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Estimation of Domain of Attraction for Discrete-Time Positive Interval Type-2 Polynomial Fuzzy Systems with Input Saturation

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***Abstract*—This paper focuses on expanding the estimation of the domain of attraction (DOA) for discrete-time positive nonlinear systems subject to input saturation and parameter uncertainties. To facilitate analysis and design, the interval type-2 (IT2) polynomial fuzzy model is used to represent the nonlinear plant and capture uncertainties. Combining with the IT2 polynomial fuzzy controller, the discrete-time positive IT2 polynomial fuzzy-model-based (PIT2PFMB) control system is formed to facilitate analysis. To enlarge the estimation of DOA of the discrete-time PIT2PFMB system, polyhedron is used to characterize the DOA with the help of linear copositive Lyapunov function (LCLF). Referring to the non-convex conditions derived by LCLF, an effective convexification method is proposed in this paper. For comparison purposes, the saturation-dependent-Lyapunov-function-based method is extended to the PIT2PFMB control system by adding the corresponding positivity conditions. In addition, this paper attempts to enlarge the estimation of the DOA by improving the IT2 membership-function-dependent (IT2MFD) method and extending it to all conditions including the stability conditions and the DOA estimation conditions. Finally, an example with simulation results is given to verify the effectiveness of all the methods proposed in this paper for expanding the estimation of the DOA.**

***Index Terms*—discrete-time positive interval type-2 polynomial fuzzy-model-based (PIT2PFMB) system, linear copositive Lyapunov function (LCLF), domain of attraction (DOA), interval type-2 membership function dependent (IT2MFD) method**

I. Introduction

Positive systems are a kind of systems whose state vectors remain in positive quadrant for any nonnegative initial conditions. In some practical systems, such as population systems [1], biology [2], network systems [3], the state variables cannot adopt negative values. Therefore, the systematic research on the positive systems is a realistic demand. Based on a series of research on the positive linear systems, Farina *et al.* gave some results related to the positive linear systems, including their properties, control, and the application [4]. In addition, the prevalence of the nonlinear characteristics in the practical systems stimulates the emergence of some research results on positive nonlinear systems. Starting from literature [5] and [6], Takagi-Sugeno (T-S) fuzzy model and polynomial fuzzy model were used to describe the various types of positive nonlinear systems [7]–[9]. These two fuzzy models all make it easier to analyze positive nonlinear systems by representing positive nonlinear systems as a set of linear subsystems weighted by nonlinear membership functions (MFs). Compared with T-S fuzzy model, the polynomial fuzzy model has stronger modeling ability, because it allows polynomial terms to exist in the subsystem matrices. Also, thanks to the polynomial terms [10], the polynomial fuzzy controller demonstrating a stronger compensation capability is a good candidate controller for highly nonlinear dynamic systems [11], [12]. Following the polynomial fuzzy model based analysis method, the sum-of-squares (SOS) based conditions are obtained, and a feasible solution of these conditions can be found numerically by the third-party MATLAB toolbox SOSTOOLS [13].

The T-S fuzzy model and polynomial fuzzy model mentioned above belong to the type-1 fuzzy model. Type-1 fuzzy model has limited capabilities to directly handle uncertainties of system parameters, although it is a very useful tool to deal with nonlinearity of systems. To cope with the uncertain system parameters in practical systems [14]–[18], type-2 fuzzy sets [19] have more potential to capture uncertainties. For general type-2 fuzzy sets, the membership grades of secondary MFs are functions of the premise variables, which can represent the system uncertainties more effectively [20], [21], but the computational burden will increase accordingly. Thus, the work [22] proposed the Newton-Cotes quadrature for the α-planes integration to reduce the computational cost of general type-2 fuzzy systems. When the uncertainties of the system parameters is low, the interval type-2 (IT2) fuzzy sets [23]–[25] can replace general type-2 fuzzy sets to model the nonlinear systems, so that the computational cost is further reduced. The IT2 fuzzy sets is equivalent to one α-plane of general type-2 fuzzy sets, so it can effectively reduce the computational cost. In order to introduce IT2 fuzzy sets into control strategies of IT2 fuzzy logic systems, the stability analysis and control synthesis of IT2 fuzzy-model-based (FMB) control systems were investigated for the first time in work [26]. Whereafter, the IT2 fuzzy model was extended to the positive systems [27].

Input saturation is a common phenomenon caused by the physical constraint of the actuator, and its main feature is the capped control signal. The capping of control signals will degrade the performance and even affect the stability of

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control systems. In order to guarantee the various performance under input saturation, scholars have done extensive research on input saturation to reduce its negative impact on FMB systems performance [28]–[30]. Also, the input saturation will destroy the global stability of the system, i.e., Only local states whose set is usually called the domain of attraction (DOA) can be driven to the equilibrium point by capped control signals. To reduce the impact of saturation on the system stability, [31] designed a novel controller to expand the estimation of the DOA for general T-S FMB control systems, so that the system states that deviate further from the equilibrium point can be steered to the equilibrium point. Recently, considering the influence of the positive constraint of the system states and the type-1 MFs information on the estimation of the DOA of the continue-time FMB systems, a novel method was proposed in work [32] to enlarge the estimation of the DOA for positive polynomial fuzzy systems. However, to the best of authors' knowledge, limited works in the literature investigate the estimation of DOA for discrete-time positive interval type-2 polynomial fuzzy-model-based (PIT2PFMB) control systems with input saturation. Facing the demand for theoretical research triggered by the widespread existence of positive nonlinear systems with uncertainty and input saturation in real life, such as buck converter [33] and pest population system [34], the enlargement of the estimation of DOA for discrete-time PIT2PFMB systems is investigated in this paper.

In the following, the methods for relaxing the estimation of DOA for discrete-time PIT2PFMB control systems will be discussed. In the existing literatures, the expansion of the estimation of the DOA was generally achieved by two types of methods. The first type of methods concentrate on improving the capacity of dealing with saturation nonlinearity. In order to overcome the challenge caused by the sharp change in the saturation characteristic for stability analysis, a smooth function was adopted in the works [35], [36] to approximate the saturation function. Besides, the work [37] adopted the sector-condition-based method to handle the saturation nonlinearity. However, the controller and auxiliary controller designed by this method must be in a given ratio. In contrast, the convex hull representation method [38] proposed by Hu *et al.* seems to be less conservative, because the controller and the auxiliary controller can be designed arbitrarily. Because of this advantage, this method is still widely used today [39], [40]. Furthermore, the work [41] conducted an in-depth study on convex hull representation method, and further improved this method by designing multiple auxiliary controllers. Another type of methods to enlarge the estimation of DOA are to consider the saturation information and derive the saturation-dependent results. For example, the anti-windup control scheme which switches between two different control strategies based on whether saturation occurs was adopted in work [42] to relax the estimation of DOA. In addition, the saturation information can also be introduced into the Lyapunov function [43], [44]. As a result, the saturation-dependent Lyapunov matrices increase the analysis flexibility. However, the saturation-dependent Lyapunov function is hardly applied to the positive systems, which motivates us to develop a complete set of theoretical methods for estimating the DOA for the positive systems based on the saturation-dependent Lyapunov function.

In addition to the above two types of methods, the shape of the invariant sets also affects the conservativeness of the results to a certain extent if invariant sets theory is used to estimate the DOA [45]. However, since the shape of the invariant sets are consistent with the level sets of the Lyapunov function, the conservativeness of the result is difficult to be reduced by only changing the shape of the invariant sets without considering the type of the Lyapunov function. Generally speaking, polyhedron has stronger ability to characterize the DOA than traditional ellipsoid, because polyhedrons are often natural expressions of physical constraints on control variables [45]. However, the linear copositive Lyapunov function (LCLF) [8], [46]–[48] whose level sets are polyhedrons will lead to the non-convex resultant conditions, brings difficulties in finding numerical solutions, so the LCLF and polyhedron are hardly used in the existing literature to estimate the DOA. It inspires us to adopt the LCLF-based method, so that the polyhedron can be used to represent a more relaxed DOA. Meanwhile, an effective convexification method is needed to handle the non-convex conditions derived by LCLF.

For the fuzzy systems, although some existing papers on the estimation of DOA have made improvement by some methods, the results are still conservative, because the fuzzy system is the combination of some local linear systems weighted by MFs, but the existing methods do not consider the effect of MFs. In the past few years, several seminal results about on IT2 membership-function-dependent (IT2MFD) analysis methods were proposed in literature [49]–[51] to reduce the conservatism. The works [49] and [50] introduced the upper and lower MFs information and the basic properties of the IT2 MFs into the resultant conditions through the S-procedure. In work [51], the information of the IT2 MFs was included into stability conditions by using polynomial approximated embedded type-1 MFs, and the boundary information of the premise variables were also considered. However, these IT2MFD analysis methods were only applied on the general IT2 fuzzy system to relax stability rather than expand the estimation of DOA, and some methods are cumbersome to implement. Motivated by the above discussion, we will devote to improving the IT2MFD analysis method presented in work [51] and extending the improved IT2MFD analysis method to the positive IT2 FMB control systems with input saturation to enlarge the estimation of DOA.

This paper is dedicated to enlarge the estimation of DOA for the discrete-time PIT2PFMB control system with input saturation. To reduce the computational burden and characterize the imperfect matching MFs resulted from parameter uncertainty, the IT2 polynomial fuzzy controller and auxiliary controller are designed under the imperfect premise matching (IPC) concept [49], [52]–[55] which means that the fuzzy model and controller can have different premise MFs and/or number of rules. The main contributions of this paper are summarized as follows:

1) It is the first attempt to investigate the estimation of DOA for the discrete-time PIT2PFMB control system with

input saturation. The influences of positive constrains, nonlinearity, parameter uncertainties and input saturation on the system are considered comprehensively.

2) Two novel methods are proposed to enlarge the estimation of DOA for discrete-time PIT2PFMB system.
   - The first novel method is that the LCLF and corresponding polyhedron invariant sets are utilized to enlarge the estimation of DOA. Compared with the saturation-dependent-Lyapunov-function-based method, this LCLF-based method leads to more relaxed results and lower computational burden. Also, an effective convexification method is proposed to develop the LCLF-based DOA estimation conditions.
   - The second novel approach is to propose an improved IT2MFD analysis method based on the piecewise linear embedded type-1 MFs, and extend this method to the DOA estimation conditions for the discrete-time PIT2PFMB control system. Compared with the existing IT2MFD analysis method [51], this improved IT2MFD analysis method allows more flexibility to choose the reference embedded type-1 MFs, which abandons the complicated process of finding the reference embedded type-1 MFs.

The organization of this paper is as follows. In Section II, the notations and the discrete-time IT2 polynomial fuzzy model with input saturation, discrete-time IT2 polynomial fuzzy controller are described. Also, the saturation-dependent-Lyapunov-function-based method is adapted to estimate the DOA for the discrete-time PIT2PFMB control system. In Section III, the LCLF is adopted to perform the stability analysis and DOA estimation of PIT2PFMB control system, and an effective method is proposed to handle the non-convex conditions. Moreover, the improved IT2MFD is given and applied on the resultant conditions. In Section IV, an example is used to illustrate the effectiveness of proposed methods for discrete-time PIT2PFMB control system. In Section V, a conclusion is drawn.

II. Preliminary

A. Notation

The following notations are used throughout the paper. A monomial in $\mathbf{x}(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T$ is a function in the form of $x_1^{d_1}(k)x_2^{d_2}(k)\ldots x_n^{d_n}(k)$, where $d_i \geq 0, i \in \{1, 2, \ldots, n\}$ are nonnegative integers. The degree of a monomial is $d = \sum_{i=1}^{n} d_i$. A polynomial $f(\mathbf{x}(k))$ is an SOS if there exist polynomials $f_1(\mathbf{x}(k)), f_2(\mathbf{x}(k)), \ldots, f_m(\mathbf{x}(k))$ such that $f(\mathbf{x}(k)) = \sum_{i=1}^{m} f_i^2(\mathbf{x}(k))$, where $f_i(\mathbf{x}(k))$ is a polynomial and m is a nonnegative integer. It is clear that $f(\mathbf{x}(k))$ being an SOS naturally implies $f(\mathbf{x}(k)) \geq 0$ for all $\mathbf{x}(k) \in \Re^n$. The expressions of $\mathbf{A} \prec 0$ and $\mathbf{A} \succ 0$, mean that all elements of $\mathbf{A}$ are negative and positive, respectively; $\mathbf{A} < 0$ and $\mathbf{A} > 0$ mean that $\mathbf{A}$ is negative definite and positive definite, respectively. $\mathbf{A}^{(\alpha,\beta)}$ is the α-th row, β-th column element of $\mathbf{A}$. $\mathbf{A}^{(:,\beta)}$ is a vector denoting the β-th column of $\mathbf{A}$. $\mathbf{A}^{(\alpha,:)}$ is a vector denoting the α-th row of $\mathbf{A}$. $\underline{p}$ represents $\{1, 2, \ldots, p\}$, where p is a non-zero integer. diag$\{\cdot\}$ denotes a square diagonal matrix with the elements of argument in the diagonal.

B. IT2 Polynomial Fuzzy Plant Model

The nonlinear system is described by an IT2 polynomial fuzzy model with p rules. The i^{th} rule is of the following format:

Rule i :
IF $f_1(\mathbf{x}(k))$ is M_1^i AND $\cdots$ AND $f_\psi(\mathbf{x}(k))$ is M_ψ^i,
THEN $\mathbf{x}(k+1) = \mathbf{A}_i(\mathbf{x}(k))\mathbf{x}(k) + \mathbf{B}_i(\mathbf{x}(k))\text{sat}(\mathbf{u}(k))$

where $\mathbf{x}(k) \in \Re^n$ and $\mathbf{u}(k) \in \Re^m$ are the state vector and control input vector of the system, respectively; n, m are their dimensions, $f_\vartheta(\mathbf{x}(k))$ is the premise variable and M_ϑ^i is an IT2 fuzzy set corresponding to its premise variable in rule i, $i \in \underline{p}$, $\vartheta \in \underline{\psi}$, and ψ is a positive integer; $\mathbf{A}_i(\mathbf{x}(k)) \in \Re^{n\times n}$, $\mathbf{B}_i(\mathbf{x}(k)) \in \Re^{n\times m}$ are the known polynomial system matrices and input matrices, respectively. The firing strength of the i^{th} rule is of the following interval sets:

$$W_i(\mathbf{x}(k)) = [\underline{w}_i(\mathbf{x}(k)), \overline{w}_i(\mathbf{x}(k))], \qquad i = 1, 2, \ldots, p \quad (1)$$

where $\underline{w}_i(\mathbf{x}(k)) = \prod_{\vartheta=1}^{\psi} \underline{\mu}_{M_\vartheta^i}(f_\vartheta(\mathbf{x}(k))) \geq 0$, $\overline{w}_i(\mathbf{x}(k)) = \prod_{\vartheta=1}^{\psi} \overline{\mu}_{M_\vartheta^i}(f_\vartheta(\mathbf{x}(k))) \geq 0$, $\overline{\mu}_{M_\vartheta^i}(f_\vartheta(\mathbf{x}(k))) \geq \underline{\mu}_{M_\vartheta^i}(f_\vartheta(\mathbf{x}(k))) \geq 0$, $\overline{w}_i(\mathbf{x}(k)) \geq \underline{w}_i(\mathbf{x}(k)) \geq 0, \forall i$ in which $\underline{w}_i(\mathbf{x}(k))$, $\overline{w}_i(\mathbf{x}(k))$, $\underline{\mu}_{M_\vartheta^i}(f_\vartheta(\mathbf{x}(k)))$ and $\overline{\mu}_{M_\vartheta^i}(f_\vartheta(\mathbf{x}(k)))$ denote the lower grade of membership, upper grade of membership, lower membership function, and upper membership function, respectively. The function sat$(\cdot)$: $\Re^m \to \Re^m$ is the standard saturation function. It is defined as

$$\text{sat}(\mathbf{u}(k)) = [\text{sat}(u^{(1,1)}(k)), \text{sat}(u^{(2,1)}(k)), \ldots, \text{sat}(u^{(m,1)}(k))]^T, \quad (2)$$

where

$$\text{sat}(u^{(\iota,1)}(k)) = \begin{cases} u_{lim} & \text{if} \quad u^{(\iota,1)}(k) > u_{lim} \\ u^{(\iota,1)}(k) & \text{if} \quad -u_{lim} \leq u^{(\iota,1)}(k) \leq u_{lim} \\ -u_{lim} & \text{if} \quad u^{(\iota,1)}(k) < -u_{lim} \end{cases},$$

$u^{(\iota,1)}(k)$ is the ι^{th} element of $\mathbf{u}(k)$, $\iota \in \underline{m}$, $u_{lim} > 0$ is the control input limit. The inferred IT2 polynomial fuzzy model is represented as follows:

$$\mathbf{x}(k+1) = \sum_{i=1}^{p} \tilde{w}_i(\mathbf{x}(k)) \left(\mathbf{A}_i(\mathbf{x}(k))\mathbf{x}(k) + \mathbf{B}_i(\mathbf{x}(k))\text{sat}(\mathbf{u}(k))\right), \quad (3)$$

where

$$\tilde{w}_i(\mathbf{x}(k)) = \underline{\xi}_i(\mathbf{x}(k))\underline{w}_i(\mathbf{x}(k)) + \overline{\xi}_i(\mathbf{x}(k))\overline{w}_i(\mathbf{x}(k)) \geq 0, \quad \forall i, \quad (4)$$

in which $\underline{\xi}_i(\mathbf{x}(k))$ and $\overline{\xi}_i(\mathbf{x}(k))$ are nonlinear functions not necessarily to be known but exist with the properties that $0 \leq \underline{\xi}_i(\mathbf{x}(k)) \leq 1$, $0 \leq \overline{\xi}_i(\mathbf{x}(k)) \leq 1$ and $\underline{\xi}_i(\mathbf{x}(k)) + \overline{\xi}_i(\mathbf{x}(k)) = 1$, $\forall i \in \underline{p}$. $\tilde{w}_i(\mathbf{x}(k))$ can be regarded as the grades of membership of the embedded type-1 membership functions, and (4) defines the type reduction. The parameter uncertainties of nonlinear systems will lead to the uncertain values of $\underline{\xi}_i(\mathbf{x}(k))$ and $\overline{\xi}_i(\mathbf{x}(k))$ which are used to represent $\tilde{w}_i(\mathbf{x}(k))$ with $\underline{w}_i(\mathbf{x}(k))$ and $\overline{w}_i(\mathbf{x}(k))$. $\tilde{w}_i(\mathbf{x}(k))$ satisfies the property that $\sum_{i=1}^{p} \tilde{w}_i(\mathbf{x}(k)) = 1$.

