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Stability Analysis and Estimation of Domain of Attraction for Positive Polynomial Fuzzy Systems with Input Saturation

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***Abstract*—In this paper, the stability and positivity of positive polynomial fuzzy-model-based (PPFMB) control system are investigated, in which the positive polynomial fuzzy model and positive polynomial fuzzy controller are allowed to have different premise membership functions from each other. These mismatched premise membership functions can increase the flexibility of controller design, however, it will lead to the conservative results when the stability is analyzed based on the Lyapunov stability theory. To relax the positivity/stability conditions, the improved Taylor-series-membership-functions-dependent (ITSMFD) method is introduced by introducing the sample points information of Taylor series approximate membership functions, local error information and boundary information of substate space of premise variables into the stability/positivity conditions. Meanwhile, the ITSMFD method is extended to the PPFMB control system with input saturation to relax the estimation of domain of attraction (DOA). Finally, simulation examples are presented to verify the feasibility of this method.**

***Index Terms*—Input saturation, improved Taylor-series-membership-functions-dependent (ITSMFD) method , domain of attraction (DOA), imperfect premise matching (IPC) concept, positive polynomial fuzzy-model-based (PPFMB) control systems.**

I. Introduction

Positive systems are a class of systems whose trajectories of states remain forever positive whenever the initial conditions are non-negative. In recent years, some papers have investigated linear positive systems [1], [2]. As most of industrial plants are nonlinear systems, the research on positive nonlinear systems is meaningful and necessary. As a powerful mathematical tool for dealing with nonlinear systems, Takagi-Sugeno (T-S) fuzzy model was used in [3] to deal with positive nonlinear system for the first time. Then, linear-matrix-inequality (LMI) based stability and positivity conditions for positive fuzzy systems [4] were obtained based on Lyapunov stability theory. At present, the research on the positive T-S fuzzy-model-based (FMB) control systems have been expanded into various topics such as stability analysis for positive T-S FMB continue time systems with time delay [5], [6], positive T-S FMB Markov Jump systems [7] and positive T-S FMB switched systems [8]–[10], tracking control [11], constrained control [12], output feedback control [13], filter design [14], [15], observer design [16], observer-based control [17], etc. Beyond these, considering the uncertainty of significant data in some practical systems [18], positive interval type-2 TSK fuzzy model [19] instead of T-S fuzzy model was used to represent nonlinear systems with parameter uncertainties.

Although a large number of research results on positive T-S FMB systems have been obtained in recent years, the research on positive T-S FMB systems with input saturation is relatively less. When the system states deviate significantly from the equilibrium point, a large control signal is required to stabilize the closed loop system. However, the physical constraint of actuator with saturation will lead to capped control signal, which will degrade the performance and even can affect the stability of control system. Thus, it is important to identify which states can be steered to the equilibrium point under capped control signals. The set of these states is referred to as domain of attraction (DOA). In the existing literature, some efforts have been made to relax the estimation of DOA.

For linear systems with input saturation, most literature relaxed the estimation of DOA by improving the method of dealing with saturation nonlinearity [20]–[25]. For example, sector-condition-based method was adopted in [24] to handle the saturation nonlinearity, and convex hull representation method used in [20], [25] provided a less conservative representation for saturation nonlinearity so that a more relaxed estimation of DOA was obtained. Subsequently, in order to further relax the estimation of DOA, [23] improved the convex hull representation method by using multiple auxiliary feedback gains. In addition, [22] relaxed DOA by using composite quadratic Lyapunov function and [21] adopted the anti-windup control scheme to handle input saturation. For the T-S FMB nonlinear systems with input saturation [26]–[34], there are some membership functions in the model to weight and combine linear subsystems. Thus, some methods related to membership functions can be employed in T-S FMB nonlinear systems with input saturation to relax the estimation of DOA. In [28], non-PDC control law with parameter-dependent Lyapunov function was adopted to achieve this purpose. In addition, the pioneering work in [35]–[41] proposed the innovative

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membership-function-dependent (MFD) analysis methods for relaxing the stability analysis results using the membership-function information. Furthermore, [35]–[41] also advocated the significant imperfect premise matching (IPC) concept that the membership functions and/or number of rules between the fuzzy model and controller can be different. Inspired by it, the MFD analysis methods and IPC concept will be applied in this paper to relax the estimation of DOA of the positive nonlinear system with input saturation with the incorporation of membership function information in the analysis.

Related Works: MFD Analysis Methods

The essence of MFD analysis methods is to excavate the information of membership functions and introduce them into stability conditions. The methods of excavating information of membership functions can be roughly divided into four categories initialized and developed by H. K. Lam and his team [36], [42]. The first is to regard the membership functions and system states as symbolic variables [37], then slack matrices carrying the information of membership functions and state variables are introduced by using S-procedure technique [43]. The second is to use local or global boundary information of membership functions [35], [36], [42], [44]–[47]. The third is to use interpolation technique to approximate the original membership functions, then the approximation error and the information of sample points of the obtained approximated membership functions such as piecewise linear membership functions [38], Taylor series membership functions [41] are introduced into the stability conditions. The fourth is to approximate the original membership functions directly using staircase membership functions [39] or polynomials in every sub-domains [40], then the approximated membership functions and the approximation error are considered in stability analysis. Comparing the methods among these four categories, the first one is the simplest, but it has the weakest ability to relax results due to less information being introduced. Theoretically, the third and the fourth ones have the strongest ability to relax results, however, the fourth one is more difficult to find the appropriate polynomials to approximate the membership functions than the third one.

Proposed Approach

Considering the advantages of the above four categories methods, the improved Taylor-series-membership-functions-dependent (ITSMFD) method is proposed in this paper to relax the results by introducing more detailed information, namely the local approximation error information and the boundary information of the substate space of premise variables into the stability conditions and DOA estimation conditions.

When the ITSMFD method is used to analyze the stability of system, the LMI-based analysis method cannot be used in stability/positive analysis due to the existence of polynomial variables in resultant conditions. Instead, sum of squares (SOS) based analysis approach was then utilized to derive conditions in terms of SOS [48], whose feasible solution can be found numerically, e.g., using the third-party MATLAB toolbox SOSTOOLS [49]. In addition, considering the superior modeling capability of polynomial fuzzy model for nonlinear plants, polynomial fuzzy model instead of T-S fuzzy model is used to represent the positive nonlinear system in this paper.

Although the positive polynomial fuzzy-model-based (PPFMB) control system with input saturation is relatively less investigated, it is vital to the positive nonlinear control systems when input signal is limited by actuator saturation. Furthermore, MFD analysis methods have been shown to contribute to the relaxation of the stability region, so there is a strong incentive to use MFD analysis methods to expand the estimation of DOA of PPFMB control system with input saturation.

The contributions of this paper are listed as below:

1) IPC concept is adopted to design the polynomial fuzzy controller, which means that the membership functions and/or number of rules between the polynomial fuzzy model and controller are allowed to be different. Compared with matched premise membership functions concept, which is often called the parallel distributed compensation (PDC) concept, the IPC concept increases the flexibility of controller design, thereby offers higher potential to lower the implementation costs of controller.
2) The ITSMFD method is proposed, which not only has advantages of the Taylor-series-membership-functions-dependent (TSMFD) method mentioned in [41], but introduces further information, namely the local approximation error information and the boundary information of the substate space of premise variables into the stability conditions. Thus, the proposed ITSMFD method may lead to less conservatism than the existing MFD methods.
3) The proposed ITSMFD method is not only applied to the stability conditions, but also extended to the DOA estimation condition for the first time to relax the estimation of DOA of PPFMB control system with input saturation.

The organization of this paper is as follows. In Section II, the notations and the polynomial fuzzy model with input saturation, polynomial fuzzy controller are described. In Section III, the basic positive conditions, stability conditions and DOA estimation conditions are obtained. Moreover, these conditions with convex form are summarized in Theorem 1 and Corollary 1. In Section IV, Taylor-series-membership-function related work are given. Also, the ITSMFD method is proposed and used to obtain more relaxed analysis for PPFMB control system with and without input saturation. In Section V, two examples are used to illustrate the effectiveness of proposed method for PPFMB control system without and with input saturation, respectively. In Section VI, a conclusion is drawn.

