_\mu = \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab}$ and $\gamma_\mu = e_\mu^a \Gamma_a$ (lower case γ 's are in the curved space, while upper case Γ 's are in the tangent space). Now we distinguish between the AdS

and the sphere directions. Using the expression for the 4-form we arrive at

$$\begin{aligned} D_\mu \eta &= -\frac{\kappa}{6} \gamma_\mu \Gamma_\star \eta, & \mu \text{ along } AdS_4, \\ D_\mu \eta &= \frac{\kappa}{12} \gamma_\mu \Gamma_\star \eta, & \mu \text{ along } S^7, \end{aligned} \quad (\text{B.5})$$

with $\Gamma_\star \equiv \Gamma_{0123}$. To find the solution it is useful to use the following identities. If $[X, Y] = 2Z$ and $[X, Z] = -2Y$ then

$$e^{\frac{1}{2}\theta X} Y = (\cos \theta Y + \sin \theta Z) e^{\frac{1}{2}\theta X}. \quad (\text{B.6})$$

If, on the other hand, $[X, Y] = \pm 2Z$ and $[X, Z] = \pm 2Y$ then

$$e^{\frac{1}{2}\theta X} Y = (\cosh \theta Y \pm \sinh \theta Z) e^{\frac{1}{2}\theta X}. \quad (\text{B.7})$$

The AdS_4 part of the Killing spinor is given by

$$e^{\frac{u}{2}\Gamma_2\Gamma_\star} e^{\frac{\rho}{2}\Gamma_1\Gamma_\star} e^{\frac{t}{2}\Gamma_0\Gamma_\star} e^{\frac{\phi}{2}\Gamma_{23}} \eta_0. \quad (\text{B.8})$$

It can be readily checked that it must be $\kappa = -3/R$.

On the other hand, the vielbeins for the 7-sphere (2.117) are given by

$$\begin{aligned} e^4 &= 2R d\alpha, \\ e^5 &= -R \cos \alpha d\theta_1, \quad e^6 = -R \cos \alpha (2dx + \cos \theta_1 d\phi_1), \\ e^7 &= R \cos \alpha \sin \theta_1 d\phi_1, \quad e^8 = R \sin \alpha d\theta_2, \\ e^9 &= R \sin \alpha (2dv + \cos \theta_2 d\phi_2), \quad e^\# = -R \sin \alpha \sin \theta_2 d\phi_2, \end{aligned} \quad (\text{B.9})$$

and the non-vanishing components of the spin connection are

$$\begin{aligned} \omega^{54} &= \frac{1}{2} \sin \alpha d\theta_1, \quad \omega^{64} = \sin \alpha (dx + \frac{1}{2} \cos \theta_1 d\phi_1), \\ \omega^{74} &= -\frac{1}{2} \sin \alpha \sin \theta_1 d\phi_1, \quad \omega^{48} = -\frac{1}{2} \cos \alpha d\theta_2, \\ \omega^{49} &= -\cos \alpha (dv + \frac{1}{2} \cos \theta_2 d\phi_2), \quad \omega^{4\#} = \frac{1}{2} \cos \alpha \sin \theta_2 d\phi_2, \\ \omega^{65} &= -\frac{1}{2} \sin \theta_1 d\phi_1, \quad \omega^{75} = dx - \frac{1}{2} \cos \theta_1 d\phi_1, \quad \omega^{76} = \frac{1}{2} d\theta_1, \\ \omega^{98} &= -\frac{1}{2} \sin \theta_2 d\phi_2, \quad \omega^{\#8} = dv - \frac{1}{2} \cos \theta_2 d\phi_2, \quad \omega^{\#9} = \frac{1}{2} d\theta_2. \end{aligned} \quad (\text{B.10})$$