Definition 1: The polynomial fuzzy system (3) is said to be positive only if for every nonnegative initial state, its state variables and outputs are all nonnegative.

Lemma 1: A polynomial fuzzy system (3) is guaranteed to be positive if $\sum_{i=1}^{p} \tilde{w}_i(\mathbf{x}(k))\mathbf{A}_i(\mathbf{x}(k)) \succ 0$ when $\mathrm{sat}(\mathbf{u}(k)) = 0$.

C. IT2 Polynomial Fuzzy Controller

The IPC concept is adopted to design an IT2 polynomial fuzzy controller with c rules for the IT2 polynomial fuzzy system (3), the j^{th} rule of the polynomial fuzzy controller is as follows:

$$\begin{aligned}&\text{Rule } j: \text{IF } g_1(\mathbf{x}(k)) \text{ is } N_1^j \text{ AND} \cdots \text{AND } g_\phi(\mathbf{x}(k)) \text{ is } N_\phi^j,\\ &\qquad \text{THEN } \mathbf{u}(k) = \mathbf{G}_j(\mathbf{x}(k))\mathbf{x}(k)\end{aligned}$$

where $g_\vartheta(\mathbf{x}(k))$ is the premise variable and N_ϑ^j is an IT2 fuzzy set corresponding to its premise variable in rule j, $j \in \underline{p}$, $\vartheta \in \underline{\phi}$, and ϕ is a positive integer; $\mathbf{G}_j(\mathbf{x}(k)) \in \Re^{m\times n}$ are the polynomial fuzzy controller gains to be determined. The firing strength of the j^{th} rule is of the following interval sets:

$$M_j(\mathbf{x}(k)) = [\underline{m}_j(\mathbf{x}(k)), \overline{m}_j(\mathbf{x}(k))], \qquad j = 1, 2, \ldots, c \quad (5)$$

where $\underline{m}_j(\mathbf{x}(k)) = \prod_{\vartheta=1}^{\phi} \underline{\mu}_{N_\vartheta^j}(g_\vartheta(\mathbf{x}(k))) \geq 0$, $\overline{m}_j(\mathbf{x}(k)) = \prod_{\vartheta=1}^{\phi} \overline{\mu}_{N_\vartheta^j}(g_\vartheta(\mathbf{x}(k))) \geq 0$, $\overline{\mu}_{N_\vartheta^j}(g_\vartheta(\mathbf{x}(k))) \geq \underline{\mu}_{N_\vartheta^j}(g_\vartheta(\mathbf{x}(k))) \geq 0$, $\overline{m}_j(\mathbf{x}(k)) \geq \underline{m}_j(\mathbf{x}(k)) \geq 0, \forall j$ in which $\underline{m}_j(\mathbf{x}(k))$, $\overline{m}_j(\mathbf{x}(k))$, $\underline{\mu}_{N_\vartheta^i}(g_\vartheta(\mathbf{x}(k)))$ and $\overline{\mu}_{N_\vartheta^i}(g_\vartheta(\mathbf{x}(k)))$ denote the lower grade of membership, upper grade of membership, lower membership function, and upper membership function, respectively. The inferred IT2 polynomial fuzzy controller is represented as follows:

$$\mathbf{u}(k) = \sum_{j=1}^{c} \tilde{m}_j(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))\mathbf{x}(k), \quad (6)$$

where

$$\begin{aligned}\tilde{m}_j(\mathbf{x}(k)) &= \frac{\underline{\varsigma}_j(\mathbf{x}(k))\underline{m}_j(\mathbf{x}(k)) + \overline{\varsigma}_j(\mathbf{x}(k))\overline{m}_j(\mathbf{x}(k))}{\sum_{l=1}^{c}(\underline{\varsigma}_l(\mathbf{x}(k))\underline{m}_l(\mathbf{x}(k)) + \overline{\varsigma}_l(\mathbf{x}(k))\overline{m}_l(\mathbf{x}(k)))}\\ &\geq 0, \quad \forall j, \end{aligned} \quad (7)$$

in which $\underline{\varsigma}_j(\mathbf{x}(k))$ and $\overline{\varsigma}_j(\mathbf{x}(k))$ are predefined functions, they satisfy the properties that $0 \leq \underline{\varsigma}_j(\mathbf{x}(k)) \leq 1$, $0 \leq \overline{\varsigma}_j(\mathbf{x}(k)) \leq 1$ and $\underline{\varsigma}_j(\mathbf{x}(k)) + \overline{\varsigma}_j(\mathbf{x}(k)) = 1$, $\forall j \in \underline{c}$. $\tilde{m}_j(\mathbf{x}(k))$ can be regarded as the grades of membership of the embedded type-1 membership functions, and (7) is the type reduction.

D. Control Input Saturation

In order to handle the input saturation, the convex hull representation method is adopted in this paper.

Let $\mathbf{E}$ be the set of $m\times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathbf{E}$. Suppose that each element of $\mathbf{E}$ is labeled as $\mathbf{E}_s$, $s \in \underline{2^m}$. Then, $\mathbf{E} = \{\mathbf{E}_s : s \in \underline{2^m}\}$. Denote $\mathbf{E}_s^- = \mathbf{I} - \mathbf{E}_s$, then $\mathbf{E}_s^-$ is also an element of $\mathbf{E}$ if $\mathbf{E}_s \in \mathbf{E}$.

For matrices $\mathbf{G}_j(\mathbf{x}(k))$ and $\mathbf{H}_j(\mathbf{x}(k)) \in \Re^{m\times n}$, supposing that $|\mathbf{H}_j(\mathbf{x}(k))\mathbf{x}(k)|_\infty \leq u_{lim}$, then there exists $\eta_s(\mathbf{x}(k))$ satisfying $\eta_s(\mathbf{x}(k)) \geq 0$ and $\sum_{s=1}^{2^m} \eta_s(\mathbf{x}(k)) = 1$ so that $\mathrm{sat}(\mathbf{u}(k))$ can be represented as

$$\begin{aligned}&\mathrm{sat}(\mathbf{u}(k)) = \mathrm{sat}(\sum_{j=1}^{c} \tilde{m}_j(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))\mathbf{x}(k)) =\\ &\sum_{j=1}^{c}\sum_{s=1}^{2^m} \tilde{m}_j(\mathbf{x}(k))\eta_s(\mathbf{x}(k))[(\mathbf{E}_s\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{E}_s^-\mathbf{H}_j(\mathbf{x}(k)))\mathbf{x}(k)],\end{aligned} \quad (8)$$

where the polynomial fuzzy controller gain $\mathbf{G}_j(\mathbf{x}(k))$ and the auxiliary polynomial fuzzy controller gain $\mathbf{H}_j(\mathbf{x}(k))$ are to be determined. $\mathbf{E}_s\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{E}_s^-\mathbf{H}_j(\mathbf{x}(k)), \forall s \in \underline{2^m}, j \in \underline{c}$ is the set of matrices formed by choosing some rows from $\mathbf{G}_j(\mathbf{x}(k))$ and the rest from $\mathbf{H}_j(\mathbf{x}(k))$.

E. Saturation-dependent Lyapunov Function and Elliptical DOA

According to (3), (6) and (8), the closed-loop control system is obtained as follows:

$$\begin{aligned}&\mathbf{x}(k+1)\\ &= \sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{s=1}^{2^m} \tilde{w}_i(\mathbf{x}(k))\tilde{m}_j(\mathbf{x}(k))\eta_s(\mathbf{x}(k))[\mathbf{A}_i(\mathbf{x}(k))\\ &\quad + \mathbf{B}_i(\mathbf{x}(k))(\mathbf{E}_s\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{E}_s^-\mathbf{H}_j(\mathbf{x}(k)))]\mathbf{x}(k)\\ &= \sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{s=1}^{2^m} \tilde{h}_{ij}(\mathbf{x}(k))\eta_s(\mathbf{x}(k))[\mathbf{A}_i(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))\times\\ &\quad (\mathbf{E}_s\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{E}_s^-\mathbf{H}_j(\mathbf{x}(k)))]\mathbf{x}(k),\end{aligned} \quad (9)$$

where

$$\tilde{h}_{ij}(\mathbf{x}(k)) \equiv \tilde{w}_i(\mathbf{x}(k))\tilde{m}_j(\mathbf{x}(k)) \quad (10)$$

with $\sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{h}_{ij}(\mathbf{x}(k)) = 1$, $\underline{h}_{ij}(\mathbf{x}(k)) \leq \tilde{h}_{ij}(\mathbf{x}(k)) \leq \overline{h}_{ij}(\mathbf{x}(k))$, where $\underline{h}_{ij}(\mathbf{x}(k))$ and $\overline{h}_{ij}(\mathbf{x}(k))$ are the lower and upper grade of membership of closed-loop system, respectively.

In order to obtain the estimation of the DOA for discrete-time PIT2PFMB control system (9), the saturation-dependent Lyapunov function is used. As a result, a largest elliptical DOA is obtained by the following theorem:

Theorem 1: For the discrete-time PIT2PFMB control system (9), if there exist positive definite matrices $\mathbf{Q}_s \in \Re^{n\times n}$, a diagonal positive definite matrix $\mathbf{X} \in \Re^{n\times n}$, polynomial matrices $\mathbf{D}_j(\mathbf{x}(k)) \in \Re^{m\times n}$ and $\mathbf{O}_j(\mathbf{x}(k)) \in \Re^{m\times n}$, $\forall j \in \underline{c}$, and a scalar $\gamma > 0$ such that the following SOS-based conditions of optimization problem are satisfied:

$$
\begin{aligned}
&\min \gamma\\
&s.t. 1)\ \nu^T \begin{bmatrix} \gamma & (x_0^r)^T \\ x_0^r & \mathbf{Q}_s \end{bmatrix} \nu \text{ is SOS, } \forall r \in \underline{l}, s \in \underline{2^m};\\
&\quad 2)\ \nu^T(\mathbf{X} - \varepsilon_1 \mathbf{I})\nu \text{ is SOS;}\\
&\quad 3)\ \nu^T(\mathbf{Q}_s - \varepsilon_2 \mathbf{I})\nu \text{ is SOS, } s \in \underline{2^m};\\
&\quad 4)\ \nu^T(\mathbf{\Omega}_{ijs}^{(\alpha,\beta)}(\mathbf{x}(k)) - \varepsilon_3(\mathbf{x}(k)))\nu \text{ is SOS,}\\
&\quad\ \ \forall i \in \underline{p}, j \in \underline{c}, s \in \underline{2^m}, \alpha, \beta \in \underline{n};\\
&\quad 5)\ \nu^T(\mathbf{\Psi}_{ijs\hat{s}}(\mathbf{x}(k)) - \varepsilon_4(\mathbf{x}(k))\mathbf{I})\nu \text{ is SOS,}\\
&\quad\ \ \forall i \in \underline{p}, j \in \underline{c}, s, \hat{s} \in \underline{2^m};\\
&\quad 6)\ \nu^T \mathbf{\Phi}_{js\iota}(\mathbf{x}(k))\nu \text{ is SOS, } \forall j \in \underline{c}, s \in \underline{2^m}, \iota \in \underline{m};
\end{aligned} \tag{11}
$$

where ν is an arbitrary vector independent of $\mathbf{x}$ with appropriate dimensions; $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3(\mathbf{x}(k)) > 0$, $\varepsilon_4(\mathbf{x}(k)) > 0$ are predefined scalar polynomials; $\mathbf{\Psi}_{ijs\hat{s}}(\mathbf{x}(k))$, $\mathbf{\Phi}_{js\iota}(\mathbf{x}(k))$ and $\mathbf{\Omega}_{ijs}^{(\alpha,\beta)}(\mathbf{x}(k))$ are defined in (15), (20) and (23), then the system (9) is asymptotically stable and positive with initial conditions contained in $\Im(V)$, where $\Im(V)$ is defined in (16) and is regarded as the largest estimation of the DOA. The polynomial fuzzy controller gains can be obtained by $\mathbf{G}_j(\mathbf{x}(k)) = \mathbf{D}_j(\mathbf{x}(k))\mathbf{X}^{-1}$ and the auxiliary polynomial fuzzy controller gains can be obtained by $\mathbf{H}_j(\mathbf{x}(k)) = \mathbf{O}_j(\mathbf{x}(k))\mathbf{X}^{-1}$.

Proof 1: This proof consists of two parts: estimation of DOA for general polynomial FMB systems, positivity analysis. In the first part, saturation-dependent-Lyapunov-function-based DOA estimation method [43] is adapted to be applied to the general polynomial FMB systems. Since the results of the first part do not guarantee the positivity of system states, the positivity analysis is performed in the second part so that the results are applicable to PIT2PFMB systems (9).

Part I: Adapt saturation-dependent-Lyapunov-function-based DOA estimation method

In this part, the saturation-dependent-Lyapunov-function-based DOA estimation method [43] is adapted to derive the stability conditions and DOA estimation conditions.