II. Preliminary

A. Notation

The following notations are used throughout the paper. A monomial in $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is a function in the of form $x_1^{d_1}(t)x_2^{d_2}(t)\ldots x_n^{d_n}(t)$, where $d_n \geq 0, i \in$

$\{1,2,\ldots,n\}$ are nonnegative integers. The degree of a monomial is $d=\sum_{i=1}^{n} d_i$. A polynomial $\mathbf{f}(\mathbf{x}(t))$ is an SOS if there exist polynomials $\mathbf{f}_1(\mathbf{x}(t)),\mathbf{f}_2(\mathbf{x}(t)),\ldots,\mathbf{f}_m(\mathbf{x}(t))$ such that $\mathbf{f}(\mathbf{x}(t))=\sum_{i=1}^{m}\mathbf{f}_i^2(\mathbf{x}(t))$, where $\mathbf{f}_i(\mathbf{x}(t))$ is a polynomial and m is a nonnegative integer. It is clear that $\mathbf{f}(\mathbf{x}(t))$ being an SOS naturally implies $\mathbf{f}(\mathbf{x}(t))\geq 0$ for all $\mathbf{x}(t)\in\Re^n$. The expressions of $\mathbf{A}\prec 0$ and $\mathbf{A}\succ 0$, mean that all elements of $\mathbf{A}$ are negative and positive, respectively; $\mathbf{A}<0$ and $\mathbf{A}>0$ mean that $\mathbf{A}$ is negative definite and positive definite, respectively. $\mathbf{A}^{(\alpha,\beta)}$ is the α-th row, β-th column element of $\mathbf{A}$. $\mathbf{A}^{(:,\beta)}$ is a vector denoting the β-th column of $\mathbf{A}$. $\mathbf{A}^{(\alpha,:)}$ is a vector denoting the α-th row of $\mathbf{A}$. Matrix $\mathbf{Q}$ is called Metzler matrix [1], if its off diagonal elements are all nonnegative. $\underline{p}$ represents $\{1,2,\ldots,p\}$, where p is a nonzero integer. diag$\{\cdot\}$ denotes a square diagonal matrix with the elements of argument in the diagonal. sym$(X)=X+X^T$.

B. Polynomial Fuzzy Plant Model with Input Saturation

The nonlinear system with input saturation is described by a polynomial fuzzy model with p rules. The i^{th} rule is of the following format:

$$\text{Rule } i: \text{IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND} \cdots \text{AND } f_\psi(\mathbf{x}(t)) \text{ is } M_\psi^i,$$
$$\text{THEN } \dot{\mathbf{x}}(t)=\mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t)+\mathbf{B}_i(\mathbf{x}(t))\text{sat}(\mathbf{u}(t))$$

where $\mathbf{x}(t)\in\Re^n$ and $\mathbf{u}(t)\in\Re^m$ are the state vector and control input vector of the system, respectively; n, m are their dimensions, $f_\vartheta(\mathbf{x}(t))$ is the premise variable and M_ϑ^i is the fuzzy set corresponding to its premise variable in rule i, $i\in\underline{p}$, $\vartheta\in\underline{\psi}$, and ψ is a positive integer; $\mathbf{A}_i(\mathbf{x}(t))\in\Re^{n\times n}$, $\mathbf{B}_i(\mathbf{x}(t))\in\Re^{n\times m}$ are the known polynomial system matrices and input matrices, respectively. The function sat$(\cdot)$: $\Re^m\to\Re^m$ is the standard saturation function. It is defined as

$$\text{sat}(\mathbf{u}(t))=[\text{sat}(u^{(1,1)}(t)),\text{sat}(u^{(2,1)}(t)),\ldots,\text{sat}(u^{(m,1)}(t))]^T, \tag{1}$$

where

$$\text{sat}(u^{(\iota,1)}(t))=\begin{cases}u_{lim} & \text{if } \quad u^{(\iota,1)}(t)>u_{lim}\\ u^{(\iota,1)}(t) & \text{if } \quad -u_{lim}\leq u^{(\iota,1)}(t)\leq u_{lim}\\ -u_{lim} & \text{if } \quad u^{(\iota,1)}(t)<-u_{lim}\end{cases},$$

$u^{(\iota,1)}(t)$ is the ι^{th} element of $\mathbf{u}(t)$, $\iota\in\underline{m}$, $u_{lim}>0$ is the control input limit.

The dynamics of the nonlinear system is defined as follows:

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{p}w_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t)+\mathbf{B}_i(\mathbf{x}(t))\text{sat}(\mathbf{u}(t))), \tag{2}$$

where $w_i(\mathbf{x}(t))$ is the normalized grade of membership, $w_i(\mathbf{x}(t))=\prod_{\vartheta=1}^{\psi}\mu_{M_\vartheta^i}(f_\vartheta(\mathbf{x}(t)))/\sum_{k=1}^{p}\prod_{\vartheta=1}^{\psi}\mu_{M_\vartheta^k}(f_\vartheta(\mathbf{x}(t)))$, $w_i(\mathbf{x}(t))\geq 0$, and $\sum_{i=1}^{p}w_i(\mathbf{x}(t))=1$; $\mu_{M_\vartheta^i}(f_\vartheta(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy term M_ϑ^i.

Definition 1: The polynomial fuzzy system (2) is said to be positive only if for every nonnegative initial state, its state variables and outputs are all nonnegative.

Lemma 1: A polynomial fuzzy system (2) is guaranteed to be positive if $\sum_{i=1}^{p}w_i(\mathbf{x}(t))\mathbf{A}_i(\mathbf{x}(t))$ is a Metzler matrix; input matrices satisfy the conditions that $\sum_{i=1}^{p}w_i(\mathbf{x}(t))\mathbf{B}_i(\mathbf{x}(t))\succ 0$ when sat$(\mathbf{u}(t))$ is nonnegative.

C. Polynomial Fuzzy Controller

The IPC concept is adopted to design a polynomial fuzzy controller with c rules for the polynomial fuzzy system (2), the j^{th} rule of the polynomial fuzzy controller is as follows:

$$\text{Rule } j: \text{IF } g_1(\mathbf{x}(t)) \text{ is } N_1^j \text{ AND}\cdots\text{AND } g_\phi(\mathbf{x}(t)) \text{ is } N_\phi^j,$$
$$\text{THEN } \mathbf{u}(t)=\mathbf{G}_j(\mathbf{x}(t))\mathbf{x}(t)$$

where $g_\vartheta(\mathbf{x}(t))$ is the premise variable and N_ϑ^j is the fuzzy set corresponding to its premise variable in rule j, $j\in\underline{c}$, $\vartheta\in\underline{\phi}$, and ϕ is a positive integer; $\mathbf{G}_j(\mathbf{x}(t))\in\Re^{m\times n}$ is the polynomial fuzzy controller gain to be determined. The polynomial fuzzy controller is defined as follows:

$$\mathbf{u}(t)=\sum_{j=1}^{c}m_j(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))\mathbf{x}(t), \tag{3}$$

where $m_j(\mathbf{x}(t))=\prod_{\vartheta=1}^{\phi}\mu_{N_\vartheta^j}(g_\vartheta(\mathbf{x}(t)))/\sum_{k=1}^{c}\prod_{\vartheta=1}^{\phi}\mu_{N_\vartheta^k}(g_\vartheta(\mathbf{x}(t)))$, $m_j(\mathbf{x}(t))\geq 0$, and $\sum_{j=1}^{c}m_j(\mathbf{x}(t))=1$; $\mu_{N_\vartheta^j}(g_\vartheta(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy term N_ϑ^j.

D. Control Input Saturation

In order to handle the input saturation, the convex hull representation based on auxiliary polynomial fuzzy controller is adopted in this paper.

Let $\mathbf{E}$ be the set of $m\times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathbf{E}$. Suppose that each element of $\mathbf{E}$ is labeled as $\mathbf{E}_k$, $k\in\underline{2^m}$. Then, $\mathbf{E}=\{\mathbf{E}_k: k\in\underline{2^m}\}$. Denote $\mathbf{E}_k^-=\mathbf{I}-\mathbf{E}_k$, then $\mathbf{E}_k^-$ is also an element of $\mathbf{E}$ if $\mathbf{E}_k\in\mathbf{E}$.

For matrices $\mathbf{G}_j(\mathbf{x}(t))$ and $\mathbf{H}_j(\mathbf{x}(t))\in\Re^{m\times n}$, supposing that $|\mathbf{H}_j(\mathbf{x}(t))\mathbf{x}(t)|_\infty\leq u_{lim}$, then there exists $\eta_k(\mathbf{x}(t))$ satisfying $\eta_k(\mathbf{x}(t))\geq 0$ and $\sum_{k=1}^{2^m}\eta_k(\mathbf{x}(t))=1$ so that sat$(\mathbf{u}(t))$ can be represented as

$$\text{sat}(\mathbf{u}(t))=\text{sat}(\sum_{j=1}^{c}m_j(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))\mathbf{x}(t))=$$
$$\sum_{j=1}^{c}\sum_{k=1}^{2^m}m_j(\mathbf{x}(t))\eta_k(\mathbf{x}(t))[(\mathbf{E}_k\mathbf{G}_j(\mathbf{x}(t))+\mathbf{E}_k^-\mathbf{H}_j(\mathbf{x}(t)))\mathbf{x}(t)], \tag{4}$$

where the polynomial fuzzy controller gain $\mathbf{G}_j(\mathbf{x}(t))$ and the auxiliary polynomial fuzzy controller gain $\mathbf{H}_j(\mathbf{x}(t))$ are to be determined. $\mathbf{E}_k\mathbf{G}_j(\mathbf{x}(t))+\mathbf{E}_k^-\mathbf{H}_j(\mathbf{x}(t)),\forall k\in\underline{2^m}, j\in\underline{c}$ is the set of matrices formed by choosing some rows from $\mathbf{G}_j(\mathbf{x}(t))$ and the rest from $\mathbf{H}_j(\mathbf{x}(t))$.