The Killing spinor equations for the sphere components are

$$\begin{aligned}
\partial_\alpha \eta &= -\frac{1}{2} \Gamma_4 \Gamma_\star \eta, \\
\partial_{\theta_1} \eta &= \frac{1}{4} (\sin \alpha \Gamma_{45} + \Gamma_{67} + \cos \alpha \Gamma_5 \Gamma_\star) \eta, \\
\partial_x \eta &= \frac{1}{2} (\sin \alpha \Gamma_{46} + \Gamma_{57} + \cos \alpha \Gamma_6 \Gamma_\star) \eta, \\
\partial_{\phi_1} \eta &= -\frac{1}{4} \left(\sin \alpha (\cos \theta_1 \Gamma_{64} - \sin \theta_1 \Gamma_{74}) + \sin \theta_1 \Gamma_{56} + \cos \theta_1 \Gamma_{57} \right. \\
&\quad \left. - \cos \alpha (\cos \theta_1 \Gamma_6 \Gamma_\star - \sin \theta_1 \Gamma_7 \Gamma_\star) \right) \eta, \\
\partial_{\theta_2} \eta &= -\frac{1}{4} (\cos \alpha \Gamma_{84} + \Gamma_{9\#} + \sin \alpha \Gamma_8 \Gamma_\star) \eta, \\
\partial_v \eta &= -\frac{1}{2} (\cos \alpha \Gamma_{94} + \Gamma_{8\#} + \sin \alpha \Gamma_9 \Gamma_\star) \eta, \\
\partial_{\phi_2} \eta &= -\frac{1}{4} \left(-\cos \alpha (\cos \theta_2 \Gamma_{49} - \sin \theta_2 \Gamma_{4\#}) + \sin \theta_2 \Gamma_{89} + \cos \theta_2 \Gamma_{8\#} \right. \\
&\quad \left. + \sin \alpha (\cos \theta_2 \Gamma_9 \Gamma_\star - \sin \theta_2 \Gamma_{\#} \Gamma_\star) \right) \eta.
\end{aligned} \tag{B.11}$$

With a little bit of algebra it can be checked that $\eta = \Psi \eta_0$, with Ψ given by (2.123), is indeed a solution of these equations.

Appendix C

Taylor expansion of theta functions

For our analysis in Section 4.2.1 and Section 4.3 we need to understand the behaviour of theta functions of the form $\theta(z \pm \int_0^\lambda)$ with λ near a branch point λ_i . As can be seen from (4.18) (see also [109]), the differentials are best expressed in terms of the variables $y = -i\sqrt{\lambda - \lambda_i}$, and then have a finite limit at $\lambda = \lambda_i$. It is straightforward to see that the integral $\int_0^\lambda$ is an odd function of y . The Taylor expansion about $y = 0$ gives

$$\theta(z \pm \int_0^\lambda) = \theta(z \pm \int_0^{\lambda_i}) \pm y D_{\lambda_i} \theta(z \pm \int_0^{\lambda_i}) + \mathcal{O}(y^2). \quad (\text{C.1})$$

We are particularly interested in the following expansion about $\lambda = 0$

$$\begin{aligned} \hat{\theta}(z + \int_0^\lambda) \theta(z - \int_0^\lambda) - \theta(z + \int_0^\lambda) \hat{\theta}(z - \int_0^\lambda) \\ = 2y(\theta(z) D_0 \hat{\theta}(z) - \hat{\theta}(z) D_0 \theta(z)) + \mathcal{O}(y^3), \end{aligned} \quad (\text{C.2})$$

which we see behaves as $\sqrt{\lambda}$.

Similarly, for large λ we may expand about $w = -i/\sqrt{\lambda}$ and it is straightforward to see that

$$\begin{aligned} \hat{\theta}(z + \int_0^\lambda) \theta(z - \int_0^\lambda) - \theta(z + \int_0^\lambda) \hat{\theta}(z - \int_0^\lambda) \\ = 2w(\theta(z) D_\infty \hat{\theta}(z) - \hat{\theta}(z) D_\infty \theta(z)) + \mathcal{O}(w^3). \end{aligned} \quad (\text{C.3})$$

Appendix D

An alternative approach for genus-one curves

In Appendix D.2 we provide an alternative derivation of a Lax operator and algebraic curve for the genus-one solutions. This approach defines a Lax operator with respect to the sigma-model connection rather than the Pohlmeyer reduced connection. We firstly discuss the relationship between these two Lax approaches.