By employing the saturation-dependent Lyapunov function $V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbf{P}(\eta(k))\mathbf{x}(k)$, where $\mathbf{P}(\eta(k)) = \sum_{s=1}^{2^m} \eta_s(\mathbf{x}(k))\mathbf{P}_s \in \Re^{n\times n}$ is a positive definition matrix, the stability condition of the system (9) is derived as follows:

$$
\begin{aligned}
&\Delta V(\mathbf{x}(k))\\
&= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k))\\
&= \mathbf{x}^T(k)[\mathbf{\Gamma}_{ijs}^T(\tilde{f}_{ijs}(\mathbf{x}(k)))\mathbf{P}(\eta(k+1))\mathbf{\Gamma}_{ijs}(\tilde{f}_{ijs}(\mathbf{x}(k)))\\
&\quad - \mathbf{P}(\eta(k))]\mathbf{x}(k) < 0.
\end{aligned} \tag{12}
$$

where

$$
\mathbf{\Gamma}_{ijs}(\tilde{f}_{ijs}(\mathbf{x}(k))) = \sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{s=1}^{2^m} \tilde{h}_{ij}(\mathbf{x}(k))\eta_s(\mathbf{x}(k)) \times
[\mathbf{A}_i(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))(\mathbf{E}_s\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{E}_s^- \mathbf{H}_j(\mathbf{x}(k)))].
$$

By applying the Schur complement, (12) can be guaranteed by the following condition:

$$
\begin{bmatrix} \mathbf{P}(\eta(k)) & \mathbf{\Gamma}_{ijs}^T(\tilde{f}_{ijs}(\mathbf{x}(k))) \\ \mathbf{\Gamma}_{ijs}(\tilde{f}_{ijs}(\mathbf{x}(k))) & \mathbf{P}^{-1}(\eta(k+1)) \end{bmatrix} > 0. \tag{13}
$$

Define $\mathbf{X} \in \Re^{n\times n}$ as a diagonal positive matrix, pre-multiplying and post-multiplying diag$(\mathbf{X}, \mathbf{I})$ to both sides of (13). Assuming $\mathbf{P}^{-1}(\eta(k)) = \mathbf{Q}(\eta(k))$ is defined, we have

$$
\begin{bmatrix} \mathbf{X}^T\mathbf{Q}^{-1}(\eta(k))\mathbf{X} & \mathbf{X}^T\mathbf{\Gamma}_{ijs}^T(\tilde{f}_{ijs}(\mathbf{x}(k))) \\ \mathbf{\Gamma}_{ijs}(\tilde{f}_{ijs}(\mathbf{x}(k)))\mathbf{X} & \mathbf{Q}(\eta(k+1)) \end{bmatrix} > 0. \tag{14}
$$

we define that $\mathbf{D}_j(\mathbf{x}(k)) = \mathbf{G}_j(\mathbf{x}(k))\mathbf{X}$ and $\mathbf{O}_j(\mathbf{x}(k)) = \mathbf{H}_j(\mathbf{x}(k))\mathbf{X}$, and according to the inequation $\mathbf{X}^T\mathbf{Q}^{-1}(\eta(k))\mathbf{X} \geq \mathbf{X} + \mathbf{X}^T - \mathbf{Q}(\eta(k))$, the stability condition can be obtained as follows:

$$
\begin{aligned}
&\begin{bmatrix} \mathbf{X} + \mathbf{X}^T - \mathbf{Q}(\eta(k)) & \hat{\mathbf{\Gamma}}_{ijs}^T(\tilde{f}_{ijs}(\mathbf{x}(k))) \\ \hat{\mathbf{\Gamma}}_{ijs}(\tilde{f}_{ijs}(\mathbf{x}(k))) & \mathbf{Q}(\eta(k+1)) \end{bmatrix}\\
&= \begin{bmatrix} \mathbf{X} + \mathbf{X}^T - \sum_{s=1}^{2^m} \eta_s(\mathbf{x}(k))\mathbf{Q}_s & \hat{\mathbf{\Gamma}}_{ijs}^T(\tilde{f}_{ijs}(\mathbf{x}(k))) \\ \hat{\mathbf{\Gamma}}_{ijs}(\tilde{f}_{ijs}(\mathbf{x}(k))) & \sum_{\hat{s}=1}^{2^m} \eta_{\hat{s}}(\mathbf{x}(k+1))\mathbf{Q}_{\hat{s}} \end{bmatrix}\\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{h}_{ij}(\mathbf{x}(k)) \sum_{s=1}^{2^m} \eta_s(\mathbf{x}(k)) \sum_{\hat{s}=1}^{2^m} \eta_{\hat{s}}(\mathbf{x}(k+1)) \times\\
&\quad \begin{bmatrix} \mathbf{X} + \mathbf{X}^T - \mathbf{Q}_s & \hat{\mathbf{\Gamma}}_{ijs}^T(\mathbf{x}(k)) \\ \hat{\mathbf{\Gamma}}_{ijs}(\mathbf{x}(k)) & \mathbf{Q}_{\hat{s}} \end{bmatrix}\\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{h}_{ij}(\mathbf{x}(k)) \sum_{s=1}^{2^m} \eta_s(\mathbf{x}(k)) \sum_{\hat{s}=1}^{2^m} \eta_{\hat{s}}(\mathbf{x}(k+1)) \mathbf{\Psi}_{ijs\hat{s}}(\mathbf{x}(k))\\
&> 0,
\end{aligned} \tag{15}
$$

where

$$
\hat{\mathbf{\Gamma}}_{ijs}(\tilde{f}_{ijs}(\mathbf{x}(k))) = \sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{s=1}^{2^m} \tilde{h}_{ij}(\mathbf{x}(k))\eta_s(\mathbf{x}(k))\hat{\mathbf{\Gamma}}_{ijs}(\mathbf{x}(k)),
$$

$$
\begin{aligned}
&\hat{\mathbf{\Gamma}}_{ijs}(\mathbf{x}(k))\\
&\quad = \mathbf{A}_i(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))(\mathbf{E}_s\mathbf{D}_j(\mathbf{x}(k)) + \mathbf{E}_s^- \mathbf{O}_j(\mathbf{x}(k))).
\end{aligned}
$$

Thus, the stability of the discrete-time PIT2PFMB control system (9) can be guaranteed by $\mathbf{\Psi}_{ijs\hat{s}}(\mathbf{x}(k)) > 0$, $\forall i \in \underline{p}$, $j \in \underline{c}$, $s, \hat{s} \in \underline{2^m}$.

According to the attractive invariant sets theory, the attractive invariant sets can be used as the estimation of DOA. In order to find some attractive invariant sets for general discrete-time IT2 polynomial FMB systems, the level set $\Im(V)$ of the Lyapunov function candidate $V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbf{P}(\eta(k))\mathbf{x}(k)$ is defined as follows:

$$
\bigcap_{s=1}^{2^m} \boldsymbol{\varepsilon}(\mathbf{P}_s, 1) \subset \Im(V). \tag{16}
$$

where

$$
\boldsymbol{\varepsilon}(\mathbf{P}_s, 1) := \{\mathbf{x}(k) \in \Re^n : \mathbf{x}(k)^T\mathbf{P}_s\mathbf{x}(k) \leq 1\}. \tag{17}
$$

Define restricted area $\boldsymbol{L}(\mathbf{H}(\mathbf{x}(k)))$ as

$\boldsymbol{L}(\mathbf{H}(\mathbf{x}(k)))$

$$:= \{\mathbf{x}(k) \in \Re^n : |\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k)| \leq u_{lim}, \iota \in \underline{m}\}, \quad (18)$$

where $\mathbf{H}(\mathbf{x}(k)) = \sum_{j=1}^{c} m_j(\mathbf{x}(k))\mathbf{H}_j(\mathbf{x}(k))$, $\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))$ is the ι^{th} row of $\mathbf{H}(\mathbf{x}(k))$.

Since $\eta_s(\mathbf{x}(t)) \geq 0$, $\sum_{s=1}^{2^m} \eta_s(\mathbf{x}(k)) = 1$ and $|\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k)| \leq u_{lim}, \forall \iota \in \underline{m}$ are necessary conditions for $\text{sat}(\mathbf{u}(k))$ in (8) to satisfy (2), only $\boldsymbol{\varepsilon}(\mathbf{P}_s, 1) \subset \boldsymbol{L}(\mathbf{H}(\mathbf{x}(k)))$, $\forall s \in \underline{2^m}$ and $\boldsymbol{\Delta}V(\mathbf{x}(k)) < 0$ are satisfied at the same time, can $\Im(V)$ be an attractive invariant set.

According to the Lagrange multiplier method, $\boldsymbol{\varepsilon}(\mathbf{P}_s, 1) \subset \boldsymbol{L}(\mathbf{H}(\mathbf{x}(k)))$ means that the minimum value of $\mathbf{x}^T(k)\mathbf{P}_s\mathbf{x}(k)$ under restriction $\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k) = \pm u_{lim}$ is greater than 1. So the equivalent condition of $\boldsymbol{\varepsilon}(\mathbf{P}_s, 1) \subset \boldsymbol{L}(\mathbf{H}(\mathbf{x}(k)))$ is $\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))\mathbf{P}_s^{-1}(\mathbf{H}^{(\iota,:)}(\mathbf{x}(k)))^T \leq u_{lim}^2$. Applying Schur complement on this condition, we have

$$\begin{bmatrix} u_{lim}^2 & \mathbf{H}_j^{(\iota,:)}(\mathbf{x}(k)) \\ (\mathbf{H}_j^{(\iota,:)}(\mathbf{x}(k)))^T & \mathbf{P}_s \end{bmatrix} \geq 0. \quad (19)$$

Pre-multiplying and post-multiplying $\text{diag}(1, \mathbf{X})$ to both sides of (19), and according to the inequation $\mathbf{X}^T\mathbf{Q}_s^{-1}\mathbf{X} \geq \mathbf{X} + \mathbf{X}^T - \mathbf{Q}_s$, the DOA estimation condition can be obtained as follows:

$$\boldsymbol{\Phi}_{js\iota}(\mathbf{x}(k)) = \begin{bmatrix} u_{lim}^2 & \mathbf{O}_j^{(\iota,:)}(\mathbf{x}(k)) \\ (\mathbf{O}_j^{(\iota,:)}(\mathbf{x}(k)))^T & \mathbf{X} + \mathbf{X}^T - \mathbf{Q}_s \end{bmatrix} \geq 0. \quad (20)$$

In order to find the largest one from all $\Im(V)$'s, like reference [43], the reference set $\boldsymbol{\chi}_R$ is defined as polyhedron $\boldsymbol{\chi}_R := co\{\mathbf{x}_0^1, \mathbf{x}_0^2, \ldots, \mathbf{x}_0^l\}$. Then the condition $\acute{\alpha}\boldsymbol{\chi}_R \subset \Im(V)$ can be used as the optimization condition, the largest $\Im(V)$ is picked by maximizing the optimization parameter $\acute{\alpha}$. According to the definition of $\Im(V)$, the condition $\acute{\alpha}\boldsymbol{\chi}_R \subset \Im(V)$ can be represented by $\acute{\alpha}\boldsymbol{\chi}_R \subset \boldsymbol{\varepsilon}(\mathbf{P}_s, 1)$, $\forall s \in \underline{2^m}$, and this condition can be guaranteed by the following convex condition:

$$\begin{bmatrix} \gamma & (x_0^r)^T \\ x_0^r & \mathbf{Q}_s \end{bmatrix} \geq 0, \ \forall s \in \underline{2^m}, \quad (21)$$

where $\gamma = \frac{1}{\acute{\alpha}^2}$, $r \in \underline{l}$, and the largest estimation of DOA is obtained by minimizing the optimization parameter γ.

Part II: Positivity Analysis

Although the stability conditions and DOA estimation conditions are obtained in the last part, they can only be used to estimate the DOA for the general polynomial FMB systems. For PIT2PFMB systems, positivity conditions are required to limit the results.

The PIT2PFMB control system (9) can be regarded as a PIT2PFMB system without input matrices, with $\mathbf{A}_i(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))(\mathbf{E}_s\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{E}_s^-\mathbf{H}_j(\mathbf{x}(k)))$ being the system matrix. Thus, the system (9) can be regarded as the specific case of system (3). According to Lemma 1, the positivity conditions of (9) are as follows:

$$\begin{aligned} &\mathbf{A}_i^{(\alpha,\beta)}(\mathbf{x}(k)) + \mathbf{B}_i^{(\alpha,:)}(\mathbf{x}(k))(\mathbf{E}_s\mathbf{G}_j^{(:,\beta)}(\mathbf{x}(k)) \\ &\quad + \mathbf{E}_s^-\mathbf{H}_j^{(:,\beta)}(\mathbf{x}(k))) > 0, \forall i \in \underline{p}, j \in \underline{c}, s \in \underline{2^m}, \alpha, \beta \in \underline{n}. \end{aligned} \quad (22)$$

In order to unify the variables of positivity and stability conditions, post-multiplying diagonal positive matrix $\mathbf{X}$ to $\mathbf{A}_i(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))(\mathbf{E}_s\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{E}_s^-\mathbf{H}_j(\mathbf{x}(k)))$, the positivity conditions are obtained as follows:

$$\begin{aligned} &\boldsymbol{\Omega}_{ijs}^{(\alpha,\beta)}(\mathbf{x}(k)) \\ =&\mathbf{A}_i^{(\alpha,\beta)}(\mathbf{x}(k))\mathbf{X}^{(\beta,\beta)} + \mathbf{B}_i^{(\alpha,:)}(\mathbf{x}(k))(\mathbf{E}_s\mathbf{D}_j^{(:,\beta)}(\mathbf{x}(k)) \\ &+ \mathbf{E}_s^-\mathbf{O}_j^{(:,\beta)}(\mathbf{x}(k))) > 0, \ \forall i \in \underline{p}, j \in \underline{c}, s \in \underline{2^m}, \alpha, \beta \in \underline{n}. \end{aligned} \quad (23)$$

Remark 1: For general systems, $\mathbf{X}$ can be any symmetric matrix. However, for positive systems, $\mathbf{X}$ must be a diagonal positive matrix to ensure that the positivity of condition (22) is equivalent to the positivity of the condition (23).

The proof is completed.

III. Main Result

In the last section, the DOA is represented by ellipsoid, and the saturation-dependent Lyapunov function is adopted to perform the stability analysis of system (9). In this section, the polyhedron invariant sets are used to represent the DOA with the help of the LCLF. To cope with the non-convex DOA estimation conditions, an effective convexification method is proposed in this section. Furthermore, the IT2MFD analysis method [51] is improved and extended to the DOA estimation conditions to further enlarge the estimation of DOA for system (9).

A. Linear Copositive Lyapunov Function and Polyhedron DOA

In order to obtain as large estimation of DOA as possible for the system (9), the following theorem is used to design the polynomial fuzzy controller.

Theorem 2: For the PIT2PFMB control system (9), if there exist a positive vector $\boldsymbol{\lambda} \in \Re^{n\times 1}$, polynomial vectors $\mathbf{y}_j^{(:,\beta)}(\mathbf{x}(k)) \in \Re^{m\times 1}$ and $\mathbf{o}_j^{(:,\beta)}(\mathbf{x}(k)) \in \Re^{m\times 1}$, $\forall j \in \underline{c}$, $\beta \in \underline{n}$ and a scalar $\hat{\gamma} > 0$ such that the following SOS-based conditions of optimization problem are satisfied:

$$\begin{aligned} &\min \hat{\gamma} \\ s.t.\ &1)\ \nu^T(\hat{\gamma} - \mathbf{x}_0^r\boldsymbol{\lambda})\nu \text{ is SOS},\ \forall r \in \underline{l}; \\ &2)\ \nu^T(\boldsymbol{\lambda}^{(\alpha,1)} - \varepsilon_1)\nu \text{ is SOS};\ \alpha \in \underline{n}; \\ &3)\ \nu^T(\boldsymbol{\Theta}_{ijs}^{(\alpha,\beta)}(\mathbf{x}(k)) - \varepsilon_2(\mathbf{x}(k)))\nu \text{ is SOS}, \\ &\quad \forall i \in \underline{p}, j \in \underline{c}, s \in \underline{2^m}, \alpha, \beta \in \underline{n}; \\ &4)\ -\nu^T(\boldsymbol{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k)) + \varepsilon_3(\mathbf{x}(k)))\nu \text{ is SOS}, \\ &\quad \forall i \in \underline{p}, j \in \underline{c}, s \in \underline{2^m}, \alpha \in \underline{n}; \\ &5)\ -\nu^T\boldsymbol{\Upsilon}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))\nu \text{ is SOS}, \\ &\quad \forall z \in \{1, 2\}, j \in \underline{c}, \iota \in \underline{m}, \beta \in \underline{n}; \end{aligned} \quad (24)$$

where ν is an arbitrary vector independent of $\mathbf{x}(k)$ with appropriate dimensions; $\varepsilon_1 > 0$, $\varepsilon_2(\mathbf{x}(k)) > 0$, $\varepsilon_3(\mathbf{x}(k)) > 0$ are predefined scalar polynomials; $\boldsymbol{\Theta}_{ijs}^{(\alpha,\beta)}(\mathbf{x}(k))$, $\boldsymbol{\Xi}_{ijs}(\mathbf{x}(k))$

and $\boldsymbol{\Upsilon}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))$ are defined in (30), (28) and (37), ρ is the predefined positive scalar, then the system (9) is asymptotically stable and positive with initial conditions contained in $\aleph(\boldsymbol{\lambda},1)$, where $\aleph(\boldsymbol{\lambda},1)$ is an estimation of the DOA. The polynomial fuzzy controller gains can be obtained as $\mathbf{G}_j(\mathbf{x}(k)) = [\frac{\mathbf{y}_j^{(:,1)}(\mathbf{x}(k))}{\lambda_1}, \frac{\mathbf{y}_j^{(:,2)}(\mathbf{x}(k))}{\lambda_2}, \ldots, \frac{\mathbf{y}_j^{(:,n)}(\mathbf{x}(k))}{\lambda_n}]$ and the auxiliary polynomial fuzzy controller gains can be obtained as $\mathbf{H}_j(\mathbf{x}(k)) = [\frac{\mathbf{o}_j^{(:,1)}(\mathbf{x}(k))}{\lambda_1}, \frac{\mathbf{o}_j^{(:,2)}(\mathbf{x}(k))}{\lambda_2}, \ldots, \frac{\mathbf{o}_j^{(:,n)}(\mathbf{x}(k))}{\lambda_n}]$.