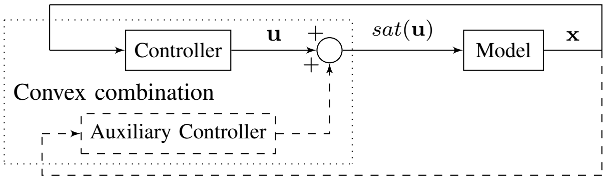


Fig. 1. A block diagram of PPFMB control system under two different situations. When no saturation is considered for control signal, the closed-loop system is the part indicated by the solid line. When input saturation is considered, the part indicated by the dashed line needs to be added to the closed-loop system, and the control signals generated by the controller and auxiliary controller are combined together.

III. Basic Stability Analysis

In the following analysis, for simplicity, the time t is dropped for the situation without ambiguity. From (2), (3) and (4), the closed-loop control system is obtained as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{k=1}^{2^m} w_i(\mathbf{x})m_j(\mathbf{x})\eta_k(\mathbf{x})\times [(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})(\mathbf{E}_k\mathbf{G}_j(\mathbf{x}) + \mathbf{E}_k^-\mathbf{H}_j(\mathbf{x})))\mathbf{x}]. \quad (5)$$

If there are no limits on control signal, the closed-loop control system is reduced to as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})[(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}))\mathbf{x}]. \quad (6)$$

In this paper, the positivity and stability of PPFMB control system will be investigated under two different situations. The first situation is that the input saturation is considered, the block diagram of the PPFMB control system in this case is represented by Fig. 1. In this case, the control objective is to guarantee that the PPFMB control system is asymptotically stable and positive with as large DOA as possible. In the second situation, where no saturation on control signal is considered, the block diagram of the PPFMB control system in this case is represented by the solid line portion of Fig. 1. The control objective of this case is to guarantee that the PPFMB control system is asymptotically stable and positive with as large stability region as possible. Since the second situation is a special case of the first, only the analysis process of the first situation will be shown in this paper and the analysis process of the second situation will be omitted. The analysis results under the first and second situations will be summarized in theorems and corollaries.

A. Stability Analysis

In order to perform stability analysis, a quadratic Lyapunov function candidate $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ is chosen, where $\mathbf{P}$ is a diagonal positive definite matrix. The time derivation of $V(\mathbf{x})$ is as follows:

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} = \sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{k=1}^{2^m} w_i(\mathbf{x})m_j(\mathbf{x})\eta_k(\mathbf{x})\mathbf{x}^T\boldsymbol{\Xi}_{ijk}(\mathbf{x})\mathbf{x} \quad (7)$$

where

$$\boldsymbol{\Xi}_{ijk}(\mathbf{x}) = \mathrm{sym}(\mathbf{P}(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})(\mathbf{E}_k\mathbf{G}_j(\mathbf{x}) + \mathbf{E}_k^-\mathbf{H}_j(\mathbf{x})))). \quad (8)$$

Thus, $\dot{V}(\mathbf{x}) < 0$ holds if $\boldsymbol{\Xi}_{ijk}(\mathbf{x}) < 0$, $\forall i \in \underline{p}; j \in \underline{c}; k \in \underline{2^m}$.

B. Positivity Analysis

The PPFMB control system (5) can be regarded as a PPFMB system without input matrices, with $\mathbf{A}_i + \mathbf{B}_i(\mathbf{E}_k\mathbf{G}_j(\mathbf{x}) + \mathbf{E}_k^-\mathbf{H}_j(\mathbf{x}))$ being the system matrix. Thus, the system (5) can be regarded as the specific case of system (2). According to Lemma 1, the positivity conditions of (5) are as follows:

$$\boldsymbol{\Theta}_{ijk}^{(\alpha,\beta)}(\mathbf{x}) > 0,\ \forall i \in \underline{p}, j \in \underline{c}, k \in \underline{2^m}, \alpha \neq \beta \in \underline{n}, \quad (9)$$

where

$$\boldsymbol{\Theta}_{ijk}^{(\alpha,\beta)}(\mathbf{x}) = \mathbf{A}_i^{(\alpha,\beta)}(\mathbf{x}) + \mathbf{B}_i^{(\alpha,:)}(\mathbf{x})(\mathbf{E}_k\mathbf{G}_j^{(:,\beta)}(\mathbf{x}) + \mathbf{E}_k^-\mathbf{H}_j^{(:,\beta)}(\mathbf{x})). \quad (10)$$

C. Estimation of Attraction Domain

Denote the state transition map by $\phi : (t, \mathbf{x}_0) \to \mathbf{x}(t)$, where $\mathbf{x}_0$ is the state vector of initial moment of this transfer map. The DOA of the equilibrium $\mathbf{x} = 0$ is defined by $\boldsymbol{S} := \{\mathbf{x}_0 \in \Re^n : \lim_{t\to\infty}\phi(t, \mathbf{x}_0) = 0\}$. Because there is no effective method to find the boundary of $\boldsymbol{S}$, attractive invariant set is used to estimate the DOA.

When $\dot{V}(\mathbf{x}) < 0$ and positivity conditions are satisfied, there exist some attractive invariant sets for PPFMB control system (5). However, since $\eta_k(\mathbf{x}(t)) \geq 0$, $\sum_{k=1}^{2^m}\eta_k(\mathbf{x}(t)) = 1$ and $|\mathbf{H}^{(\iota,:)}(\mathbf{x})\mathbf{x}| \leq u_{lim}, \forall \iota \in \underline{m}$ are necessary conditions for $\mathrm{sat}(\mathbf{u}(t))$ in (4) to satisfy (1), only the subset of $\boldsymbol{S}$ lying between $\mathbf{H}^{(\iota,:)}(\mathbf{x})\mathbf{x} = u_{lim}$ and $\mathbf{H}^{(\iota,:)}(\mathbf{x})\mathbf{x} = -u_{lim}$ can be detected. In order to detect the attractive invariant set that can be served as an estimation of DOA for the PPFMB control system, the level set of the Lyapunov function candidate $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ is defined as follows:

$$\boldsymbol{\varepsilon}(\mathbf{P}, 1) := \{\mathbf{x} \in \Re^n : \mathbf{x}^T\mathbf{P}\mathbf{x} \leq 1\}. \quad (11)$$

For the matrix $\mathbf{H}(\mathbf{x}) \in \Re^{m\times n}$, define

$$\boldsymbol{L}(\mathbf{H}(\mathbf{x})) := \{\mathbf{x} \in \Re^n : |\mathbf{H}^{(\iota,:)}(\mathbf{x})\mathbf{x}| \leq u_{lim}, \iota \in \underline{m}\}, \quad (12)$$

where $\mathbf{H}(\mathbf{x}) = \sum_{j=1}^{c} m_j(\mathbf{x})\mathbf{H}_j(\mathbf{x})$, $\mathbf{H}^{(\iota,:)}(\mathbf{x})$ is the ιth row of $\mathbf{H}(\mathbf{x})$.

The ellipsoid $\boldsymbol{\varepsilon}(\mathbf{P}, 1)$ can serve as an estimation of DOA for the PPFMB control system if $\boldsymbol{\varepsilon}(\mathbf{P}, 1) \subset \boldsymbol{L}(\mathbf{H}(\mathbf{x}))$, $\dot{V}(\mathbf{x}) < 0$ and positivity conditions can be satisfied.

Let $\boldsymbol{\chi}_R \subset \Re^n$ be a bounded convex set of some desired shape. We call it a shape reference set. Suppose that $0 \in \boldsymbol{\chi}_R$. For a positive real number α, denote $\alpha\boldsymbol{\chi}_R = \{\alpha\mathbf{x} : \mathbf{x} \in \boldsymbol{\chi}_R\}$.

For a set $\boldsymbol{S} \subset \Re^n$, define the size of $\boldsymbol{S}$ with respect to $\boldsymbol{\chi}_R$ as

$$\alpha_R(\boldsymbol{S}) := \sup\{\alpha > 0 : \alpha\boldsymbol{\chi}_R \subset \boldsymbol{S}\}. \quad (13)$$

If $\alpha_R(\boldsymbol{S}) \geq 1$, then $\boldsymbol{\chi}_R \subset \boldsymbol{S}$.