D.1 Equivalence sigma-model and Pohlmeyer reduced Lax operators

It is straightforward to see that the currents (4.34) are given in terms of the reduced currents (4.7) via

$$\mathcal{J}(x) = \frac{1}{1-x} (h^\dagger)^{-1} (j + \bar{j}^\dagger) h^\dagger, \quad \bar{\mathcal{J}}(x) = \frac{1}{1+x} (h^\dagger)^{-1} (\bar{j} + j^\dagger) h^\dagger. \quad (\text{D.1})$$

with h defined by (4.3). Taking the Lax connection M in (4.29) to be given by the generalized Maurer-Cartan forms (4.34) the holomorphic part of the Lax equation may be written as

$$\partial L = \frac{1}{1-x} [L, (h^\dagger)^{-1} (j + \bar{j}^\dagger) h^\dagger]. \quad (\text{D.2})$$

Let us define $\mathcal{L} \equiv h^\dagger L (h^\dagger)^{-1}$. Then (D.2) is equivalent to

$$\partial \mathcal{L} = \frac{1}{1-x} ([\mathcal{L}, j] + x[\mathcal{L}, \bar{j}^\dagger]), \quad (\text{D.3})$$

which is none other than

$$\partial \mathcal{L} = [\mathcal{L}, J]. \quad (\text{D.4})$$

Therefore, if we solve for $\mathcal{L}$ we may obtain L by simple gauge transformation and furthermore the characteristic polynomials of L and $\mathcal{L}$ are identical.

D.2 Factorized string method for genus-one

We have constructed the algebraic curve for all-genus. This was done for a Lax operator with respect to the Pohlmeyer connection. We now construct the Lax operator with respect to the sigma-model connection, for the case of genus-one. The resulting expression will be given in terms of the sigma-model spectral parameter x , rather than the Pohlmeyer spectral parameter $(1+x)/(1-x)$.

The factorized string method of [83] separates the functional dependence of the connections on the two world-sheet coordinates σ and τ (where $w = \sigma + is$). This can be implemented for the genus-one solutions where as we will show, the Maurer-Cartan forms may be expressed as

$$\mathcal{J}_\sigma(\sigma, \tau) = S^{-1}(\sigma) \mathcal{J}_\sigma^0(\tau) S(\sigma), \quad \mathcal{J}_\tau(\sigma, \tau) = S^{-1}(\sigma) \mathcal{J}_\tau^0(\tau) S(\sigma). \quad (\text{D.5})$$

In the case of interest $S \in SL(2; \mathbb{C})/SU(2)$. As a consequence of this factorization, a Lax operator may be defined as $L \equiv \mathcal{J}_\sigma(x) - \mathcal{J}_S^L$. Here $\mathcal{J}_\sigma(x) \equiv \frac{1}{1-x^2}(\mathcal{J}_\sigma - ix\mathcal{J}_\tau)$ and $\mathcal{J}_S^L \equiv S^{-1}\partial_\tau S = \text{constant}$. Note that this Lax operator is defined with respect to the generalized Maurer-Cartan connection, *i.e.*, it satisfies

$$\partial_\sigma L = [L, \mathcal{J}_\sigma(x)], \quad \partial_\tau L = [L, \mathcal{J}_\tau(x)]. \quad (\text{D.6})$$