Proof 2: The proof consists of three parts: stability analysis, positivity analysis and estimation of DOA. These three parts give the convex stability conditions, positivity conditions and DOA estimation conditions, respectively.

Part I: Stability Analysis

A system whose system matrices are obtained by transposing the system matrices of original system is called dual system of the original system. For discrete-time system, the stability of dual system is equivalent to that of original system, but the stability analysis results of dual system can avoid non-convex stability conditions. So the stability analysis of system (9) is studied by using the following dual system:

$$\begin{aligned}\mathbf{x}(k+1) = &\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{s=1}^{2^m}\tilde{h}_{ij}(\mathbf{x}(k))\eta_s(\mathbf{x}(k))(\mathbf{A}_i(\mathbf{x}(k)) \\ &+ \mathbf{B}_i(\mathbf{x}(k))(\mathbf{E}_s\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{E}_s^{-}\mathbf{H}_j(\mathbf{x}(k))))^T\mathbf{x}(k).\end{aligned} \tag{25}$$

Lemma 2: With symmetric matrices $\mathbf{P}$ and $\mathbf{R}$ of appropriate dimensions, $\mathbf{R} > 0$ and a scalar ρ, the following inequality holds [56]:

$$-\mathbf{P}\mathbf{R}^{-1}\mathbf{P} \leq \rho^2\mathbf{R} - 2\rho\mathbf{P}.$$

In order to perform stability analysis, the LCLF candidate $V(\mathbf{x}(k)) = \mathbf{x}^T(k)\boldsymbol{\lambda}$ is chosen, where every element of $\boldsymbol{\lambda} \in \Re^{n\times 1}$ is positive. From (25) and LCLF candidate, we have:

$$\begin{aligned}\Delta V(\mathbf{x}(k)) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\ &= \mathbf{x}^T\{\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{s=1}^{2^m}\tilde{h}_{ij}(\mathbf{x}(k))\eta_s(\mathbf{x}(k))[\mathbf{A}_i(\mathbf{x}(k)) \\ &\quad + \mathbf{B}_i(\mathbf{x}(k))(\mathbf{E}_s\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{E}_s^{-}\mathbf{H}_j(\mathbf{x}(k)))]\}\boldsymbol{\lambda} \\ &\quad - \mathbf{x}^T\boldsymbol{\lambda}.\end{aligned} \tag{26}$$

Defining $\mathbf{G}_j(\mathbf{x}(k)) = [\frac{\mathbf{y}_j^{(:,1)}(\mathbf{x}(k))}{\lambda_1}, \frac{\mathbf{y}_j^{(:,2)}(\mathbf{x}(k))}{\lambda_2}, \ldots, \frac{\mathbf{y}_j^{(:,n)}(\mathbf{x}(k))}{\lambda_n}]$, $\mathbf{H}_j(\mathbf{x}(k)) = [\frac{\mathbf{o}_j^{(:,1)}(\mathbf{x}(k))}{\lambda_1}, \frac{\mathbf{o}_j^{(:,2)}(\mathbf{x}(k))}{\lambda_2}, \ldots, \frac{\mathbf{o}_j^{(:,n)}(\mathbf{x}(k))}{\lambda_n}]$, where $\mathbf{y}_j^{(:,1)}(\mathbf{x}(k))$, $\mathbf{y}_j^{(:,2)}(\mathbf{x}(k))$, $\ldots$, $\mathbf{y}_j^{(:,n)}(\mathbf{x}(k)) \in \Re^{m\times 1}$ and $\mathbf{o}_j^{(:,1)}(\mathbf{x}(k))$, $\mathbf{o}_j^{(:,2)}(\mathbf{x}(k))$, $\ldots$, $\mathbf{o}_j^{(:,n)}(\mathbf{x}(k)) \in \Re^{m\times 1}$ for $j \in c$ are to be determined, λ_n is the n^{th} element of $\boldsymbol{\lambda}$. Then (26) can be derived by follows:

$$\begin{aligned}&\Delta V(\mathbf{x}(k)) \\ &= \mathbf{x}^T\{\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{s=1}^{2^m}\tilde{h}_{ij}(\mathbf{x}(k))\eta_s(\mathbf{x}(k))[\mathbf{A}_i(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))\times \\ &\quad(\mathbf{E}_s\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{E}_s^{-}\mathbf{H}_j(\mathbf{x}(k)))]\boldsymbol{\lambda} - \boldsymbol{\lambda}\} \\ &= \mathbf{x}^T\{\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{s=1}^{2^m}\tilde{h}_{ij}(\mathbf{x}(k))\eta_s(\mathbf{x}(k))[\mathbf{A}_i(\mathbf{x}(k))\boldsymbol{\lambda} + \mathbf{B}_i(\mathbf{x}(k))\times \\ &\quad(\mathbf{E}_s\sum_{q=1}^{n}\mathbf{y}_j^{(:,q)}(\mathbf{x}(k)) + \mathbf{E}_s^{-}\sum_{q=1}^{n}\mathbf{o}_j^{(:,q)}(\mathbf{x}(k)))] - \boldsymbol{\lambda}\} \\ &= \mathbf{x}^T\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{s=1}^{2^m}\tilde{h}_{ij}(\mathbf{x}(k))\eta_s(\mathbf{x}(k))\boldsymbol{\Xi}_{ijs}(\mathbf{x}(k)),\end{aligned} \tag{27}$$

where

$$\begin{aligned}\boldsymbol{\Xi}_{ijs}(\mathbf{x}(k)) = &\mathbf{A}_i(\mathbf{x}(k))\boldsymbol{\lambda} + \mathbf{B}_i(\mathbf{x}(k))(\mathbf{E}_s\sum_{q=1}^{n}\mathbf{y}_j^{(:,q)}(\mathbf{x}(k)) + \\ &\mathbf{E}_s^{-}\sum_{q=1}^{n}\mathbf{o}_j^{(:,q)}(\mathbf{x}(k)))) - \boldsymbol{\lambda}.\end{aligned} \tag{28}$$

Then, the stability of system (9) can be guaranteed by $V(\mathbf{x}(k)) > 0$ and $\Delta V(\mathbf{x}(k)) < 0$ which is obtained in the above. Therefore, the stability conditions can be concluded as $\boldsymbol{\lambda} \succ 0$ and $\boldsymbol{\Xi}_{ijs}(\mathbf{x}(k)) \prec 0$.

Part II: Positivity Analysis

According to the definition of $\mathbf{G}_j(\mathbf{x}(k))$ and $\mathbf{H}_j(\mathbf{x}(k))$, $\mathbf{G}_j^{(:,\beta)}(\mathbf{x}(k))$ in (22) is replaced by $\frac{\mathbf{y}_j^{(:,\beta)}(\mathbf{x}(k))}{\lambda_\beta}$ and $\mathbf{H}_j^{(:,\beta)}(\mathbf{x}(k))$ in (22) is replaced by $\frac{\mathbf{o}_j^{(:,\beta)}(\mathbf{x}(k))}{\lambda_\beta}$. Then, the positivity conditions of system (9) also can be denoted as following form:

$$\begin{aligned}&\mathbf{A}_i^{(\alpha,\beta)}(\mathbf{x}(k)) + \mathbf{B}_i^{(\alpha,:)}(\mathbf{x}(k))(\mathbf{E}_s\frac{\mathbf{y}_j^{(:,\beta)}(\mathbf{x}(k))}{\lambda_\beta} \\ &+ \mathbf{E}_s^{-}\frac{\mathbf{o}_j^{(:,\beta)}(\mathbf{x}(k))}{\lambda_\beta}) > 0, \forall i \in \underline{p}, j \in \underline{c}, s \in \underline{2^m}, \alpha, \beta \in \underline{n}.\end{aligned} \tag{29}$$

Post-multiplying both sides of (29) with λ_β, then the positive conditions (29) can be guaranteed by the following conditions:

$$\begin{aligned}&\boldsymbol{\Theta}_{ijs}^{(\alpha,\beta)}(\mathbf{x}(k)) \\ =&\mathbf{A}_i^{(\alpha,\beta)}(\mathbf{x}(k))\lambda_\beta \\ &+ \mathbf{B}_i^{(\alpha,:)}(\mathbf{x}(k))(\mathbf{E}_s\mathbf{y}_j^{(:,\beta)}(\mathbf{x}(k)) + \mathbf{E}_s^{-}\mathbf{o}_j^{(:,\beta)}(\mathbf{x}(k))) \\ >&0,\ \forall i \in \underline{p}, j \in \underline{c}, s \in \underline{2^m}, \alpha, \beta \in \underline{n}.\end{aligned} \tag{30}$$

Part III: Estimation of DOA

In order to estimate the DOA for PIT2PFMB control system (9) through the invariant sets theory, the level set of the LCLF candidate $V(\mathbf{x}(k)) = \mathbf{x}(k)^T\boldsymbol{\lambda}$ is defined as follows:

$$\aleph(\boldsymbol{\lambda},1) := \{\mathbf{x}(k) \in \Re^n : \mathbf{x}(k)^T\boldsymbol{\lambda} \leq 1\}. \tag{31}$$

Since $\eta_s(\mathbf{x}(t)) \geq 0$, $\sum_{s=1}^{2^m}\eta_s(\mathbf{x}(k)) = 1$ and $|\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k)| \leq u_{lim}, \forall\iota \in \underline{m}$ are necessary conditions for $\mathrm{sat}(\mathbf{u}(k))$ in (8) to satisfy (2), only $\aleph(\boldsymbol{\lambda},1) \subset \boldsymbol{L}(\mathbf{H}(\mathbf{x}(k)))$, $\boldsymbol{\Delta}V(\mathbf{x}(k)) < 0$ and positivity conditions are satisfied at the same time, can polyhedron $\aleph(\boldsymbol{\lambda},1)$ be regarded an attractive invariant set.

The shape reference set $\boldsymbol{\chi}_R \subset \Re^n$ is chosen as polyhedron $\boldsymbol{\chi}_R := co\{\mathbf{x}_0^1, \mathbf{x}_0^2, \ldots, \mathbf{x}_0^l\}$. Denote $\acute{\alpha}\boldsymbol{\chi}_R = \{\acute{\alpha}\mathbf{x}(k) : \mathbf{x}(k) \in \boldsymbol{\chi}_R\}$. The condition $\acute{\alpha}\boldsymbol{\chi}_R \subset \aleph(\boldsymbol{\lambda}, 1)$ is used to pick the largest one $\aleph(\boldsymbol{\lambda}, 1)$ from all $\aleph(\boldsymbol{\lambda}, 1)$'s as the least conservative estimate of DOA, which is achieved by maximizing the value of $\acute{\alpha}$.

The least conservative estimation of the DOA can be obtained by solving the following conditions:

$$
\begin{aligned}
&\sup \acute{\alpha} \\
s.t.1)\ &\acute{\alpha}\boldsymbol{\chi}_R \subset \aleph(\boldsymbol{\lambda}, 1); \\
2)\ &\boldsymbol{\lambda}^{(\alpha,1)} > 0;\ \alpha \in \underline{n}; \\
3)\ &\boldsymbol{\Theta}_{ijs}^{(\alpha,\beta)}(\mathbf{x}(k)) > 0, \quad \forall i \in \underline{p}, j \in \underline{c}, s \in \underline{2^m}, \alpha, \beta \in \underline{n}; \\
4)\ &\boldsymbol{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k)) < 0, \quad \forall i \in \underline{p}, j \in \underline{c}, s \in \underline{2^m}; \\
5)\ &\aleph(\boldsymbol{\lambda}, 1) \subset \boldsymbol{L}(\mathbf{H}(\mathbf{x}(k)));
\end{aligned} \tag{32}
$$

where $\boldsymbol{\Xi}_{ijs}(\mathbf{x})$ and $\boldsymbol{\Theta}_{ijs}^{(\alpha,\beta)}(\mathbf{x})$ are defined in (28) and (30).

In (32), sub-conditions 1) and 5) are non-convex conditions which cannot be solved by the toolbox SOSTOOLS. In the following, an effective convexification method will be proposed to obtain the convex form of (32) which is shown in Theorem 2.

Referring to sub-condition 1) of (32), $\acute{\alpha}\boldsymbol{\chi}_R \subset \aleph(\boldsymbol{\lambda}, 1)$ means that the point $\acute{\alpha}\mathbf{x}$ on the bound of $\acute{\alpha}\boldsymbol{\chi}_R$ is inside the polyhedron $\aleph(\boldsymbol{\lambda}, 1)$, where $\acute{\alpha}\boldsymbol{\chi}_R := co\{\acute{\alpha}\mathbf{x}_0^1, \acute{\alpha}\mathbf{x}_0^2, \ldots, \acute{\alpha}\mathbf{x}_0^{\hat{l}}\}$, $\aleph(\boldsymbol{\lambda}, 1) := \{\mathbf{x} \in \Re^n : \mathbf{x}^T\boldsymbol{\lambda} \le 1\}$. By substituting boundary point of $\acute{\alpha}\boldsymbol{\chi}_R$ into $\aleph(\boldsymbol{\lambda}, 1)$, $\hat{\gamma} - \mathbf{x}_0^r\boldsymbol{\lambda} \ge 0$ is obtained as the equivalent condition of $\acute{\alpha}\boldsymbol{\chi}_R \subset \aleph(\boldsymbol{\lambda}, 1)$, $r \in \underline{\hat{l}}$, where $\hat{\gamma} = \frac{1}{\acute{\alpha}}$.

Sub-condition 5) of (32) implies that all the curves $\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k) = \pm u_{lim}$ lie completely outside of the polyhedron $\aleph(\boldsymbol{\lambda}, 1)$.

For the case that $\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k) = u_{lim}$ lies outside of the polyhedron $\aleph(\boldsymbol{\lambda}, 1)$, the sub-condition 5) of (32) implies the following inequation:

$$
\boldsymbol{\lambda}^T\mathbf{x}(k) \ge \frac{\mathbf{H}_j^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k)}{u_{lim}},\ \forall j \in \underline{c}, \iota \in \underline{m}. \tag{33}
$$

Based on the definition $\mathbf{H}_j(\mathbf{x}(k)) = [\frac{\mathbf{o}_j^{(:,1)}(\mathbf{x}(k))}{\lambda_1}, \frac{\mathbf{o}_j^{(:,2)}(\mathbf{x}(k))}{\lambda_2}, \ldots, \frac{\mathbf{o}_j^{(:,n)}(\mathbf{x}(k))}{\lambda_n}]$, the inequation (33) can be derived as:

$$
\mathbf{o}_j^{(\iota,\beta)}(\mathbf{x}(k)) - (\lambda_\beta)^2 u_{lim} \le 0,\ \forall j \in \underline{c}, \iota \in \underline{m}, \beta \in \underline{n}. \tag{34}
$$

For the case that $\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k) = -u_{lim}$ lies outside of the polyhedron $\aleph(\boldsymbol{\lambda}, 1)$, the sub-condition 5) of (32) implies the following inequation:

$$
\boldsymbol{\lambda}^T\mathbf{x}(k) \ge \frac{\mathbf{H}_j^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k)}{-u_{lim}},\ \forall j \in \underline{c}. \tag{35}
$$

When $\mathbf{H}_j(\mathbf{x}(k))$ is replaced by $[\frac{\mathbf{o}_j^{(:,1)}(\mathbf{x}(k))}{\lambda_1}, \frac{\mathbf{o}_j^{(:,2)}(\mathbf{x}(k))}{\lambda_2} \ldots, \frac{\mathbf{o}_j^{(:,n)}(\mathbf{x}(k))}{\lambda_n}]$, (35) is equivalent to the following inequation:

$$
\mathbf{o}_j^{(\iota,\beta)}(\mathbf{x}(k)) + (\lambda_\beta)^2 u_{lim} \ge 0,\ \forall j \in \underline{c}, \iota \in \underline{m}, \beta \in \underline{n}. \tag{36}
$$

The lemma 2 is applied to non-convex conditions (34) and (36), the following conditions are obtained:

$$
\begin{aligned}
\boldsymbol{\Upsilon}_{1j}^{(\iota,\beta)}(\mathbf{x}(k)) &= \mathbf{o}_j^{(\iota,\beta)}(\mathbf{x}(k)) + \frac{\rho^2}{u_{lim}} - 2\rho\lambda_\beta \le 0, \\
\boldsymbol{\Upsilon}_{2j}^{(\iota,\beta)}(\mathbf{x}(k)) &= -\mathbf{o}_j^{(\iota,\beta)}(\mathbf{x}(k)) + \frac{\rho^2}{u_{lim}} - 2\rho\lambda_\beta \le 0.
\end{aligned} \tag{37}
$$

where ρ is predefined positive scalar.