When estimating the DOA of PPFMB control system, we pick the largest one from all the $\boldsymbol{\varepsilon}(\mathbf{P},1)$'s that satisfies the set invariance conditions as the least conservative estimate of DOA, which is achieved by maximizing the value of $\alpha_R(\boldsymbol{\varepsilon}(\mathbf{P},1))$, where $\alpha_R(\boldsymbol{\varepsilon}(\mathbf{P},1))$ is the size of $\boldsymbol{\varepsilon}(\mathbf{P},1)$ with respect to the shape reference set $\boldsymbol{\chi}_R$. The least conservative estimation of the DOA can be obtained by solving the following conditions:

$$\begin{aligned}
&\sup \alpha \\
&s.t. 1)\ \alpha\boldsymbol{\chi}_R \subset \boldsymbol{\varepsilon}(\mathbf{P},1); \quad 2)\ \mathbf{P} > 0; \\
&\quad 3)\ \boldsymbol{\Theta}_{ijk}^{(\alpha,\beta)}(\mathbf{x}) > 0, \quad \forall i \in \underline{p}, j \in \underline{c}, k \in \underline{2^m}, \alpha \neq \beta; \\
&\quad 4)\ \boldsymbol{\Xi}_{ijk}(\mathbf{x}) < 0, \quad \forall i \in \underline{p}, j \in \underline{c}, k \in \underline{2^m}; \\
&\quad 5)\ \boldsymbol{\varepsilon}(\mathbf{P},1) \subset \boldsymbol{L}(\mathbf{H}(\mathbf{x}));
\end{aligned} \tag{14}$$

where $\boldsymbol{\Xi}_{ijk}(\mathbf{x})$ and $\boldsymbol{\Theta}_{ijk}^{(\alpha,\beta)}(\mathbf{x})$ are defined in (8) and (10).

Remark 1: Referring to (14), the sub-condition 1) is used to find the largest one from all $\boldsymbol{\varepsilon}(\mathbf{P},1)$'s that satisfies the sub-conditions 2) to 5) as the least conservative estimate of DOA. The sub-condition 2) guarantees $V(\mathbf{x}) > 0$. The sub-condition 3) is used to guarantee that the PPFMB system (5) is positive, which is proved in Subsection III-B. The sub-condition 4) guarantees $\dot{V}(\mathbf{x}) < 0$, which is proved in Subsection III-A. When sub-conditions 2) to 4) are satisfied, the sub-condition 5) guarantees $\boldsymbol{\varepsilon}(\mathbf{P},1)$ can be an estimation of DOA of the PPFMB system (5), because $|\mathbf{H}^{(\iota,:)}(\mathbf{x})\mathbf{x}| \leq u_{lim}, \forall \iota \in \underline{m}$ is the necessary condition for $\mathrm{sat}(\mathbf{u}(t))$ in (4) to satisfy (1).

D. Basic Analysis Results

Because the non-convex conditions (14) cannot be applied to obtain controller gains $\mathbf{G}_j(\mathbf{x})$ and auxiliary controller gains $\mathbf{H}_j(\mathbf{x})$ numerically, in this subsection, some transformation techniques are applied to obtain convex form of (14) which is shown in Theorem 1. The analysis results of PPFMB system without input saturation are summarized in Corollary 1.

Theorem 1: For the PPFMB control system (5), if there exist a diagonal positive definite matrix $\mathbf{X} \in \Re^{n\times n}$, polynomial matrices $\mathbf{D}_j(\mathbf{x}) \in \Re^{m\times n}$ and $\mathbf{O}_j(\mathbf{x}) \in \Re^{m\times n}$ $\forall j \in \underline{c}$, and scalar $\gamma > 0$ such that the following SOS-based conditions of optimization problem are satisfied:

$$\begin{aligned}
&\min \gamma \\
&s.t. 1)\ \nu^T \begin{bmatrix} \gamma & (x_0^s)^T \\ x_0^s & \mathbf{X} \end{bmatrix} \nu \text{ is SOS},\ \forall s \in \underline{l}; \\
&\quad 2)\ \nu^T(\mathbf{X} - \varepsilon_1\mathbf{I})\nu \text{ is SOS}; \\
&\quad 3)\ \nu^T(\tilde{\boldsymbol{\Theta}}_{ijk}^{(\alpha,\beta)}(\mathbf{x}) - \varepsilon_2(\mathbf{x}))\nu \text{ is SOS}, \\
&\qquad \forall i \in \underline{p}, j \in \underline{c}, k \in \underline{2^m}, \alpha \neq \beta; \\
&\quad 4)\ -\nu^T(\tilde{\boldsymbol{\Xi}}_{ijk}(\mathbf{x}) - \varepsilon_3(\mathbf{x})\mathbf{I})\nu \text{ is SOS},\ \forall i \in \underline{p}, j \in \underline{c}, k \in \underline{2^m}; \\
&\quad 5)\ \nu^T\boldsymbol{\Upsilon}_{j\iota}(\mathbf{x})\nu \text{ is SOS},\ \forall j \in \underline{c}, \iota \in \underline{m};
\end{aligned} \tag{15}$$

where ν is an arbitrary vector independent of $\mathbf{x}$ with appropriate dimensions; $\varepsilon_1 > 0$, $\varepsilon_2(\mathbf{x}) > 0$, $\varepsilon_3(\mathbf{x}) > 0$ are predefined scalar polynomials; $\tilde{\boldsymbol{\Theta}}_{ijk}^{(\alpha,\beta)}(\mathbf{x})$, $\tilde{\boldsymbol{\Xi}}_{ijk}(\mathbf{x})$ and $\boldsymbol{\Upsilon}_{j\iota}(\mathbf{x})$ are defined in (17), (18) and (19), then the system (5) is asymptotically stable and positive with initial conditions contained in $\boldsymbol{\varepsilon}(\mathbf{P},1)$, where $\boldsymbol{\varepsilon}(\mathbf{P},1)$ is an estimation of the DOA. The polynomial fuzzy controller gains can be obtained by $\mathbf{G}_j(\mathbf{x}) = \mathbf{D}_j(\mathbf{x})\mathbf{X}^{-1}$ and the auxiliary polynomial fuzzy controller gains can be obtained by $\mathbf{H}_j(\mathbf{x}) = \mathbf{O}_j(\mathbf{x})\mathbf{X}^{-1}$, $\mathbf{X} = \mathbf{P}^{-1}$.

Proof: (i) Referring to sub-condition 1) of (14), the reference set $\boldsymbol{\chi}_R$ is defined as polyhedron $\boldsymbol{\chi}_R := co\{\mathbf{x}_0^1, \mathbf{x}_0^2, \ldots, \mathbf{x}_0^l\}$, where $\mathbf{x}_0^1, \mathbf{x}_0^2, \ldots, \mathbf{x}_0^l$ are vertices of polyhedron and are also the given initial states, l is positive integer. $\alpha\boldsymbol{\chi}_R \subset \boldsymbol{\varepsilon}(\mathbf{P},1)$ means that the point $\alpha\mathbf{x}$ on the bound of $\alpha\boldsymbol{\chi}_R$ is inside the ellipse $\boldsymbol{\varepsilon}(\mathbf{P},1)$, where $\alpha\boldsymbol{\chi}_R := co\{\alpha\mathbf{x}_0^1, \alpha\mathbf{x}_0^2, \ldots, \alpha\mathbf{x}_0^l\}$, $\boldsymbol{\varepsilon}(\mathbf{P},1) := \{\mathbf{x} \in \Re^n : \mathbf{x}^T\mathbf{P}\mathbf{x} \leq 1\}$. By substituting boundary point of $\alpha\boldsymbol{\chi}_R$ into $\boldsymbol{\varepsilon}(\mathbf{P},1)$, $\alpha^2(\mathbf{x}_0^s)^T\mathbf{P}\mathbf{x}_0^s \leq 1$ is obtained as the equivalent condition for $\alpha\boldsymbol{\chi}_R \subset \boldsymbol{\varepsilon}(\mathbf{P},1)$, $s \in \underline{l}$. Then $\begin{bmatrix} \gamma & (\mathbf{x}_0^s)^T \\ \mathbf{x}_0^s & \mathbf{X} \end{bmatrix} \geq 0$ is obtained by using Schur complement, where $\gamma = \frac{1}{\alpha^2}$ and $\mathbf{X} = \mathbf{P}^{-1}$.