The first step is to write down the explicit expressions for the Maurer-Cartan forms for these solutions. It follows from the definitions of the currents and (4.28) that

$$\begin{aligned} \mathcal{J}_\sigma^* &= \frac{\vartheta_3^2(\int_0^\lambda)}{D_\infty \vartheta_1(0) \vartheta_1(2 \int_0^\lambda)} \frac{1}{\vartheta_1^2(z)} \\ &\times \begin{pmatrix} \lambda \vartheta_{3,+} \vartheta_{3,-} - \vartheta_{1,+} \vartheta_{1,-} & (\lambda \vartheta_{3,-}^2 + \vartheta_{1,-}^2) e^{-2\tilde{\mu}\sigma - 2\tilde{\nu}\tau} \\ (-\lambda \vartheta_{3,+}^2 - \vartheta_{1,+}^2) e^{2\tilde{\mu}\sigma + 2\tilde{\nu}\tau} & -\lambda \vartheta_{3,+} \vartheta_{3,-} + \vartheta_{1,+} \vartheta_{1,-} \end{pmatrix}, \\ \mathcal{J}_\tau^* &= \frac{\vartheta_3^2(\int_0^\lambda)}{D_\infty \vartheta_1(0) \vartheta_1(2 \int_0^\lambda)} \frac{i}{\vartheta_1^2(z)} \\ &\times \begin{pmatrix} -\lambda \vartheta_{3,+} \vartheta_{3,-} - \vartheta_{1,+} \vartheta_{1,-} & (-\lambda \vartheta_{3,-}^2 + \vartheta_{1,-}^2) e^{-2\tilde{\mu}\sigma - 2\tilde{\nu}\tau} \\ (\lambda \vartheta_{3,+}^2 - \vartheta_{1,+}^2) e^{2\tilde{\mu}\sigma + 2\tilde{\nu}\tau} & \lambda \vartheta_{3,+} \vartheta_{3,-} + \vartheta_{1,+} \vartheta_{1,-} \end{pmatrix}, \end{aligned} \quad (\text{D.7})$$

where we have written the complex conjugates of the currents for presentational purposes and we defined $\tilde{\mu}(\lambda)\sigma + \tilde{\nu}(\lambda)\tau \equiv \mu(\lambda)z + \nu(\lambda)\bar{z}$,¹ and we have introduced the shorthand, $\vartheta_{3,\pm} \equiv \theta(z \pm \int_0^\lambda)$ and $\vartheta_{1,\pm} \equiv \hat{\theta}(z \pm \int_0^\lambda)$. We treat λ as a constant here.

We have thus far failed to mention anything about the form of $z(w, \bar{w})$. It is

¹Again, the specific forms of $\tilde{\mu}$ and $\tilde{\nu}$ are not important, we only note that they are functions of the spectral parameter.

related to the world-sheet coordinates by [1]

$$z \equiv 2 [\omega(\infty)w + \omega(0)\bar{w}] , \quad (\text{D.8})$$

which for the genus-one case, reduces to

$$z = 4i\omega(\infty)\tau . \quad (\text{D.9})$$

In particular, this ensures that none of the theta functions are a function of σ . This may be exploited to factorize the currents with respect to σ . Consider the matrix

$$S(\sigma) = \begin{pmatrix} e^{\tilde{\mu}\sigma} & 0 \\ 0 & e^{-\tilde{\mu}\sigma} \end{pmatrix} \in SL(2; \mathbb{C})/SU(2) . \quad (\text{D.10})$$

It is apparent that this matrix satisfies (D.5), with

$$\begin{aligned} \mathcal{J}_\sigma^{0\star} &= \frac{\vartheta_3^2(\int_0^\lambda)}{D_\infty \vartheta_1(0) \vartheta_1(2 \int_0^\lambda)} \frac{1}{\vartheta_1^2(z)} \\ &\times \begin{pmatrix} \lambda \vartheta_{3,+} \vartheta_{3,-} - \vartheta_{1,+} \vartheta_{1,-} & (\lambda \vartheta_{3,-}^2 + \vartheta_{1,-}^2) e^{-2\tilde{\nu}\tau} \\ (-\lambda \vartheta_{3,+}^2 - \vartheta_{1,+}^2) e^{2\tilde{\nu}\tau} & -\lambda \vartheta_{3,+} \vartheta_{3,-} + \vartheta_{1,+} \vartheta_{1,-} \end{pmatrix} , \end{aligned} \quad (\text{D.11})$$