Since $\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k) = u_{lim}$ and $\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k) = -u_{lim}$ are two boundaries of the ribbon area $\boldsymbol{L}(\mathbf{H}(\mathbf{x}(k)))$, the sub-condition 5) of (32) can be guaranteed by integrating the above two cases. Thus (37) can be regarded as the convex form of sub-condition 5) of (32).

The proof is completed.

B. IT2 Membership Function Dependent Analysis

In the last subsection, the basic stability conditions are derived from LCLF, and the polyhedron is used to estimate the DOA. Although the above analysis strategy relaxes the conservatism of the estimation of the DOA derived by the saturation-dependent-Lyapunov-function and the elliptical DOA to some extent, the analysis results are still conservative due to their independence on MFs. For example, if the IT2 MFs are considered, the sub-conditions 5) of (32) for the case that $\mathbf{H}^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k) = u_{lim}$ lies outside of the polyhedron $\aleph(\boldsymbol{\lambda}, 1)$ implies $\boldsymbol{\lambda}^T\mathbf{x}(k) \ge \frac{\sum_{j=1}^{c}\tilde{m}_j\mathbf{H}_j^{(\iota,:)}(\mathbf{x}(k))\mathbf{x}(k)}{u_{lim}}, \forall \iota \in \underline{m}$. If the IT2 MFs are not considered, then the sub-conditions 5) of (32) implies (33). It is obvious that (33) imposes more strict restriction on the decision variables. Thus, the information of IT2 MFs will be considered in this subsection. Inspired by paper [51], the information of the IT2 MFs is included by approximating the embedded type-1 MFs. Unlike [51], the embedded type-1 MFs are approximated by piecewise linear MFs which are easy to be obtained compared with the polynomial MFs. Also, the piecewise linear MFs are interpolation approximation functions which allows the stability conditions to be met only at the interpolation points. In addition, the IT2MFD analysis method of this paper provides the freedom to choose the reference embedded type-1 MFs by introducing the lower approximation error.

As shown in Fig. 1, the upper MF $\overline{w}_1(x_1(k))$ and lower MF $\underline{w}_1(x_1(k))$ of IT2 MFs are represented by the black dotted line and black dash-dotted line, respectively. The area between the upper MF and lower MF is denoted as footprint of uncertainty (FOU) which can characterize the IT2 MFs [57]. This FOU is composed of an infinite number of embedded type-1 MFs. We randomly choose one embedded type-1 MF $\breve{w}_1(x_1(k))$ and plot it as the red dashed line. We call the red dashed line as reference embedded type-1 MF, which is obtained by $\breve{w}_1(\mathbf{x}(k)) = \underline{\xi}_1(\mathbf{x}(k))\underline{w}_1(\mathbf{x}(k)) + \overline{\xi}_1(\mathbf{x}(k))\overline{w}_1(\mathbf{x}(k))$ with the arbitrarily predefined functions $\underline{\xi}_1(\mathbf{x}(k))$ and $\overline{\xi}_1(\mathbf{x}(k))$. This reference embedded type-1 MF $\breve{w}_1(\mathbf{x}(k))$ is approximated by the piecewise linear embedded type-1 MF $\hat{\breve{w}}_1(\mathbf{x}(k))$, which can be adopted in the SOS-based or LMI-based conditions. Then, the maximum and minimum approximation error of IT2 MFs are obtained by $\overline{w}_1(x_1(k)) - \hat{\breve{w}}_1(\mathbf{x}(k))$ and $\underline{w}_1(x_1(k)) - \hat{\breve{w}}_1(\mathbf{x}(k))$, respectively. In this way, the parameter uncertainties

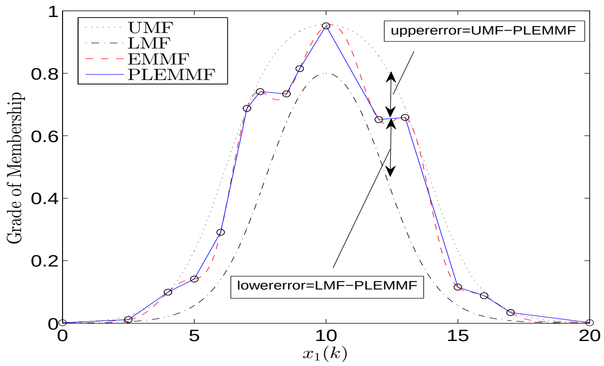

Fig. 1. The upper membership function (UMF), the lower membership function (LMF), the embedded type-1 membership function (EMMF), the piecewise linear embedded type-1 membership function (PLEMMF) and the interpolation points are represented by black dotted line, black dash-dotted line, red dashed line, blue solid line and '∘'. It should be noted that these abbreviations only are used in the legend of this figure.

are transferred to the approximation error of IT2 MFs, and the approximation error can be any value instead of being positive like [51], i.e., $\underline{\xi}_1(\mathbf{x}(k))$ and $\overline{\xi}_1(\mathbf{x}(k))$ can be any value, as long as $0 \le \underline{\xi}_1(\mathbf{x}(k)) \le 1$, $0 \le \overline{\xi}_1(\mathbf{x}(k)) \le 1$ and $\underline{\xi}_1(\mathbf{x}(k)) + \overline{\xi}_1(\mathbf{x}(k)) = 1$.

In (10), $\tilde{h}_{ij}(\mathbf{x}(k)) \equiv \tilde{w}_i(\mathbf{x}(k))\tilde{m}_j(\mathbf{x}(k))$ is defined. $\mathbf{\Psi}$ is denoted as the whole state space, which is divided into K connected substate spaces $\mathbf{\Psi}_\epsilon$, $\epsilon \in \underline{K}$, $K = \prod_{r=1}^{n} d_r$, d_r is the number of substate spaces of x_r. Now, we randomly choose an embedded type-1 MF $\check{h}_{ij}(\mathbf{x}(k))$ as the reference embedded type-1 MF. $\hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k))$ denotes a piecewise linear approximation function of $\check{h}_{ij}(\mathbf{x}(k))$ in substate space $\mathbf{\Psi}_\epsilon$. The piecewise linear embedded type-1 MF $\hat{\check{h}}_{ij}(\mathbf{x}(k))$ is represented as follows:

$$
\begin{aligned}
&\hat{\check{h}}_{ij}(\mathbf{x}(k)) \\
&= \sum_{\epsilon=1}^{K} \varphi_\epsilon(\mathbf{x}(k))\hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k)) \\
&= \sum_{\epsilon=1}^{K} \varphi_\epsilon(\mathbf{x}(k)) \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_{\hat{n}}=1}^{2}\prod_{r=1}^{\hat{n}} \upsilon_{ri_r\epsilon}(x_r(k))\check{\zeta}_{iji_1i_2\ldots i_{\hat{n}}},
\end{aligned} \tag{38}
$$

where $\varphi_\epsilon(\mathbf{x}(k)) = 1$ if $\mathbf{x}(k) \in \mathbf{\Psi}_\epsilon$; $\varphi_\epsilon(\mathbf{x}(k)) = 0$ if $\mathbf{x}(k) \notin \mathbf{\Psi}_\epsilon$. $\upsilon_{ri_r\epsilon}(x_r(k))$ are predefined interpolation functions, which satisfy the properties that $0 \le \upsilon_{ri_r\epsilon}(x_r(k)) \le 1$ and $\upsilon_{r1\epsilon}(x_r(k)) + \upsilon_{r2\epsilon}(x_r(k)) = 1$ for $i_r \in \{1, 2\}$, $r \in \underline{\hat{n}}$, $\hat{n}$ denotes the number of systems variables which the stability condition depends on. $\check{\zeta}_{iji_1i_2\ldots i_{\hat{n}}}$ denotes the value of the embedded type-1 MF $\check{h}_{ij}(\mathbf{x}(k))$ at the interpolation point $\mathbf{x} = [x_{1i_1}, x_{2i_2}, \ldots, x_{\hat{n}i_{\hat{n}}}]$.

Then the approximation error of IT2 MFs in substate space $\mathbf{\Psi}_\epsilon$ is obtained by $\tilde{h}_{ij\epsilon}(\mathbf{x}(k)) - \hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k))$. Since $\underline{h}_{ij\epsilon}(\mathbf{x}(k)) \le \tilde{h}_{ij\epsilon}(\mathbf{x}(k)) \le \overline{h}_{ij\epsilon}(\mathbf{x}(k))$, the following inequality can be obtained:

$$
\begin{aligned}
&\underline{h}_{ij\epsilon}(\mathbf{x}(k)) - \hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k)) \\
&\quad \le \tilde{h}_{ij\epsilon}(\mathbf{x}(k)) - \hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k)) \\
&\quad \le \overline{h}_{ij\epsilon}(\mathbf{x}(k)) - \hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k))
\end{aligned} \tag{39}
$$

The maximum approximation error in substate space $\mathbf{\Psi}_\epsilon$ is denoted as $\overline{\sigma}_{ij\epsilon} = \overline{h}_{ij\epsilon}(\mathbf{x}(k)) - \hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k))$. The minimum approximated error is denoted as $\underline{\sigma}_{ij\epsilon} = \underline{h}_{ij\epsilon}(\mathbf{x}(k)) - \hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k))$. Then,

$$
\underline{\sigma}_{ij\epsilon} \le \tilde{h}_{ij\epsilon}(\mathbf{x}(k)) - \hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k)) \le \overline{\sigma}_{ij\epsilon} \tag{40}
$$

In order to introduce the information of IT2 MFs into the stability conditions, the positive defined slack matrix $\mathbf{W}_{ijs}(\mathbf{x}(k))$ is defined, which needs to satisfy the conditions $\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k)) \ge 0$ and $\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k)) \ge \mathbf{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))$, $\forall \alpha \in \underline{n}, i \in \underline{p}, j \in \underline{c}, s \in \underline{2^m}$. Then the basic stability condition $\mathbf{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k)) < 0$ can be relaxed by the follows:

$$
\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{s=1}^{2^m} \tilde{h}_{ij}(\mathbf{x}(k))\eta_s(\mathbf{x}(k))\mathbf{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k)) \\
&= \sum_{s=1}^{2^m}\eta_s(\mathbf{x}(k))\sum_{\epsilon=1}^{K}\varphi_\epsilon(\mathbf{x}(k))\sum_{i=1}^{p}\sum_{j=1}^{c}[(\hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k)) + \underline{\sigma}_{ij\epsilon})\times \\
&\quad \mathbf{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k)) + (\tilde{h}_{ij\epsilon}(\mathbf{x}(k)) - \hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k)) - \underline{\sigma}_{ij\epsilon})\times \\
&\quad \mathbf{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))] \\
&\le \sum_{s=1}^{2^m}\eta_s(\mathbf{x}(k))\sum_{\epsilon=1}^{K}\varphi_\epsilon(\mathbf{x}(k))\sum_{i=1}^{p}\sum_{j=1}^{c}[(\hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k)) + \underline{\sigma}_{ij\epsilon})\times \\
&\quad \mathbf{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k)) + (\overline{\sigma}_{ij\epsilon} - \underline{\sigma}_{ij\epsilon})\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))] \\
&< 0.
\end{aligned} \tag{41}
$$

Remark 2: In [51], the positivity of weight coefficient of $\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))$ is guaranteed by positive approximation error of IT2 MFs. In this paper, the lower approximation error $\underline{\sigma}_{ij\epsilon}$ is included into the stability conditions. In this way, weight coefficient of $\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))$ becomes $\overline{\sigma}_{ij\epsilon} - \underline{\sigma}_{ij\epsilon}$ which always be guaranteed to be non-negative, regardless of whether the approximation error of IT2 MFs is positive or negative. As a result, the reference embedded type-1 MF can be chosen randomly instead of searching through complicated procedure.

In order to introduce the boundary information of substate space $\mathbf{\Psi}_\epsilon$ into the stability conditions, the threshold function $\pi_{\hat{r}\epsilon}(\mathbf{x}(k)) = \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_{\hat{n}}=1}^{2}\prod_{r=1}^{\hat{n}}\upsilon_{ri_r\epsilon}(x_r(k))(x_{\hat{r}}(k) - x_{\hat{r}\epsilon min})(x_{\hat{r}\epsilon max} - x_{\hat{r}}(k))$ is defined, where $x_{\hat{r}\epsilon min}$ and $x_{\hat{r}\epsilon max}$ are the lower boundary and upper boundary of substate space $\mathbf{\Psi}_\epsilon$ on the $x_{\hat{r}}$ dimension, $\hat{r} \in \underline{\hat{n}}$, $\hat{n}$ denotes the number of systems variables which the stability condition depends on. This threshold function $\pi_{\hat{r}\epsilon}(\mathbf{x}(k))$ is positive if $x_{\hat{r}}$ is in the substate space $\mathbf{\Psi}_\epsilon$, otherwise it is negative. According to the S-procedure concepts [58], with the help of the positive slack vectors $\mathbf{R}_{\hat{r}s\epsilon}(\mathbf{x}(k))$, the condition (41) can be further relaxed by the follows:

$$
\begin{aligned}
&\sum_{s=1}^{2^m}\eta_s(\mathbf{x}(k))\sum_{\epsilon=1}^{K}\varphi_\epsilon(\mathbf{x}(k))\{\sum_{i=1}^{p}\sum_{j=1}^{c}[(\hat{\check{h}}_{ij\epsilon}(\mathbf{x}(k)) + \underline{\sigma}_{ij\epsilon})\times \\
&\mathbf{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k)) + (\overline{\sigma}_{ij\epsilon} - \underline{\sigma}_{ij\epsilon})\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))]
\end{aligned}
$$

$$+\sum_{\hat{r}=1}^{\hat{n}}\pi_{\hat{r}\epsilon}(\mathbf{x}(k))\mathbf{R}_{\hat{r}s\epsilon}^{(\alpha,1)}(\mathbf{x}(k))\}<0. \tag{42}$$

Referring to the expression of $\hat{\breve{h}}_{ij\epsilon}(\mathbf{x}(k))$ and $\pi_{\hat{r}\epsilon}(\mathbf{x}(k))$, $\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_{\hat{n}}=1}^{2}\prod_{r=1}^{\hat{n}}\upsilon_{ri_r\epsilon}(x_r(k)) = 1$ and $\upsilon_{ri_r\epsilon}(x_r(k))$ are independent of rule i, j in the substate space $\mathbf{\Psi}_\epsilon$. Defining $\hat{\pi}_{\hat{r}\epsilon}(\mathbf{x}(k))=(x_{\hat{r}}(k)-x_{\hat{r}\epsilon min})(x_{\hat{r}\epsilon max}-x_{\hat{r}}(k))$, whose properties are the same as that of $\pi_{\hat{r}\epsilon}(\mathbf{x}(k))$, the above inequation is equivalent to the following inequality:

$$\sum_{s=1}^{2^m}\eta_s(\mathbf{x}(k))\sum_{\epsilon=1}^{K}\varphi_\epsilon(\mathbf{x}(k))\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_{\hat{n}}=1}^{2}\prod_{r=1}^{\hat{n}}\upsilon_{ri_r\epsilon}(x_r(k))$$
$$\{\sum_{i=1}^{p}\sum_{j=1}^{c}[(\breve{\zeta}_{iji_1i_2\ldots i_{\hat{n}}}+\underline{\sigma}_{ij\epsilon})\mathbf{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))+(\overline{\sigma}_{ij\epsilon}-\underline{\sigma}_{ij\epsilon})\times$$
$$\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))]+\sum_{\hat{r}=1}^{\hat{n}}\hat{\pi}_{\hat{r}\epsilon}(\mathbf{x}(k))\mathbf{R}_{\hat{r}s\epsilon}^{(\alpha,1)}(\mathbf{x}(k))\}<0. \tag{43}$$

Due to $\eta_s(\mathbf{x}(k))\geq 0$ and $\upsilon_{ri_r\epsilon}(x_r(k))\geq 0$ for $\forall r\in\underline{\hat{n}}, i_r\in\{1,2\},\epsilon\in\underline{K},s\in\underline{2^m}$, $\Delta V(x(k))<0$ is guaranteed by $\sum_{i=1}^{p}\sum_{j=1}^{c}[(\breve{\zeta}_{iji_1i_2\ldots i_n}+\underline{\sigma}_{ij\epsilon})\mathbf{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))+(\overline{\sigma}_{ij\epsilon}-\underline{\sigma}_{ij\epsilon})\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))]+\sum_{\hat{r}=1}^{\hat{n}}\hat{\pi}_{\hat{r}\epsilon}(\mathbf{x}(k))\mathbf{R}_{\hat{r}s\epsilon}^{(\alpha,1)}(\mathbf{x}(k))<0$, $\forall s\in\underline{2^m}$, $\mathbf{x}(k)\in\mathbf{\Psi}_\epsilon,\epsilon\in\underline{K}$, which means that the stability conditions only need to be satisfied on the interpolation points.