(ii) Define $\mathbf{D}_j(\mathbf{x}) = \mathbf{G}_j(\mathbf{x})\mathbf{X}$ and $\mathbf{O}_j(\mathbf{x}) = \mathbf{H}_j(\mathbf{x})\mathbf{X}$. Due to all elements of positive definite diagonal matrix $\mathbf{X}$ are positive, performing transformation to sub-condition 3) of (14) by post-multiplying $\mathbf{X}$ to both sides leads to $\boldsymbol{\Theta}_{ijk}^{(\alpha,\beta)}(\mathbf{x}) > 0$ if

$$\tilde{\boldsymbol{\Theta}}_{ijk}^{(\alpha,\beta)}(\mathbf{x}) > 0,\ \forall i \in \underline{p}, j \in \underline{c}, k \in \underline{2^m}, \alpha \neq \beta \in \underline{n}, \tag{16}$$

where

$$\begin{aligned}
\tilde{\boldsymbol{\Theta}}_{ijk}^{(\alpha,\beta)}(\mathbf{x}) = &\ \mathbf{A}_i^{(\alpha,\beta)}(\mathbf{x})\mathbf{X}^{(\beta,\beta)} \\
&+ \mathbf{B}_i^{(\alpha,:)}(\mathbf{x})(\mathbf{E}_k\mathbf{D}_j^{(:,\beta)}(\mathbf{x}) + \mathbf{E}_k^-\mathbf{O}_j^{(:,\beta)}(\mathbf{x})).
\end{aligned} \tag{17}$$

(iii) Performing congruence transformation to sub-condition 4) of (14) by pre-multiplying and post-multiplying $\mathbf{X}$ to both sides, we have

$$\begin{aligned}
&\tilde{\boldsymbol{\Xi}}_{ijk}(\mathbf{x}) = \\
&\mathrm{sym}(\mathbf{A}_i(\mathbf{x})\mathbf{X} + \mathbf{B}_i(\mathbf{x})(\mathbf{E}_k\mathbf{D}_j(\mathbf{x}) + \mathbf{E}_k^-\mathbf{O}_j(\mathbf{x}))) < 0.
\end{aligned} \tag{18}$$

Thus, $\dot{V}(t) < 0$ holds if $\tilde{\boldsymbol{\Xi}}_{ijk}(\mathbf{x}) < 0$ for $\forall i \in \underline{p}, j \in \underline{c}, k \in \underline{2^m}$.

(iv) Sub-condition 5) of (14) implies that all the curves $\mathbf{H}^{(\iota,:)}(\mathbf{x})\mathbf{x} = \pm u_{lim}$, where the vector $\mathbf{H}^{(\iota,:)}(\mathbf{x})$ is the ι-th row of $\mathbf{H}(\mathbf{x})$, $\iota \in \underline{m}$, lie completely outside of the ellipsoid $\boldsymbol{\varepsilon}(\mathbf{P},1)$, i.e., for each point on the curves $\mathbf{H}^{(\iota,:)}(\mathbf{x})\mathbf{x} = \pm u_{lim}$, we have $\min\{\mathbf{x}^T\mathbf{P}\mathbf{x} : \mathbf{H}^{(\iota,:)}(\mathbf{x})\mathbf{x} = u_{lim}\} \geq 1$. By using the Lagrange multiplier method, we obtain $\min\{\mathbf{x}^T\mathbf{P}\mathbf{x} : \mathbf{H}^{(\iota,:)}(\mathbf{x})\mathbf{x} = u_{lim}\} = u_{lim}^2(\mathbf{H}^{(\iota,:)}(\mathbf{x})\mathbf{P}^{-1}(\mathbf{H}^{(\iota,:)}(\mathbf{x}))^T)^{-1} \geq 1$. Therefore, sub-condition 5) is equivalent to $\mathbf{H}^{(\iota,:)}(\mathbf{x})\mathbf{P}^{-1}(\mathbf{H}^{(\iota,:)}(\mathbf{x}))^T \leq u_{lim}^2$, which is equivalent to $\mathbf{O}^{(\iota,:)}(\mathbf{x})\mathbf{P}(\mathbf{O}^{(\iota,:)}(\mathbf{x}))^T \leq u_{lim}^2$ with $\mathbf{H}(\mathbf{x}) = \mathbf{O}(\mathbf{x})\mathbf{P}^{-1}$. Applying Scur complement on $\mathbf{O}^{(\iota,:)}(\mathbf{x})\mathbf{P}(\mathbf{O}^{(\iota,:)}(\mathbf{x}))^T \leq u_{lim}^2$, we have

$$\boldsymbol{\Upsilon}_{j\iota}(\mathbf{x}) = \begin{bmatrix} u_{lim}^2 & \mathbf{O}_j^{(\iota,:)}(\mathbf{x}) \\ (\mathbf{O}_j^{(\iota,:)}(\mathbf{x}))^T & \mathbf{X} \end{bmatrix} \geq 0. \tag{19}$$

■

Corollary 1: For the PPFMB control system (6), if there exist a diagonal positive definite matrix $\mathbf{X} \in \Re^{n\times n}$, polynomial matrix $\mathbf{D}_j(\mathbf{x}) \in \Re^{m\times n}$ $\forall j \in \underline{c}$ such that the following SOS-based conditions are satisfied:

$$\nu^T(\mathbf{X} - \varepsilon_1\mathbf{I})\nu \text{ is SOS}; \tag{20}$$

$$\nu^T(\hat{\boldsymbol{\Theta}}_{ij}^{(\alpha,\beta)}(\mathbf{x})-\varepsilon_2(\mathbf{x}))\nu \text{ is SOS}, \ \forall i\in \underline{p}, j\in \underline{c}, \alpha\neq\beta; \tag{21}$$

$$-\nu^T(\hat{\boldsymbol{\Xi}}_{ij}(\mathbf{x})-\varepsilon_3(\mathbf{x})\mathbf{I})\nu \text{ is SOS}, \ \forall i\in \underline{p}, j\in \underline{c}; \tag{22}$$

where

$$\hat{\boldsymbol{\Theta}}_{ij}^{(\alpha,\beta)}(\mathbf{x})=\mathbf{A}_i^{(\alpha,\beta)}(\mathbf{x})\mathbf{X}^{(\beta,\beta)}+\mathbf{B}_i^{(\alpha,:)}(\mathbf{x})\mathbf{D}_j^{(:,\beta)}(\mathbf{x}); \tag{23}$$

$$\hat{\boldsymbol{\Xi}}_{ij}(\mathbf{x})=\text{sym}(\mathbf{A}_i(\mathbf{x})\mathbf{X}+\mathbf{B}_i(\mathbf{x})\mathbf{D}_j(\mathbf{x})); \tag{24}$$

ν is an arbitrary vector independent of $\mathbf{x}$ with appropriate dimensions; $\varepsilon_1>0$, $\varepsilon_2(\mathbf{x})>0$, $\varepsilon_3(\mathbf{x})>0$ are predefined scalar polynomials, then the system (6) is asymptotically stable and positive. The polynomial fuzzy controller gains can be obtained by $\mathbf{G}_j(\mathbf{x})=\mathbf{D}_j(\mathbf{x})\mathbf{X}^{-1}$, $\mathbf{X}=\mathbf{P}^{-1}$.

IV. Stability Analysis with Taylor series Membership Functions

As a tool for approximating membership functions, Taylor-series membership functions (TSMFs) are recalled in this section. And following the same line of [41], the TSMFD method will be applied to the stability conditions of PPFMB control system with input saturation. Then, this MFD method will be improved by introducing the local approximation error information and the boundary information of substate space of premise variables. Finally, this improved MFD method, namely ITSMFD method will be applied to the DOA estimation conditions. Meanwhile, the case that there no limits on the control signal is considered in this section, the analysis results based on the TSMFD method and ITSMFD method are shown in Corollary 2 and Corollary 3, respectively.