$$\begin{aligned} \mathcal{J}_\tau^{0\star} &= \frac{\vartheta_3^2(\int_0^\lambda)}{D_\infty \vartheta_1(0) \vartheta_1(2 \int_0^\lambda)} \frac{i}{\vartheta_1^2(z)} \\ &\times \begin{pmatrix} -\lambda \vartheta_{3,+} \vartheta_{3,-} - \vartheta_{1,+} \vartheta_{1,-} & (-\lambda \vartheta_{3,-}^2 + \vartheta_{1,-}^2) e^{-2\tilde{\nu}\tau} \\ (\lambda \vartheta_{3,+}^2 - \vartheta_{1,+}^2) e^{2\tilde{\nu}\tau} & \lambda \vartheta_{3,+} \vartheta_{3,-} + \vartheta_{1,+} \vartheta_{1,-} \end{pmatrix} . \end{aligned} \quad (\text{D.12})$$

Additionally

$$\mathcal{J}_S^L = \tilde{\mu} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \text{constant} . \quad (\text{D.13})$$

As L is traceless, the spectral curve may be given by

$$y^2 = \frac{1}{2} \text{tr} L^2, \quad (\text{D.14})$$

or more explicitly

$$\begin{aligned} y^2 &= \frac{1}{2(1-x^2)^2} \text{tr} \left[(\mathcal{J}_\sigma - \mathcal{J}_S^L)^2 + x \{2i(\mathcal{J}_S^L - \mathcal{J}_\sigma)\mathcal{J}_\tau\} \right. \\ &\quad \left. + x^2 \{2(\mathcal{J}_\sigma - \mathcal{J}_S^L)\mathcal{J}_S^L - \mathcal{J}_\tau^2\} + x^3 \{-2i\mathcal{J}_S^L \mathcal{J}_\tau\} + x^4 \{(\mathcal{J}_S^L)^2\} \right] . \end{aligned} \quad (\text{D.15})$$

The calculation of the coefficients is straightforward and requires the elliptic theta

function identities given in Section 4.4. The result is

$$y^2 = \frac{1}{(1-x^2)^2} \left\{ 2 \left(\lambda - \frac{1}{\lambda} \right) - 2 \left(\lambda_1 - \frac{1}{\lambda_1} \right) + x \left[-4 \left(\lambda - \frac{1}{\lambda} \right) \right] \right. \\ \left. + x^2 \left[4 \left(\lambda_1 - \frac{1}{\lambda_1} \right) \right] + x^3 \left[4 \left(\lambda - \frac{1}{\lambda} \right) \right] \right. \\ \left. + x^4 \left[-2 \left(\lambda - \frac{1}{\lambda} \right) - 2 \left(\lambda_1 - \frac{1}{\lambda_1} \right) \right] \right\}, \quad (\text{D.16})$$

where λ_1 is one of the finite branch points of the genus-one Riemann surface defining the elliptic theta functions (recall that other finite branch point is $-1/\lambda_1$). This is birationally equivalent to

$$y^2 = \lambda \frac{1+x}{1-x} - \frac{1}{\lambda} \frac{1-x}{1+x} - \lambda_1 + \frac{1}{\lambda_1}. \quad (\text{D.17})$$

We see that solutions with different values of λ lead to birationally equivalent curves. Let us absorb $\tilde{\lambda} \equiv \lambda \frac{1+x}{1-x}$ and $\tilde{y} \equiv y/\tilde{\lambda}$ which brings the curve to the standard form

$$\tilde{y}^2 = \tilde{\lambda} \left(\tilde{\lambda} - \lambda_1 \right) \left(\tilde{\lambda} + \frac{1}{\lambda_1} \right). \quad (\text{D.18})$$

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