In the above analysis, the IT2MFD method is applied on the stability conditions to enlarge the estimation of DOA. In the following, the same line will be followed to obtain IT2MFD DOA estimation conditions. The piecewise linear reference embedded type-1 controller MF $\hat{\breve{m}}_j(\mathbf{x}(k))$ is represented as follows:

$$\begin{aligned}&\hat{\breve{m}}_j(\mathbf{x}(k))\\&=\sum_{\epsilon=1}^{K}\varphi_\epsilon(\mathbf{x}(k))\hat{\breve{m}}_{j\epsilon}(\mathbf{x}(k))\\&=\sum_{\epsilon=1}^{K}\varphi_\epsilon(\mathbf{x}(k))\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\ldots\sum_{i_{\hat{n}}=1}^{2}\prod_{r=1}^{\hat{n}}\upsilon_{ri_r\epsilon}(x_r(k))\breve{\vartheta}_{ji_1i_2\ldots i_{\hat{n}}}.\end{aligned} \tag{44}$$

In substate space $\mathbf{\Psi}_\epsilon$, the maximum and minimum approximated error of IT2 controller MF $\tilde{m}_{j\epsilon}(\mathbf{x}(k))$ are denoted as $\overline{\varpi}_{j\epsilon}$ and $\underline{\varpi}_{j\epsilon}$. Define the slack matrices $\mathbf{Y}_{zj}(\mathbf{x}(k))$ and $\mathbf{V}_{\hat{r}z\epsilon}(\mathbf{x}(k))$ satisfying the conditions $\mathbf{Y}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))\geq 0$, $\mathbf{V}_{\hat{r}z\epsilon}^{(\iota,\beta)}(\mathbf{x}(k))\geq 0$, $\mathbf{Y}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))\geq\mathbf{\Upsilon}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))$, $\forall z\in\{1,2\},\hat{r}\in\underline{\hat{n}},j\in\underline{c}$. Then, the DOA estimation condition (37) can be relaxed as follows:

$$\begin{aligned}&\sum_{j=1}^{c}\tilde{m}_j(\mathbf{x}(k))\mathbf{\Upsilon}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))\\&\leq\sum_{\epsilon=1}^{K}\varphi_\epsilon(\mathbf{x}(k))\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_{\hat{n}}=1}^{2}\prod_{r=1}^{\hat{n}}\upsilon_{ri_r\epsilon}(x_r(k))\\&\{\sum_{j=1}^{c}[(\breve{\vartheta}_{ji_1i_2\ldots i_{\hat{n}}}+\underline{\varpi}_{j\epsilon})\mathbf{\Upsilon}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))\\&+(\overline{\varpi}_{j\epsilon}-\underline{\varpi}_{j\epsilon})\mathbf{Y}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))]+\sum_{\hat{r}=1}^{\hat{n}}\hat{\pi}_{\hat{r}\epsilon}(\mathbf{x}(k))\mathbf{V}_{\hat{r}z\epsilon}^{(\iota,\beta)}(\mathbf{x}(k))\}\\&\leq 0,\quad\forall z\in\{1,2\},\epsilon\in\underline{K}.\end{aligned} \tag{45}$$

Due to $\upsilon_{ri_r\epsilon}(x_r(k))\geq 0$ for $\forall r\in\underline{\hat{n}}, i_r\in\{1,2\},\epsilon\in\underline{K}$, the sub-condition 5) of (32) can be guaranteed by $\sum_{j=1}^{c}[(\breve{\vartheta}_{ji_1i_2\ldots i_{\hat{n}}}+\underline{\varpi}_{j\epsilon})\mathbf{\Upsilon}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))+(\overline{\varpi}_{j\epsilon}-\underline{\varpi}_{j\epsilon})\mathbf{Y}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))]+\sum_{\hat{r}=1}^{\hat{n}}\hat{\pi}_{\hat{r}\epsilon}(\mathbf{x}(k))\mathbf{V}_{\hat{r}z\epsilon}^{(\iota,\beta)}(\mathbf{x}(k))\leq 0$, $\forall z\in\{1,2\}$, $\epsilon\in\underline{K}$.

Theorem 3: For the PIT2PFMB control system (9), if there exist a positive vector $\boldsymbol{\lambda}\in\Re^{n\times 1}$, polynomial vectors $\mathbf{y}_j^{(:,\beta)}(\mathbf{x}(k))\in\Re^{m\times 1}$ and $\mathbf{o}_j^{(:,\beta)}(\mathbf{x}(k))\in\Re^{m\times 1}$, $\forall j\in\underline{c}$, positive polynomial scalars $\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))$, $\mathbf{R}_{\hat{r}s\epsilon}^{(\alpha,1)}(\mathbf{x}(k))$, $\mathbf{Y}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))$, $\mathbf{V}_{\hat{r}z\epsilon}^{(\iota,\beta)}(\mathbf{x}(k))$ and a scalar $\hat{\gamma}>0$ such that the following SOS-based conditions of optimization problem are satisfied:

$$\begin{aligned}&\min\hat{\gamma}\\
s.t.\;&1)\ \nu^T(\hat{\gamma}-\mathbf{x}_0^r\boldsymbol{\lambda})\nu\text{ is SOS},\ \forall r\in\underline{l};\\
&2)\ \nu^T(\boldsymbol{\lambda}^{(\alpha,1)}-\varepsilon_1)\nu\text{ is SOS},\ \alpha\in\underline{n};\\
&3)\ \nu^T(\mathbf{\Theta}_{ijs}^{(\alpha,\beta)}(\mathbf{x}(k))-\varepsilon_2(\mathbf{x}(k)))\nu\text{ is SOS},\\
&\quad\forall i\in\underline{p},j\in\underline{c},s\in\underline{2^m},\alpha,\beta\in\underline{n};\\
&4)\ \nu^T(\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))-\varepsilon_3(\mathbf{x}(k)))\nu\text{ is SOS},\\
&\quad\forall i\in\underline{p},j\in\underline{c},s\in\underline{2^m},\alpha\in\underline{n};\\
&5)\ \nu^T(\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))-\mathbf{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))-\varepsilon_4(\mathbf{x}(k)))\nu\\
&\quad\text{is SOS},\ \forall i\in\underline{p},j\in\underline{c},s\in\underline{2^m},\alpha\in\underline{n};\\
&6)\ \nu^T(\mathbf{R}_{\hat{r}s\epsilon}^{(\alpha,1)}(\mathbf{x}(k))-\varepsilon_5(\mathbf{x}(k)))\nu\text{ is SOS},\\
&\quad\forall s\in\underline{2^m},\hat{r}\in\underline{\hat{n}},\epsilon\in\underline{K},\alpha\in\underline{n};\\
&7)\ -\nu^T\{\sum_{i=1}^{p}\sum_{j=1}^{c}[(\breve{\zeta}_{iji_1i_2\ldots i_{\hat{n}}}+\underline{\sigma}_{ij\epsilon})\mathbf{\Xi}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))\\
&\quad+(\overline{\sigma}_{ij\epsilon}-\underline{\sigma}_{ij\epsilon})\mathbf{W}_{ijs}^{(\alpha,1)}(\mathbf{x}(k))]+\sum_{\hat{r}=1}^{\hat{n}}\hat{\pi}_{\hat{r}\epsilon}(\mathbf{x}(k))\times\\
&\quad\mathbf{R}_{\hat{r}s\epsilon}^{(\alpha,1)}(\mathbf{x}(k))+\varepsilon_6(\mathbf{x}(k))\}\nu\text{ is SOS},\ \forall i\in\underline{p},j\in\underline{c},\\
&\quad s\in\underline{2^m},\hat{r}\in\underline{\hat{n}},\alpha\in\underline{n},\epsilon\in\underline{K},\mathbf{x}(k)\in\mathbf{\Psi}_\epsilon,\\
&\quad i_1,i_2,\ldots,i_{\hat{n}}\in\{1,2\};\\
&8)\ \nu^T(\mathbf{Y}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))-\varepsilon_7(\mathbf{x}(k)))\nu\text{ is SOS},\\
&\quad\forall z\in\{1,2\},j\in\underline{c},\iota\in\underline{m},\beta\in\underline{n};\\
&9)\ \nu^T(\mathbf{Y}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))-\mathbf{\Upsilon}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))-\varepsilon_8(\mathbf{x}(k)))\nu\\
&\quad\text{is SOS},\ \forall z\in\{1,2\},j\in\underline{c},\iota\in\underline{m},\beta\in\underline{n};\\
&10)\ \nu^T(\mathbf{V}_{\hat{r}z\epsilon}^{(\iota,\beta)}(\mathbf{x}(k))-\varepsilon_9(\mathbf{x}(k)))\nu\text{ is SOS},\\
&\quad\forall\hat{r}\in\underline{\hat{n}},z\in\{1,2\},\epsilon\in\underline{K},\iota\in\underline{m},\beta\in\underline{n};\\
&11)\ -\nu^T\{\sum_{j=1}^{c}[(\breve{\vartheta}_{ji_1i_2\ldots i_{\hat{n}}}+\underline{\varpi}_{j\epsilon})\mathbf{\Upsilon}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))\\
&\quad+(\overline{\varpi}_{j\epsilon}-\underline{\varpi}_{j\epsilon})\mathbf{Y}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))]+\sum_{\hat{r}=1}^{\hat{n}}\hat{\pi}_{\hat{r}\epsilon}(\mathbf{x}(k))\times\\
&\quad\mathbf{V}_{\hat{r}z\epsilon}^{(\iota,\beta)}(\mathbf{x}(k))\}\nu\text{ is SOS},\ \forall j\in\underline{c},\hat{r}\in\underline{\hat{n}},z\in\{1,2\},\\
&\quad\iota\in\underline{m},\beta\in\underline{n},\epsilon\in\underline{K},\mathbf{x}(k)\in\mathbf{\Psi}_\epsilon,i_1,i_2,\ldots,i_{\hat{n}}\in\{1,2\};\end{aligned} \tag{46}$$

where ν is an arbitrary vector independent of $\mathbf{x}(k)$ with appropriate dimensions; $\varepsilon_1 > 0$, $\varepsilon_2(\mathbf{x}(k)) > 0$, …, $\varepsilon_9(\mathbf{x}(k)) > 0$ are predefined scalar polynomials; $\mathbf{\Xi}_{ijs}(\mathbf{x}(k))$, $\mathbf{\Theta}_{ijs}^{(\alpha,\beta)}(\mathbf{x}(k))$ and $\mathbf{\Upsilon}_{zj}^{(\iota,\beta)}(\mathbf{x}(k))$ are defined in (28), (30) and (37), ρ is the predefined positive scalar, then the system (9) is asymptotically stable and positive with initial conditions contained in $\aleph(\boldsymbol{\lambda}, 1)$, where $\aleph(\boldsymbol{\lambda}, 1)$ is an estimation of the DOA. The polynomial fuzzy controller gains can be obtained as $\mathbf{G}_j(\mathbf{x}(k)) = [\frac{\mathbf{y}_j^{(:,1)}(\mathbf{x}(k))}{\lambda_1}, \frac{\mathbf{y}_j^{(:,2)}(\mathbf{x}(k))}{\lambda_2}, \dots, \frac{\mathbf{y}_j^{(:,n)}(\mathbf{x}(k))}{\lambda_n}]$ and the auxiliary polynomial fuzzy controller gains can be obtained as $\mathbf{H}_j(\mathbf{x}(k)) = [\frac{\mathbf{o}_j^{(:,1)}(\mathbf{x}(k))}{\lambda_1}, \frac{\mathbf{o}_j^{(:,2)}(\mathbf{x}(k))}{\lambda_2}, \dots, \frac{\mathbf{o}_j^{(:,n)}(\mathbf{x}(k))}{\lambda_n}]$.

IV. Simulation Example

A three-rules PIT2PFMB system is considered. The system and input matrices are as follows:

$$
\begin{aligned}
\mathbf{A}_1(x_1(k)) &= \begin{bmatrix} 0.1 - 0.001x_1^2(k) & 0.4 \\ 0.24 + 0.01x_1(k) & 0.06 \end{bmatrix}, \\
\mathbf{A}_2(x_1(k)) &= \begin{bmatrix} 0.4 & 0.1 + 0.01x_1(k) \\ 0.2 & 0.08 \end{bmatrix}, \\
\mathbf{A}_3(x_1(k)) &= \begin{bmatrix} 0.3 & 0.4 \\ 0.24 + 0.01x_1(k) & 0.06 \end{bmatrix}, \\
\mathbf{B}_1(x_1(k)) &= \begin{bmatrix} 1 + 0.001x_1^2(k) \\ 0.0015x_1^2(k) \end{bmatrix}, \\
\mathbf{B}_2(x_1(k)) &= \begin{bmatrix} 1 + 0.003x_1^2(k) \\ 0.001x_1^2(k) \end{bmatrix}, \\
\mathbf{B}_3(x_1(k)) &= \begin{bmatrix} 1 + 0.001x_1^2(k) \\ 0.003x_1^2(k) \end{bmatrix}.
\end{aligned} \tag{47}
$$

In this example, the saturation value $u_{lim} = 1$. The lower and upper MFs of the PIT2PFMB system are chosen as $\overline{w}_1(x_1(k)) = 1 - \frac{1}{1+e^{-\frac{x_1(k)-22.2-d_1}{3}}}$, $\overline{w}_3(x_1(k)) = \frac{1}{1+e^{-\frac{x_1(k)-37.8+d_1}{3}}}$, $\underline{w}_1(x_1(k)) = 1 - \frac{1}{1+e^{-\frac{x_1(k)-22.2+d_1}{3}}}$, $\underline{w}_3(x_1(k)) = \frac{1}{1+e^{-\frac{x_1(k)-37.8-d_1}{3}}}$, $\underline{w}_2(x_1(k)) = 1 - \overline{w}_1(x_1(k)) - \overline{w}_3(x_1(k))$. $\overline{w}_2(x_1(k)) = 1 - \underline{w}_1(x_1(k)) - \underline{w}_3(x_1(k))$, where $d_1 = 0.6$. The reference embedded type-1 model MF $\breve{w}_i(x_1(k))$, $i = 1, 3$ is obtained by $\breve{w}_i(x_1(k)) = \underline{\xi}_i(x_1(k))\underline{w}_i(x_1(k)) + \overline{\xi}_i(x_1(k))\overline{w}_i(x_1(k))$, where $\underline{\xi}_1(x_1(k)) = (\sin(2x_1(k)) + 2)/3$, $\overline{\xi}_1(x_1(k)) = 1 - \underline{\xi}_1(x_1(k))$, $\underline{\xi}_3(x_1(k)) = (\cos(2x_1(k)) + 2)/3$, $\overline{\xi}_3(x_1(k)) = 1 - \underline{\xi}_3(x_1(k))$, $\breve{w}_2(x_1(k))$ is obtained by $\breve{w}_2(x_1(k)) = 1 - \breve{w}_1(x_1(k)) - \breve{w}_3(x_1(k))$. In this paper, we adopt IPC concept [49], [52] to design the polynomial fuzzy controller, which means that the MFs and/or number of rules between the polynomial fuzzy model and controller are allowed to be different. In order to reduce the complexity of implementing controller, the number of rules for the fuzzy controller is chosen as 2 and the lower and upper MFs are chosen as

$$
\underline{m}_1(x_1(k)) = \begin{cases} 0 & \text{if} \quad x_1(k) > 55 - d_2 \\ -\frac{x_1(k)+55-d_2}{50} & \text{if } 5 - d_2 \le x_1(k) \le 55 - d_2 \\ 1 & \text{if} \quad x_1(k) < 5 - d_2 \end{cases},
$$

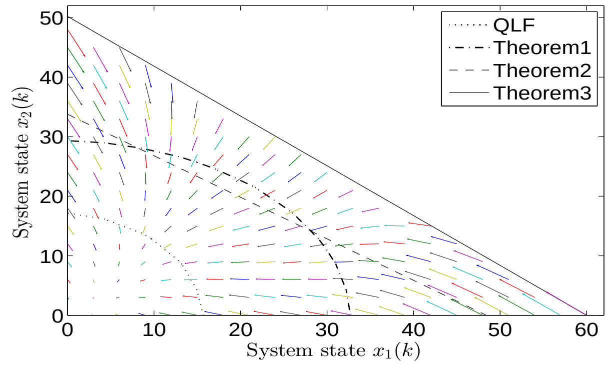

Fig. 2. Comparison of DOAs with quiver plot, where the dotted-border elliptical area is obtained by quadratic Lyapunov function, the dash-dotted-border elliptical area is obtained by Theorem 1, the dashed-border triangular area is obtained by Theorem 2 and the solid-border triangular area is obtained by Theorem 3.

$$
\overline{m}_1(x_1(k)) = \begin{cases} 1 & \text{if} \quad x_1(k) > 55 + d_2 \\ -\frac{x_1(k)+55+d_2}{50} & \text{if } 5 + d_2 \le x_1(k) \le 55 + d_2 \\ 0 & \text{if} \quad x_1(k) < 5 + d_2 \end{cases},
$$

$\overline{m}_2(x_1(k)) = 1 - \underline{m}_1(x_1(k))$, and $\underline{m}_2(x_1(k)) = 1 - \overline{m}_1(x_1(k))$, where $d_2 = 0.4$. The reference embedded type-1 controller MF $\breve{m}_j(x_1(k))$ is obtained by $\breve{m}_j(x_1(k)) = \frac{\underline{\varsigma}_j(x_1(k))\underline{m}_j(x_1(k)) + \overline{\varsigma}_j(x_1(k))\overline{m}_j(x_1(k))}{\sum_{l=1}^{c}(\underline{\varsigma}_l(x_1(k))\underline{m}_l(x_1(k)) + \overline{\varsigma}_l(\mathbf{x}(k))\overline{m}_l(x_1(k)))}$, where $\overline{\varsigma}_j(x_1(k)) = \underline{\varsigma}_j(x_1(k)) = 0.5$, $j = 1, 2$.