A. Taylor-Series Membership Functions

To facilitate the stability analysis, TSMFs are recalled in this subsection. The membership functions $f_{ij}(\mathbf{x})=w_i(\mathbf{x})m_j(\mathbf{x})$ are approximated by TSMFs $\tilde{f}_{ij}(\mathbf{x})$, which are as follows [36], [47]:

$$\tilde{f}_{ij}(\mathbf{x})=\sum_{\varsigma=1}^{\sigma}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n}\upsilon_{ri_r\varsigma}(x_r)\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x}), \tag{25}$$

where n is the number of state variables on which the membership functions depend. The overall state space $\boldsymbol{\Psi}$ is divided into σ connected substate spaces which are denoted as $\boldsymbol{\Psi}_\varsigma$, $\varsigma\in\underline{\sigma}$, $\sigma=\prod_{r=1}^{n}d_r$, d_r is the number of substate spaces of x_r. The predefined interpolation functions $\upsilon_{ri_r\varsigma}(x_r)$ are used to combine the local approximating functions $\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})$, where $0\le\upsilon_{ri_r\varsigma}(x_r)\le 1$ and $\upsilon_{r1\varsigma}(x_r)+\upsilon_{r2\varsigma}(x_r)=1$ for $r\in\underline{n}$, $i_r=\{1,2\}$, $\mathbf{x}\in\boldsymbol{\Psi}_\varsigma$; otherwise, $\upsilon_{ri_r\varsigma}(x_r)=0$. Substituting the endpoints of substate space as expansion points into the Taylor series expansion, we obtain $\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})$ as follows:

$$\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})=\sum_{\ell=0}^{\lambda-1}\frac{1}{\ell!}\left(\sum_{r=1}^{n}(x_r-x_{ri_r\varsigma})\frac{\partial}{\partial x_r}\right)^{\ell} \times f_{ij}(\mathbf{x})|_{(x_r=x_{ri_r\varsigma},r\in\underline{n},i_r\in\{1,2\},\varsigma\in\underline{\sigma})},$$
$$\forall i,j,i_1,i_2,\dots,i_n,\varsigma,\mathbf{x}\in\boldsymbol{\Psi}_\varsigma, \tag{26}$$

where $x_{ri_r\varsigma}$ is the endpoints when x_r falls into the substate space $\boldsymbol{\Psi}_\varsigma$. For example, when $i_r=1$ and $i_r=2$, it means that $x_{ri_r\varsigma}$ is the endpoint when x_r takes the minimum and maximum value in substate space $\boldsymbol{\Psi}_\varsigma$, respectively; $\frac{\partial}{\partial x_r}f_{ij}(\mathbf{x})|_{(x_r=x_{ri_r\varsigma},r\in\underline{n},i_r\in\{1,2\},\varsigma\in\underline{\sigma})}$ is a constant calculated by taking the partial derivative of $f_{ij}(\mathbf{x})$ with respective to x_r and then substituting x_r by $x_{ri_r\varsigma}$; λ is the predefined truncation order, which means that the highest order of the TSMFs is $\lambda-1$. The TSMFs reduce to the piecewise linear membership functions when $\lambda=1$.

B. MFD Stability Conditions with Global Approximation Error Information and Lower Bound of TSMFs

Considering the conservativeness of Theorem 1 caused by the lack of the information of membership functions $w_i(\mathbf{x})$ and $m_j(\mathbf{x})$, following the same line of [41], the TSMFD method is applied to relax the stability analysis. Based on TSMFs mentioned in Subsection IV-A, we denote $\Delta f_{ij}(\mathbf{x})=f_{ij}(\mathbf{x})-\tilde{f}_{ij}(\mathbf{x})$ as the approximation error where $\underline{\delta}_{ij}$ and $\overline{\delta}_{ij}$ are the minimum and maximum values of $\Delta f_{ij}(\mathbf{x})$ in global state space $\boldsymbol{\Psi}$, respectively, and $\underline{\beta}_{ij}$ as the minimal value of TSMFs $\tilde{f}_{ij}(\mathbf{x})$ in global state space $\boldsymbol{\Psi}$. In addition, we introduce the slack matrices $0\le\mathbf{Y}_{ijk}(\mathbf{x})=\mathbf{Y}_{ijk}^T(\mathbf{x})\in\Re^{n\times n}$ and $0\le\mathbf{W}_{ijk}(\mathbf{x})=\mathbf{W}_{ijk}^T(\mathbf{x})\in\Re^{n\times n}$, where $\mathbf{Y}_{ijk}(\mathbf{x})\ge\tilde{\boldsymbol{\Xi}}_{ijk}(\mathbf{x})$. Then, it follows from (7) that

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{k=1}^{2^m}w_i(\mathbf{x})m_j(\mathbf{x})\eta_k(\mathbf{x})\tilde{\boldsymbol{\Xi}}_{ijk}(\mathbf{x})\\
&\le\sum_{k=1}^{2^m}\eta_k(\mathbf{x})\sum_{i=1}^{p}\sum_{j=1}^{c}[(\tilde{f}_{ij}(\mathbf{x})+\underline{\delta}_{ij})\tilde{\boldsymbol{\Xi}}_{ijk}(\mathbf{x})\\
&\quad+(\overline{\delta}_{ij}-\underline{\delta}_{ij})\mathbf{Y}_{ijk}(\mathbf{x})+(\tilde{f}_{ij}(\mathbf{x})-\underline{\beta}_{ij})\mathbf{W}_{ijk}(\mathbf{x})]\\
&=\sum_{k=1}^{2^m}\eta_k(\mathbf{x})\sum_{\varsigma=1}^{\sigma}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n}\upsilon_{ri_r\varsigma}(x_r)\times\\
&\quad\sum_{i=1}^{p}\sum_{j=1}^{c}[(\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})+\underline{\delta}_{ij})\tilde{\boldsymbol{\Xi}}_{ijk}(\mathbf{x})+(\overline{\delta}_{ij}-\underline{\delta}_{ij})\mathbf{Y}_{ijk}(\mathbf{x})\\
&\quad+(\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})-\underline{\beta}_{ij})\mathbf{W}_{ijk}(\mathbf{x})]<0.
\end{aligned}\tag{27}$$

Due to $\eta_k(\mathbf{x})\ge0$, $\upsilon_{ri_r\varsigma}(x_r)\ge0$ are independent of the rule i, j, and $\sum_{\varsigma=1}^{\eta}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n}\upsilon_{ri_r\varsigma}(x_r)=1$, $\dot{V}(\mathbf{x})<0$ holds if $\sum_{i=1}^{p}\sum_{j=1}^{c}[(\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})+\underline{\delta}_{ij})\tilde{\boldsymbol{\Xi}}_{ijk}(\mathbf{x})+(\overline{\delta}_{ij}-\underline{\delta}_{ij})\mathbf{Y}_{ijk}(\mathbf{x})+(\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})-\underline{\beta}_{ij})\mathbf{W}_{ijk}(\mathbf{x})]<0$ for all $r,i_1,i_2,\dots,i_n,\varsigma,k$.

Theorem 2: For the PPFMB control system (5), if there exist a diagonal positive definite matrix $\mathbf{X}\in\Re^{n\times n}$, polynomial matrices $\mathbf{D}_j(\mathbf{x})\in\Re^{m\times n}$ and $\mathbf{O}_j(\mathbf{x})\in\Re^{m\times n}$, polynomial symmetric matrices $\mathbf{Y}_{ijk}(\mathbf{x})\in\Re^{n\times n}$ and $\mathbf{W}_{ijk}(\mathbf{x})\in\Re^{n\times n}$ $\forall i\in\underline{p}$, $j\in\underline{c}$, $k\in\underline{2^m}$, and scalar $\gamma>0$ such that the following SOS-based conditions of optimization problem are satisfied:

$\min\gamma$

$s.t.$ 1) Sub-equations $1)-3)$ of (15);

2) $\nu^T(\mathbf{Y}_{ijk}(\mathbf{x})-\varepsilon_3(\mathbf{x})\mathbf{I})\nu$ is SOS, $\forall i\in\underline{p}, j\in\underline{c}, k\in\underline{2^m}$;

3) $\nu^T(\mathbf{Y}_{ijk}(\mathbf{x}) - \tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x}) - \varepsilon_4(\mathbf{x})\mathbf{I})\nu$ is SOS, $\forall i \in \underline{p}, j \in \underline{c}, k \in \underline{2^m}$;

4) $\nu^T(\mathbf{W}_{ijk}(\mathbf{x}) - \varepsilon_5(\mathbf{x})\mathbf{I})\nu$ is SOS, $\forall i \in \underline{p}, j \in \underline{c}, k \in \underline{2^m}$;

5) $-\nu^T\{\sum_{i=1}^{p}\sum_{j=1}^{c}[(\chi_{iji_1i_2\ldots i_n\varsigma}(\mathbf{x}) + \underline{\delta}_{ij})\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x}) + (\bar{\delta}_{ij} - \underline{\delta}_{ij})\mathbf{Y}_{ijk}(\mathbf{x}) + (\chi_{iji_1i_2\ldots i_n\varsigma}(\mathbf{x}) - \underline{\beta}_{ij})\mathbf{W}_{ijk}(\mathbf{x})] + \varepsilon_6(\mathbf{x})\mathbf{I}\}\nu$ is SOS, $\forall i \in \underline{p}, j \in \underline{c}, k \in \underline{2^m}, \varsigma \in \underline{\sigma}$, $i_1, i_2, \ldots, i_n = \{1, 2\}$;

6) $\nu^T\mathbf{\Upsilon}_{j\iota}(\mathbf{x})\nu$ is SOS, $\forall j \in \underline{c}, \iota \in \underline{m}$; (28)

where ν is an arbitrary vector independent of $\mathbf{x}$; $\varepsilon_1 > 0$, $\varepsilon_2(\mathbf{x}) > 0, \ldots, \varepsilon_6(\mathbf{x}) > 0$ are predefined scalar polynomials; $\tilde{\mathbf{\Theta}}_{ijk}^{(\alpha,\beta)}(\mathbf{x})$, $\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x})$ and $\mathbf{\Upsilon}_{j\iota}(\mathbf{x})$ are defined in (17), (18) and (19), then the system (5) is asymptotically stable and positive with initial conditions contained in $\boldsymbol{\varepsilon}(\mathbf{P}, 1)$, where $\boldsymbol{\varepsilon}(\mathbf{P}, 1)$ is an estimation of the DOA. The polynomial fuzzy controller gains can be obtained by $\mathbf{G}_j(\mathbf{x}) = \mathbf{D}_j(\mathbf{x})\mathbf{X}^{-1}$ and the auxiliary polynomial fuzzy controller gains can be obtained by $\mathbf{H}_j(\mathbf{x}) = \mathbf{O}_j(\mathbf{x})\mathbf{X}^{-1}$, $\mathbf{X} = \mathbf{P}^{-1}$.