In order to verify that the LCLF with the polyhedron DOA leads to more relaxed results than the saturation-dependent Lyapunov function with elliptical DOA, Theorem 1 and Theorem 2 are applied on this example. Inspired by [43], the reference set is chosen as $\boldsymbol{\chi}_R := co\{(\sin(\theta), \cos(\theta))\}$, $\theta \in [0, \frac{\pi}{2}]$.

For Theorem 1, all ε's are chosen as 1×10^{-3} and the degree of $\mathbf{D}_j(x_1(k))$, $\mathbf{O}_j(x_1(k))$ are chosen as 4 in $x_1(k)$.

All conditions in Theorem 1 are calculated by the MATLAB toolbox SOSTOOLS. We obtain the minimal value of γ to be 0.001 when $\theta = \frac{\pi}{2}$, and the controller gains and auxiliary controller gains are obtained as follows:

$\mathbf{G}_1(x_1(k)) = [1.9941 \times 10^{-6}x_1^4(k) + 1.0741 \times 10^{-7}x_1^3(k) + 1.5049 \times 10^{-4}x_1^2(k) - 1.3406 \times 10^{-3}x_1(k) + 0.008853, 2.0322 \times 10^{-6}x_1^4(k) + 5.3242 \times 10^{-6}x_1^3(k) + 2.7607 \times 10^{-5}x_1^2(k) - 3.8899 \times 10^{-3}x_1(k) + 0.13392]$;

$\mathbf{G}_2(x_1(k)) = [1.9941 \times 10^{-6}x_1^4(k) + 1.0741 \times 10^{-7}x_1^3(k) + 1.5049 \times 10^{-4}x_1^2(k) - 1.3406 \times 10^{-3}x_1(k) + 0.008853, 2.0322 \times 10^{-6}x_1^4(k) + 5.3242 \times 10^{-6}x_1^3(k) + 2.7607 \times 10^{-5}x_1^2(k) - 3.8899 \times 10^{-3}x_1(k) + 0.13392]$;

$\mathbf{H}_1(x_1(k)) = [3.1085 \times 10^{-9}x_1^4(k) - 2.0076 \times 10^{-9}x_1^3(k) - 6.3980 \times 10^{-6}x_1^2(k) - 1.8965 \times 10^{-5}x_1(k) + 0.006491, 3.4294 \times 10^{-9}x_1^4(k) + 4.1793 \times 10^{-8}x_1^3(k) - 1.5411 \times 10^{-5}x_1^2(k) - 3.3208 \times 10^{-4}x_1(k) + 0.008052]$;

$\mathbf{H}_2(x_1(k)) = [2.9185 \times 10^{-9}x_1^4(k) - 2.0076 \times 10^{-9}x_1^3(k) - 6.3980 \times 10^{-6}x_1^2(k) - 1.8965 \times 10^{-5}x_1(k) + 0.006491, 3.4294 \times 10^{-9}x_1^4(k) + 4.1793 \times 10^{-8}x_1^3(k) - 1.5411 \times 10^{-5}x_1^2(k) - 3.3208 \times 10^{-4}x_1(k) + 0.008052]$;

The Lyapunov matrices $\mathbf{X} = \text{diag}\{1031.0325, 958.9043\}$, $\mathbf{Q}_1 = \begin{bmatrix} 1074.2031 & -26.2345 \\ -26.2332 & 863.7751 \end{bmatrix}$ and $\mathbf{Q}_2 =$

$\begin{bmatrix} 1106.3186 & 12.7971 \\ 12.7975 & 863.7749 \end{bmatrix}$ are obtained. The estimated DOA can be represented by the intersection of $\mathbf{x}^T\mathbf{Q}_1^{-1}\mathbf{x} \leq 1$ and $\mathbf{x}^T\mathbf{Q}_2^{-1}\mathbf{x} \leq 1$, the obtained elliptical DOA is shown in Fig. 2 and the border is marked with a dash-dotted line.

When Theorem 2 is applied on this example, all ε's are chosen as 1×10^{-3} and the degree of $\mathbf{y}_j^{(:,\beta)}(x_1(k))$ and $\mathbf{o}_j^{(:,\beta)}(x_1(k))$ are chosen as 4 in $x_1(k)$, ρ is chosen as 1×10^{-6}. With the help of the MATLAB SOSTOOLS toolbox, we have the minimal value of $\hat{\gamma}$ to be 0.0218 when $\theta = \frac{\pi}{2}$, and the controller gains and auxiliary controller gains are as follows:

$\mathbf{G}_1(x_1(k)) = [1.9278 \times 10^{-5}x_1^4(k) - 3.3602 \times 10^{-5}x_1^3(k) + 4.40502 \times 10^{-5}x_1^2(k) + 3.7305 \times 10^{-3}x_1(k) + 0.02978, -8.7613 \times 10^{-6}x_1^4(k) + 9.3401 \times 10^{-6}x_1^3(k) + 5.7036 \times 10^{-5}x_1^2(k) - 6.1749 \times 10^{-3}x_1(k) - 0.02524]$;

$\mathbf{G}_2(x_1(k)) = [1.9278 \times 10^{-5}x_1^4(k) - 3.3602 \times 10^{-5}x_1^3(k) + 4.40501 \times 10^{-5}x_1^2(k) + 3.7305 \times 10^{-3}x_1(k) + 0.02978, -8.7613 \times 10^{-6}x_1^4(k) + 9.3401 \times 10^{-6}x_1^3(k) + 5.7036 \times 10^{-5}x_1^2(k) - 6.1749 \times 10^{-3}x_1(k) - 0.02524]$;

$\mathbf{H}_1(x_1(k)) = [2.7031 \times 10^{-9}x_1^4(k) + 6.8547 \times 10^{-10}x_1^3(k) - 3.7107 \times 10^{-8}x_1^2(k) - 1.7516 \times 10^{-9}x_1(k) - 7.7117 \times 10^{-9}, 3.1274 \times 10^{-9}x_1^4(k) - 2.6153 \times 10^{-10}x_1^3(k) + 1.6206 \times 10^{-8}x_1^2(k) + 1.8181 \times 10^{-8}x_1(k) + 6.7047 \times 10^{-8}]$;

$\mathbf{H}_2(x_1(k)) = [2.7031 \times 10^{-9}x_1^4(k) + 6.8547 \times 10^{-10}x_1^3(k) - 3.7107 \times 10^{-8}x_1^2(k) - 1.7516 \times 10^{-9}x_1(k) - 7.7117 \times 10^{-9}, 3.1274 \times 10^{-9}x_1^4(k) - 2.6153 \times 10^{-10}x_1^3(k) + 1.6206 \times 10^{-8}x_1^2(k) + 1.8181 \times 10^{-8}x_1(k) + 6.7047 \times 10^{-8}]$;

The Lyapunov function variable $\boldsymbol{\lambda} = \begin{bmatrix} 0.02066 \\ 0.02961 \end{bmatrix}$ is obtained, the DOA can be represented by $\mathbf{x}^T\boldsymbol{\lambda} \leq 1$. The obtained dashed-border triangular DOA is shown in Fig. 2. The area of the dash-dotted-border elliptical DOA is about 728 and the area of the dashed-border triangular DOA is about 803, which means that the LCLF-based analysis strategy proposed in this paper is less conservative than the saturation-dependent-Lyapunov-function-based analysis strategy.

When all $\mathbf{P}_s$ in the saturation-dependent Lyapunov function are designed as the same variable, the saturation-dependent Lyapunov function will degenerate into quadratic Lyapunov function. Following the same derivation process as Theorem 1, the quadratic-Lyapunov-function-based results are obtained. When these results are applied on this simulation example, the obtained DOA is drawn as the dotted-border elliptical DOA in Fig. 2. It can be seen that the dash-dotted-border elliptical DOA and dashed-border triangular DOA all are larger than the dotted-border elliptical DOA. The dash-dotted-border elliptical DOA is larger than the dotted-border elliptical DOA because the saturation information is considered in Theorem 1 and the different $\mathbf{P}_s$ provide the analysis flexibility. However, the dashed-border triangular DOA is larger than the dotted-border elliptical DOA because polyhedron has stronger ability to characterize the DOA than traditional ellipsoid [45]. Compared with the Theorem 1, Theorem 2 requires fewer decision variables and symbolic variables. Therefore, Theorem 2 has lower computational burden. The number of the decision variables and symbolic variables in Theorem 1 and Theorem 2 are shown in Table I. For example, Theorem 1 is applied to (47) with $\mathbf{A}_i \in \Re^{2\times 2}$ and $\mathbf{B}_i \in \Re^{2\times 1}$, $i \in \underline{3}$, $j \in \underline{2}$. Then the dimensions of decision variables $\mathbf{X}$ and $\mathbf{Q_s}$ are 2×2, the number of them is $1 + s = 1 + 2^m = 3$; the symbolic variables of Theorem 1 are $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and vector ν, where the number of variables in vector ν is related to the dimensions of the biggest matrix. For instance, the number of variables in ν of Theorem 1 is 4, because the biggest matrices in (11) are $\mathbf{\Psi}_{ijs\hat{s}}(\mathbf{x}(k))$ whose dimensions are 4×4. The simulation of this example is performed by the 2014a MATLAB installed on a computer with 32G RAM and i5-4590 CPU. The simulation times corresponding to Theorems 1 and 2 are also shown in Table I.

All the results of the above quadratic-Lyapunov-function-based method, Theorem 1 and Theorem 2 are independent on the MFs, resulting in conservatism to some extent. In order to verify that the improved IT2MFD method of this paper can relax the estimation of DOA by introducing some information of IT2 MFs, Theorem 3 is applied on this example. $\mathbf{W}_{ijs}^{(\alpha,1)}(x_1(k))$, $\mathbf{R}_{\hat{r}s\epsilon}^{(\alpha,1)}(x_1(k))$, $\mathbf{Y}_{zj}^{(\iota,\beta)}(x_1(k))$ and $\mathbf{V}_{\hat{r}z\epsilon}^{(\iota,\beta)}(x_1(k))$ are all of degrees 0, all ε's are chosen as 1×10^{-3} and the degree of $\mathbf{y}_j^{(:,\beta)}(x_1(k))$ and $\mathbf{o}_j^{(:,\beta)}(x_1(k))$ are designed as 4 in $x_1(k)$, ρ is chosen as 3×10^{-2}. The interpolation points are chosen as $x_1 = \{0, 6, 12, 15, 16.5, 18, 19.5, 21, 22.5, 24, 25.5, 26.1, 27, 28.5, 30, 31.5, 33, 34.5, 35.1, 36, 37.5, 39, 40.5, 42, 43.5, 45, 48, 54, 60\}$. By using the MTALAB toolbox SOSTOOLS to calculate the conditions in Theorem 3, the minimal value of $\hat{\gamma}$ is obtained as 0.0167 when $\theta = \frac{\pi}{2}$. The controller gains and auxiliary controller gains are obtained as follows:

$\mathbf{G}_1(x_1(k)) = [-2.8884 \times 10^{-5}x_1^4(k) + 9.2718 \times 10^{-6}x_1^3(k) + 1.5813 \times 10^{-3}x_1^2(k) - 1.6421 \times 10^{-3}x_1(k) + 0.1591, -1.8807 \times 10^{-5}x_1^4(k) + 1.9729 \times 10^{-5}x_1^3(k) + 4.6362 \times 10^{-4}x_1^2(k) - 1.0057 \times 10^{-2}x_1(k) + 0.1093]$;

$\mathbf{G}_2(x_1(k)) = [-2.7114 \times 10^{-5}x_1^4(k) + 1.0556 \times 10^{-7}x_1^3(k) + 4.7285 \times 10^{-4}x_1^2(k) + 3.3223 \times 10^{-5}x_1(k) + 0.3047, -1.4694 \times 10^{-5}x_1^4(k) + 6.8754 \times 10^{-6}x_1^3(k) - 7.6073 \times 10^{-4}x_1^2(k) - 7.6579 \times 10^{-3}x_1(k) + 0.1973]$;

$\mathbf{H}_1(x_1(k)) = [1.0352 \times 10^{-9}x_1^4(k) + 2.8267 \times 10^{-7}x_1^3(k) + 1.0477 \times 10^{-6}x_1^2(k) + 1.3127 \times 10^{-5}x_1(k) - 1.7555 \times 10^{-4}, -2.6302 \times 10^{-9}x_1^4(k) + 5.8434 \times 10^{-7}x_1^3(k) + 1.5434 \times 10^{-5}x_1^2(k) - 9.9488 \times 10^{-4}x_1(k) + 2.2591 \times 10^{-3}]$;

$\mathbf{H}_2(x_1(k)) = [4.7252 \times 10^{-9}x_1^4(k) + 3.6424 \times 10^{-7}x_1^3(k) - 3.0846 \times 10^{-5}x_1^2(k) - 3.4201 \times 10^{-4}x_1(k) + 5.2530 \times 10^{-3}, 7.8078 \times 10^{-9}x_1^4(k) + 2.0765 \times 10^{-7}x_1^3(k) - 6.1491 \times$

TABLE I
COMPARISON OF THEOREM 1 AND THEOREM 2

	Number of decision variables	Decision variables	Number of symbolic variables $\mathbf{x}, \nu$	Times of obtaining results
Theorem 1	3	$\mathbf{X}$, $\mathbf{Q}_s \in \Re^{2\times 2}$	6	86s
	4	$\mathbf{D}_j$, $\mathbf{O}_j \in \Re^{1\times 2}$		
	1	$\gamma \in \Re^{1\times 1}$		
Theorem 2	1	$\boldsymbol{\lambda} \in \Re^{2\times 1}$	3	34s
	4	$\mathbf{y}_j$, $\mathbf{o}_j \in \Re^{1\times 2}$		
	1	$\hat{\gamma} \in \Re^{1\times 1}$		

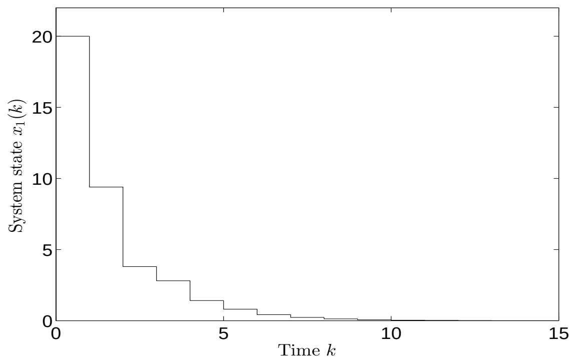

Fig. 3. Time response of system state $x_1(k)$ with initial condition $x_1(0) = 20$.

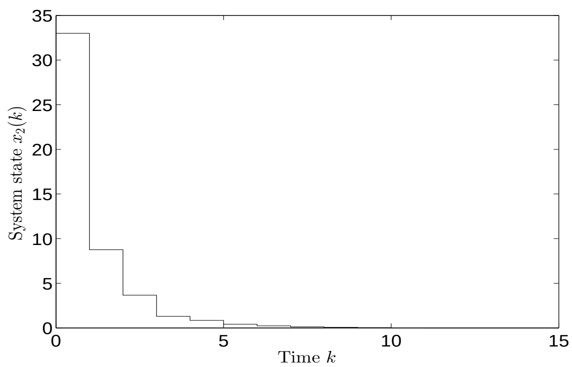

Fig. 4. Time response of system state $x_2(k)$ with initial condition $x_2(0) = 33$.

$10^{-5}x_1^2(k) + 1.1752 \times 10^{-4}x_1(k) + 2.8970 \times 10^{-2}]$;

The Lyapunov function variable $\boldsymbol{\lambda} = \begin{bmatrix} 0.01666 \\ 0.01991 \end{bmatrix}$ is obtained, the DOA can be represented by $\mathbf{x}^T\boldsymbol{\lambda} \leq 1$. The obtained solid-border triangular DOA is shown in Fig. 2. It can be seen that this solid-border triangular DOA is larger than any other DOAs, which means that the improved IT2MFD method of this paper can effectively relax the estimation of DOA for the PIT2PFMB control system. The stability of the system states in this DOA is demonstrated by the quiver plot in Fig. 2.

In order to visually show that the designed controller can steer the system state to the equilibrium point (origin) with the points in the DOA as the initial condition, the boundary point $\mathbf{x}(0) = \begin{bmatrix} 20 \\ 33 \end{bmatrix}$ of the solid-border triangular DOA is chosen as the initial condition. Under the control of the controller designed by Theorem 3, the time responses of the system states with this initial condition are shown in Figs. 3 and 4. The Fig. 5 shows the control signals, and the little figure shows the time response curves for a certain period of time. The control input signal $\mathbf{G}(x_1(k))\mathbf{x}(k)$ is denoted by solid line, the auxiliary control input signal $\mathbf{H}(x_1(k))\mathbf{x}(k)$ is denoted by dotted line and saturation control signal $\text{sat}(\mathbf{u}(k))$ is denoted by dashed line. It can be seen that the control input signal $\mathbf{G}(x_1(k))\mathbf{x}(k)$ is larger than the control limit u_{lim} for an initial period of time, which means that the control signal will be capped when it controls the system through the actuator. Also, the auxiliary control signal $\mathbf{H}(x_1(k))\mathbf{x}(k)$ in Fig. 5 is always lower than u_{lim}, which provides the possibility of satisfying (2) for $\text{sat}(\mathbf{u}(k))$ in (8), where $\eta_s(\mathbf{x}(k))$ needs to change in real time so that the convex combination of the control signal $\mathbf{G}(x_1(k))\mathbf{x}(k)$ and the auxiliary control signal $\mathbf{H}(x_1(k))\mathbf{x}(k)$ always meets condition (2).

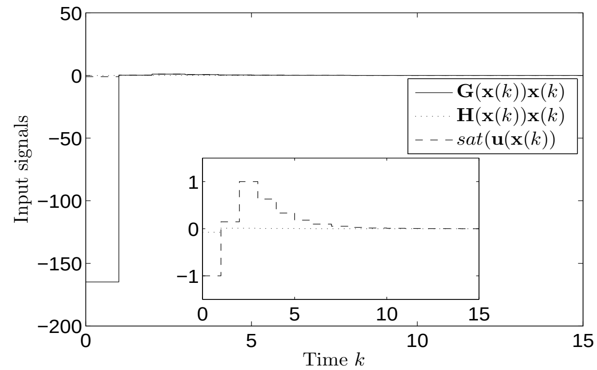

Fig. 5. Time response of control input signal $\mathbf{G}(x_1(k))\mathbf{x}(k)$ ("solid line"), auxiliary control input signal $\mathbf{H}(x_1(k))\mathbf{x}(k)$ ("dotted line") and saturation control signal $\text{sat}(\mathbf{u}(k))$ ("dashed line") with initial condition $\mathbf{x}(0) = [20, 33]^T$.

V. Conclusion

The estimation of the DOA of the discrete-time PIT2PFMB control system has been investigated in this paper. Under the comprehensive consideration of positive constrains, nonlinearity, parameter uncertainties and input saturation, a pair of IT2 polynomial fuzzy controller and auxiliary controller have been designed by using the LCLF-based analysis method, and the obstacle such as non-convex conditions encountered during the analysis has been eliminated in this paper. In addition, the upper and lower bound information of the IT2 MFs and local boundary information of the premise variables have been introduced to stability conditions and DOA estimation conditions by the improved IT2MFD analysis method. As a result, the estimation of the DOA of the discrete-time PIT2PFMB control system gets remarkable relaxation by changing the shape of the attractive invariant sets and including the IT2 MFs information into resultant conditions. In the future, saturation-dependent LCLF which utilizes the saturation information and polyhedron invariant sets at the same time can be used to further expand the estimation of DOA if the non-convex conditions can be handled effectively. Also, enlarging the estimation of DOA by introducing MFs information is an open topic, and more relaxed results could be obtained by introducing more useful MFs information.

References

[1] X. Liu, W. Yu, and L. Wang, "Stability analysis for continuous-time positive systems with time-varying delays," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 1024–1028, 2010.

[2] J. A. Jacquez, "Compartmental analysis in biology and medicine," *The Quarterly Review of Biology*, vol. 48, no. 4, pp. 661–661, 1973.

[3] R. Shorten, D. Leith, J. Foy, and R. Kilduff, "Analysis and design of aimd congestion control algorithms in communication networks," *Automatica*, vol. 41, no. 4, pp. 725 – 730, 2005.

[4] L. Farina and S. Rinaldi, *Positive linear systems: theory and applications*. John Wiley & Sons, 2011, vol. 50.

[5] A. Benzaouia, A. Hmamed, and A. E. Hajjaji, "Stabilization of controlled positive discrete-time T–S fuzzy systems by state feedback control," *International Journal of Adaptive Control & Signal Processing*, vol. 24, no. 12, pp. 1091–1106, 2010.

[6] X. Li, C. Liu, H. K. Lam, F. Liu, and X. Zhao, "Stability analysis of fuzzy polynomial positive systems with time delay," in *2014 International Conference on Fuzzy Theory and Its Applications (iFUZZY2014)*, 2014, pp. 24–28.

[7] S. Huang, Z. Xiang, and H. R. Karimi, "Mixed L_-/L_1 fault detection filter design for fuzzy positive linear systems with time-varying delays," *IET Control Theory & Applications*, vol. 8, no. 12, pp. 1023–1031, 2014.

[8] J. Lian, S. Li, and J. Liu, "T–S fuzzy control of positive Markov Jump nonlinear systems," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 4, pp. 2374–2383, Aug 2018.

[9] M. Han, H. Lam, Y. Li, F. Liu, and C. Zhang, "Observer-based control of positive polynomial fuzzy systems with unknown time delay," *Neurocomputing*, vol. 349, pp. 77 – 90, 2019.

[10] K. Tanaka, H. Yoshida, H. Ohtake, and H. O. Wang, "A sum–of–squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 4, pp. 911–922, Aug 2009.

[11] T. Haidegger, L. Kovacs, S. Preitl, R. E. Precup, and Z. Benyo, "Controller design solutions for long distance telesurgical applications," *International journal on artificial intelligence*, vol. 6, no. S11, pp. 48–71, Mar. 2011.

[12] R. E. Precup and M. L. Tomescu, "Stable fuzzy logic control of a general class of chaotic systems," *Neural Computing & Applications*, vol. 26, no. 3, pp. 541–550, Apr. 2015.

[13] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOSTOOLS: a general purpose sum of squares programming solver," in *Decision and Control, 2002, Proceedings of the IEEE Conference on*, 2002, pp. 741–746 vol.1.

[14] J. Cao, P. Li, and H. Liu, "An interval fuzzy controller for vehicle active suspension systems," *IEEE Transactions on Intelligent Transportation Systems*, vol. 11, no. 4, pp. 885–895, 2010.

[15] E. Kayacan and okyay kaynak, "Sliding mode control theory-based algorithm for online learning in type-2 fuzzy neural networks: Application to velocity control of an electro hydraulic servo system," *International Journal of Adaptive Control & Signal Processing*, vol. 26, no. 7, pp. 645–659, 2012.

[16] O. Castillo, L. Cervantes, J. Soria, M. Sanchez, and J. R. Castro, "A generalized type-2 fuzzy granular approach with applications to aerospace," *Information Sciences*, vol. 354, pp. 165 – 177, 2016.

[17] L. Cervantes and O. Castillo, "Type-2 fuzzy logic aggregation of multiple fuzzy controllers for airplane flight control," *Information Sciences*, vol. 324, pp. 247 – 256, 2015.

[18] E. Ontiveros, P. Melin, and O. Castillo, "Comparative study of interval type-2 and general type-2 fuzzy systems in medical diagnosis," *Information Sciences*, vol. 525, pp. 37 – 53, 2020.

[19] J. M. Mendel, R. I. John, and F. Liu, "Interval type-2 fuzzy logic systems made simple," *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 6, pp. 808–821, 2006.

[20] E. Ontiveros-Robles, P. Melin, and O. Castillo, "Comparative analysis of noise robustness of type 2 fuzzy logic controllers," *Kybernetika*, vol. 54, no. 1, pp. 175 – 201, 2018.

[21] O. Castillo, L. Amador-Angulo, J. R. Castro, and M. Garcia-Valdez, "A comparative study of type-1 fuzzy logic systems, interval type-2 fuzzy logic systems and generalized type-2 fuzzy logic systems in control problems," *Information Sciences*, vol. 354, pp. 257 – 274, 2016.

[22] E. Ontiveros, P. Melin, and O. Castillo, "High order α-planes integration: A new approach to computational cost reduction of general type-2 fuzzy systems," *Engineering Applications of Artificial Intelligence*, vol. 74, pp. 186 – 197, 2018.

[23] Q. Liang and J. M. Mendel, *Interval type-2 fuzzy logic systems: theory and design*. IEEE Press, 2000.

[24] O. Castillo and P. Melin, "Type-2 fuzzy logic systems. in: Recent advances in interval type-2 fuzzy systems," *Springerbriefs in Applied Sciences & Technology*, vol. 1, 2012.

[25] O. Castillo, P. Melin, A. Alanis, O. Montiel, and R. Sepulveda, "Optimization of interval type-2 fuzzy logic controllers using evolutionary algorithms," *Soft Computing*, vol. 15, no. 6, pp. 1145–1160, 2011.

[26] H. K. Lam and L. D. Seneviratne, "Stability analysis of interval type-2 fuzzy-model-based control systems," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 38, no. 3, pp. 617–628, 2008.

[27] M. S. Fadali and S. Jafarzadeh, "Stability analysis of positive interval Type-2 TSK systems with application to energy markets," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 1031–1038, 2014.

[28] H. Li, J. Wang, and P. Shi, "Output-feedback based sliding mode control for fuzzy systems with actuator saturation," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 6, pp. 1282–1293, 2016.

[29] L. Zha, J. Liu, and J. Cao, "Security control for T–S fuzzy systems with multi-sensor saturations and distributed event-triggered mechanism," *Journal of the Franklin Institute*, vol. 357, no. 5, pp. 2851 – 2867, 2020.

[30] X. Han, Y. Ma, and L. Fu, "Finite-time dynamic output-feedback dissipative control for singular uncertainty T–S fuzzy systems with actuator saturation and output constraints," *Journal of the Franklin Institute*, vol. 357, no. 8, pp. 4543 – 4573, 2020.

[31] L. Zhao and L. Li, "Robust stabilization of T–S fuzzy discrete systems with actuator saturation via PDC and non-PDC law," *Neurocomputing*, vol. 168, pp. 418 – 426, 2015.

[32] M. Han, H. Lam, F. Liu, L. Wang, and Y. Tang, "Stability analysis and estimation of domain of attraction for positive polynomial fuzzy systems with input saturation," *IEEE Transactions on Fuzzy Systems*, vol. 28, no. 8, pp. 1723–1736, 2020.

[33] A. Benzaouia, F. Mesquine, M. Benhayoun, H. Schulte, and S. Georg, "Stabilization of positive constrained T–S fuzzy systems: Application to a Buck converter," *Journal of the Franklin Institute*, vol. 351, no. 8, pp. 4111–4123, 2014.

[34] X. Wang, Y. Tao, and X. Song, "Mathematical model for the control of a pest population with impulsive perturbations on diseased pest," *Applied Mathematical Modelling*, vol. 33, no. 7, pp. 3099 – 3106, 2009.

[35] Y. Li, S. Tong, and T. Li, "Hybrid fuzzy adaptive output feedback control design for uncertain mimo nonlinear systems with time-varying delays and input saturation," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 4, pp. 841–853, 2016.

[36] ——, "Composite adaptive fuzzy output feedback control design for uncertain nonlinear strict-feedback systems with input saturation," *IEEE Transactions on Cybernetics*, vol. 45, no. 10, pp. 2299–2308, 2015.

[37] J. M. G. D. Silva, S. Tarbouriech, and R. Reginatto, "Analysis of regions of stability for linear systems with saturating inputs through an anti-windup scheme," in *International Conference on Control Applications*, 2002, pp. 1106–1111 vol.2.

[38] T. Hu and Z. Lin, *Control Systems with Actuator Saturation*. Birkhauser, 2001.

[39] J. Wang and J. Zhao, "Stabilisation of switched positive systems with actuator saturation," *Iet Control Theory & Applications*, vol. 10, no. 6, pp. 717–723, 2016.

[40] P. Li, X. Sun, and X. Zhao, "Stabilization of linear systems with input saturation and large delay," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, pp. 1–10, 2018.

[41] Y. Li and Z. Lin, "Improvements to the linear differential inclusion approach to stability analysis of linear systems with saturated linear feedback," *Automatica*, vol. 49, no. 3, pp. 821–828, 2013.

[42] P. Selvaraj, R. Sakthivel, and O. M. Kwon, "Finite-time synchronization of stochastic coupled neural networks subject to Markovian switching and input saturation," *Neural Networks*, vol. 105, pp. 154–165, 2018.

[43] Y. Y. Cao and Z. Lin, "Stability analysis of discrete-time systems with actuator saturation by a saturation-dependent Lyapunov function," *Automatica*, vol. 39, no. 7, pp. 1235–1241, 2003.

[44] L. I. Chuang, K. Chen, L. U. Junguo, and R. Tang, "Stability and stabilization analysis of fractional-order linear systems subject to actuator saturation and disturbance," vol. 50, no. 1, pp. 9718–9723, 2017.

[45] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747 – 1767, 1999.

[46] A. Benzaouia, R. Oubah, and A. E. Hajjaji, "Stabilization of positive Takagi–Sugeno fuzzy discrete-time systems with multiple delays and bounded controls," *Journal of the Franklin Institute*, vol. 351, no. 7, pp. 3719–3733, 2014.

[47] X. Chen, L. Wang, M. Chen, and J. Shen, "L_∞ -induced output-feedback controller synthesis for positive nonlinear systems via T-S fuzzy model approach," *Fuzzy Sets and Systems*, 2019.

[48] J. Lian and S. Li, "Fuzzy control of uncertain positive Markov jump fuzzy systems with input constraint," *IEEE Transactions on Cybernetics*, pp. 1–10, 2019.

[49] H. K. Lam, H. Li, C. Deters, E. L. Secco, H. A. Wurdemann, and K. Althoefer, "Control design for interval type-2 fuzzy systems under imperfect premise matching," *IEEE Transactions on Industrial Electronics*, vol. 61, no. 2, pp. 956–968, 2014.

[50] G. Song, H. Lam, and X. Yang, "Membership-function-dependent stability analysis of interval type-2 polynomial fuzzy-model-base control systems," *IET Control Theory Applications*, vol. 11, no. 17, pp. 3156–3170, 2017.

[51] B. Xiao, H. Lam, Y. Yu, and Y. Li, "Sampled-data output-feedback tracking control for interval type-2 polynomial fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 28, no. 3, pp. 424–433, 2020.

[52] H. K. Lam, "A review on stability analysis of continuous-time fuzzy-model-based control systems: From membership-function-independent to membership-function-dependent analysis," *Engineering Applications of Artificial Intelligence*, vol. 67, pp. 390 – 408, 2018.

[53] X. Li, H. K. Lam, F. Liu, and X. Zhao, "Stability and stabilization analysis of positive polynomial fuzzy systems with time delay considering piecewise membership functions," *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 4, pp. 958–971, 2017.

[54] A. Meng, H. K. Lam, Y. Yu, X. Li, and F. Liu, "Static output feedback stabilization of positive polynomial fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 3, pp. 1600–1612, June 2018.

[55] H. K. Lam, C. Liu, L. Wu, and X. Zhao, "Polynomial fuzzy-model-based control systems: Stability analysis via approximated membership functions considering sector nonlinearity of control input," *IEEE Transactions on Fuzzy Systems*, vol. 23, no. 6, pp. 2202–2214, 2015.

[56] L. Xie and C. E. De Souza, "Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty," *IEEE Transactions on Automatic Control*, vol. 37, no. 8, pp. 1188–1191, 1992.

[57] J. M. Mendel, "Type-2 fuzzy sets and systems: An overview," *IEEE Computational Intelligence Magazine*, vol. 2, no. 1, pp. 20–29, 2007.

[58] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA, USA: SIAM, 1994, vol. 15.