Corollary 2: For the PPFMB control system (6), if there exist a diagonal positive definite matrix $\mathbf{X} \in \Re^{n\times n}$, polynomial matrix $\mathbf{D}_j(\mathbf{x}) \in \Re^{m\times n}$, polynomial symmetric matrices $\hat{\mathbf{Y}}_{ij}(\mathbf{x}) \in \Re^{n\times n}$ and $\hat{\mathbf{W}}_{ij}(\mathbf{x}) \in \Re^{n\times n}$ $\forall i \in \underline{p}$, $j \in \underline{c}$ such that (20), (21) and the following SOS-based conditions are satisfied:

$$\nu^T(\hat{\mathbf{Y}}_{ij}(\mathbf{x}) - \varepsilon_3(\mathbf{x})\mathbf{I})\nu \text{ is SOS}, \ \forall i \in \underline{p}, j \in \underline{c}; \quad (29)$$

$$\nu^T(\hat{\mathbf{Y}}_{ij}(\mathbf{x}) - \hat{\mathbf{\Xi}}_{ij}(\mathbf{x}) - \varepsilon_4(\mathbf{x})\mathbf{I})\nu \text{ is SOS}, \ \forall i \in \underline{p}, j \in \underline{c}; \quad (30)$$

$$\nu^T(\hat{\mathbf{W}}_{ij}(\mathbf{x}) - \varepsilon_5(\mathbf{x})\mathbf{I})\nu \text{ is SOS}, \ \forall i \in \underline{p}, j \in \underline{c}; \quad (31)$$

$$-\nu^T\{\sum_{i=1}^{p}\sum_{j=1}^{c}[(\chi_{iji_1i_2\ldots i_n\varsigma}(\mathbf{x}) + \underline{\delta}_{ij})\hat{\mathbf{\Xi}}_{ij}(\mathbf{x}) + (\bar{\delta}_{ij} - \underline{\delta}_{ij})\hat{\mathbf{Y}}_{ij}(\mathbf{x}) + (\chi_{iji_1i_2\ldots i_n\varsigma}(\mathbf{x}) - \underline{\beta}_{ij})\hat{\mathbf{W}}_{ij}(\mathbf{x})] + \varepsilon_6(\mathbf{x})\mathbf{I}\}\nu \text{ is SOS}, \ \forall i \in \underline{p}, j \in \underline{c}, \varsigma \in \underline{\sigma}, i_1, i_2, \ldots, i_n = \{1, 2\}; \quad (32)$$

where ν is an arbitrary vector independent of $\mathbf{x}$; $\varepsilon_1 > 0$, $\varepsilon_2(\mathbf{x}) > 0, \ldots, \varepsilon_6(\mathbf{x}) > 0$ are predefined scalar polynomials; $\hat{\mathbf{\Theta}}_{ij}^{(\alpha,\beta)}(\mathbf{x})$ and $\hat{\mathbf{\Xi}}_{ij}(\mathbf{x})$ are defined in (23) and (24), respectively, then the system (6) is asymptotically stable and positive. The polynomial fuzzy controller gains can be obtained by $\mathbf{G}_j(\mathbf{x}) = \mathbf{D}_j(\mathbf{x})\mathbf{X}^{-1}$, $\mathbf{X} = \mathbf{P}^{-1}$.

C. MFD Stability Conditions with Local Approximation Error Information and Boundary Information of Substate Space of Premise Variables

In this subsection, the TSMFD analysis method used in Subsection IV-B is improved by introducing the local approximation error information and the boundary information of substate space of premise variables into the stability conditions.

In Subsection IV-A, all $\upsilon_{ri_r\varsigma}(x_r)$ exhibit the properties: $0 \le \upsilon_{ri_r\varsigma}(x_r) \le 1$ and $\upsilon_{r1\varsigma}(x_r) + \upsilon_{r2\varsigma}(x_r) = 1$ for $r \in \underline{n}$, $i_r \in \{1, 2\}$, $\mathbf{x} \in \mathbf{\Psi}_\varsigma, \varsigma \in \underline{\sigma}$; otherwise, $\upsilon_{ri_r\varsigma}(x_r) = 0$. In order to facilitate the introduction of local approximation error information, $\upsilon_{ri_r\varsigma}(x_r)$'s nature that distinguishing different substate spaces is transferred into variable $\varphi_\varsigma(\mathbf{x})$, which has the following properties [36], [47]:

$$\begin{cases} \varphi_\varsigma(\mathbf{x}) = 1 & \forall \mathbf{x} \in \mathbf{\Psi}_\varsigma, \varsigma \in \underline{\sigma} \\ \varphi_\varsigma(\mathbf{x}) = 0 & \forall \mathbf{x} \notin \mathbf{\Psi}_\varsigma, \varsigma \in \underline{\sigma} \end{cases}. \quad (33)$$

Then, $\upsilon_{ri_r\varsigma}(x_r) = \varphi_\varsigma(\mathbf{x})\hat{\upsilon}_{ri_r\varsigma}(x_r)$ are defined, where $0 \le \hat{\upsilon}_{ri_r\varsigma}(x_r) \le 1$ and $\hat{\upsilon}_{r1\varsigma}(x_r) + \hat{\upsilon}_{r2\varsigma}(x_r) = 1$ for $r \in \underline{n}$, $i_r \in \{1, 2\}$; otherwise, $\hat{\upsilon}_{ri_r\varsigma}(x_r) = 0$.

The TSMFs are defined as

$$\tilde{f}_{ij}(\mathbf{x}) = \sum_{\varsigma=1}^{\sigma}\varphi_\varsigma(\mathbf{x})\tilde{f}_{ij\varsigma}(\mathbf{x}), \quad (34)$$

where

$$\tilde{f}_{ij\varsigma}(\mathbf{x}) = \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n}\hat{\upsilon}_{ri_r\varsigma}(x_r)\chi_{iji_1i_2\ldots i_n\varsigma}(\mathbf{x}), \quad (35)$$

$\chi_{iji_1i_2\ldots i_n\varsigma}(\mathbf{x})$ are defined in (26).

Different from the TSMFD method in the Subsection IV-B, we conduct the stability analysis for every single substate space $\mathbf{\Psi}_\varsigma$, $\varsigma \in \underline{\sigma}$ separately by activating every $\varphi_\varsigma(\mathbf{x})$ in the corresponding substate space $\mathbf{\Psi}_\varsigma$. Defining the approximation error in the substate space $\mathbf{\Psi}_\varsigma$ as $\Delta f_{ij\varsigma} = f_{ij\varsigma}(\mathbf{x}) - \tilde{f}_{ij\varsigma}(\mathbf{x})$, where $f_{ij\varsigma}(\mathbf{x}) = w_i(\mathbf{x})m_j(\mathbf{x})$ in the substate space $\mathbf{\Psi}_\varsigma$, the lower and upper bounds of $\Delta f_{ij\varsigma}$ are denoted as $\underline{\delta}_{ij\varsigma}$ and $\bar{\delta}_{ij\varsigma}$, respectively. In addition, defining $0 \le \mathbf{Y}_{ijk}(\mathbf{x}) = \mathbf{Y}_{ijk}(\mathbf{x})^T \in \Re^{n\times n}$ and $\mathbf{Y}_{ijk}(\mathbf{x}) \ge \tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x})$. Then, we have

$$\begin{aligned} &\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{k=1}^{2^m} w_i(\mathbf{x})m_j(\mathbf{x})\eta_k(\mathbf{x})\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x}) \\ &= \sum_{\varsigma=1}^{\sigma}\varphi_\varsigma(\mathbf{x})\sum_{k=1}^{2^m}\eta_k(\mathbf{x})\sum_{i=1}^{p}\sum_{j=1}^{c}((\tilde{f}_{ij\varsigma}(\mathbf{x}) + \underline{\delta}_{ij\varsigma})\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x}) \\ &\quad + (f_{ij\varsigma}(\mathbf{x}) - \tilde{f}_{ij\varsigma}(\mathbf{x}) - \underline{\delta}_{ij\varsigma})\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x})) \\ &\le \sum_{\varsigma=1}^{\sigma}\varphi_\varsigma(\mathbf{x})\sum_{k=1}^{2^m}\eta_k(\mathbf{x})\sum_{i=1}^{p}\sum_{j=1}^{c}[(\tilde{f}_{ij\varsigma}(\mathbf{x}) + \underline{\delta}_{ij\varsigma})\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x}) \\ &\quad + (\bar{\delta}_{ij\varsigma} - \underline{\delta}_{ij\varsigma})\mathbf{Y}_{ijk}(\mathbf{x})]. \end{aligned} \quad (36)$$

In order to distinguish different substate spaces, $\xi_\varsigma(\mathbf{x}) = \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n}\hat{\upsilon}_{ri_r\varsigma}(x_r)(x_r - x_{r\varsigma min})(x_{r\varsigma max} - x_r)$ is defined for $\mathbf{\Psi}_\varsigma$, $\forall\varsigma \in \underline{\sigma}$, where $x_{r\varsigma min}$ and $x_{r\varsigma max}$ are the minimal and maximal value of the system state x_r in substate space $\mathbf{\Psi}_\varsigma$, respectively. The properties of $\xi_\varsigma(\mathbf{x})$ are as follows:

$$\begin{cases} \xi_\varsigma(\mathbf{x}) \ge 0, & \forall \mathbf{x} \in \mathbf{\Psi}_\varsigma, \varsigma \in \underline{\sigma} \\ \xi_\varsigma(\mathbf{x}) < 0, & \forall \mathbf{x} \notin \mathbf{\Psi}_\varsigma, \varsigma \in \underline{\sigma}, \end{cases} \quad (37)$$

The approximated membership functions (26) and (35) are introduced into (36) to replace $\tilde{f}_{ij\varsigma}(\mathbf{x})$. Define slack matrix $\mathbf{\Omega}_{k\varsigma}(\mathbf{x}) \in \Re^n$, which satisfies $\mathbf{\Omega}_{k\varsigma}(\mathbf{x}) = \mathbf{\Omega}_{k\varsigma}^T(\mathbf{x}) \ge 0$. From (37), we can get $\xi_\varsigma(\mathbf{x})\mathbf{\Omega}_{k\varsigma}(\mathbf{x}) \ge 0$ in substate space $\mathbf{\Psi}_\varsigma$. Then, based on S-procedure concepts [43], (36) is followed by

$$\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{k=1}^{2^m} w_i(\mathbf{x})m_j(\mathbf{x})\eta_k(\mathbf{x})\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x}) \le \sum_{\varsigma=1}^{\sigma}\varphi_\varsigma(\mathbf{x})\sum_{k=1}^{2^m}\eta_k(\mathbf{x})$$

$$\times \{\sum_{i=1}^{p}\sum_{j=1}^{c}[(\sum_{i_1}^{2}\sum_{i_2}^{2}\cdots\sum_{i_n}^{2}\prod_{r=1}^{n}\hat{\upsilon}_{ri_r\varsigma}(x_r)\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})$$
$$+\underline{\delta}_{ij\varsigma})\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x})+(\bar{\delta}_{ij\varsigma}-\underline{\delta}_{ij\varsigma})\mathbf{Y}_{ijk}(\mathbf{x})]+\xi_\varsigma(\mathbf{x})\mathbf{\Omega}_{k\varsigma}(\mathbf{x})\}. \quad (38)$$

Due to $\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n}\hat{\upsilon}_{ri_r\varsigma}(x_r)=1$ in the substate space $\mathbf{\Psi}_\varsigma$, and $\hat{\upsilon}_{ri_r\varsigma}(x_r)$ are independent of rule i, j, k, defining $\hat{\xi}_\varsigma(\mathbf{x})=\prod_{r=1}^{n}(x_r-x_{r\varsigma min})(x_{r\varsigma max}-x_r)$, whose properties are the same as that of $\xi_\varsigma(\mathbf{x})$, then, $\dot{V}(\mathbf{x})<0$ is achieved if

$$\sum_{\varsigma=1}^{\sigma}\varphi_\varsigma(\mathbf{x})\sum_{k=1}^{2^m}\eta_k(\mathbf{x})\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n}\hat{\upsilon}_{ri_r\varsigma}(x_r)\{\sum_{i=1}^{p}\sum_{j=1}^{c}$$
$$[(\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})+\underline{\delta}_{ij\varsigma})\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x})+(\bar{\delta}_{ij\varsigma}-\underline{\delta}_{ij\varsigma})\mathbf{Y}_{ijk}(\mathbf{x})]$$
$$+\hat{\xi}_\varsigma(\mathbf{x})\mathbf{\Omega}_{k\varsigma}(\mathbf{x})\}<0. \quad (39)$$

Since $\hat{\upsilon}_{ri_r\varsigma}(x_r)\geq 0$ and $\eta_k(\mathbf{x})\geq 0$ for $\forall r\in\underline{n}, i_r\in\{1,2\}, \varsigma\in\underline{\sigma}, k\in\underline{2^m}$, where η_k are unknown functions related to saturation, $\dot{V}(\mathbf{x})<0$ is achieved if $\sum_{i=1}^{p}\sum_{j=1}^{c}[(\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})+\underline{\delta}_{ij\varsigma})\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x})+(\bar{\delta}_{ij\varsigma}-\underline{\delta}_{ij\varsigma})\mathbf{Y}_{ijk}(\mathbf{x})]+\xi_\varsigma\hat{(}\mathbf{x})\mathbf{\Omega}_{k\varsigma}(\mathbf{x})<0$ is satisfied, $\forall k\in\underline{\eta}$, $\mathbf{x}\in\mathbf{\Psi}_\varsigma, \varsigma\in\underline{\sigma}$.

Theorem 3: For the positive polynomial fuzzy system (5), if there exist positive definite matrix $\mathbf{X}\in\Re^{n\times n}$, polynomial matrices $\mathbf{D}_j(\mathbf{x})\in\Re^{m\times n}$ and $\mathbf{O}_j(\mathbf{x})\in\Re^{m\times n}$, polynomial symmetric matrices $\mathbf{Y}_{ijk}(\mathbf{x})\in\Re^{n\times n}$ and $\mathbf{\Omega}_{k\varsigma}(\mathbf{x})\in\Re^{n\times n}$ $\forall i\in\underline{p}$, $j\in\underline{c}$, $k\in\underline{2^m}$, $\varsigma\in\underline{\sigma}$, and scalar $\gamma>0$ such that the following SOS-based conditions of optimization problem are satisfied:

$$\min\gamma$$

$s.t.1)$ Sub-equations $1)-3)$ of (28);

2) $\nu^T(\mathbf{\Omega}_{k\varsigma}(\mathbf{x})-\varepsilon_5(\mathbf{x})\mathbf{I})\nu$ is SOS, $\forall k\in\underline{2^m},\varsigma\in\underline{\sigma}$;

3) $-\nu^T\{\sum_{i=1}^{p}\sum_{j=1}^{c}[(\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})+\underline{\delta}_{ij\varsigma})\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x})+(\bar{\delta}_{ij\varsigma}-\underline{\delta}_{ij\varsigma})\mathbf{Y}_{ijk}(\mathbf{x})]+\hat{\xi}_\varsigma(\mathbf{x})\mathbf{\Omega}_{k\varsigma}(\mathbf{x})+\varepsilon_6(\mathbf{x})\mathbf{I}\}\nu$ is SOS, $\forall i\in\underline{p}, j\in\underline{c}, k\in\underline{2^m}, \varsigma\in\underline{\sigma}, \mathbf{x}\in\mathbf{\Psi}_k, i_1,i_2,\dots,i_n\in\{1,2\}$;

4) $\nu^T\mathbf{\Upsilon}_{j\iota}(\mathbf{x})\nu>0$, $\forall j\in\underline{c},\iota\in\underline{m}$; (40)

where ν is an arbitrary vector independent of $\mathbf{x}$; $\varepsilon_1>0$, $\varepsilon_2(\mathbf{x})>0,\dots,\varepsilon_6(\mathbf{x})>0$ are predefined scalar polynomials; $\tilde{\mathbf{\Theta}}_{ijk}^{(\alpha,\beta)}(\mathbf{x})$, $\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x})$ and $\mathbf{\Upsilon}_{j\iota}(\mathbf{x})$ are defined in (17), (18) and (19), then the system (5) is asymptotically stable and positive with initial conditions contained in $\boldsymbol{\varepsilon}(\mathbf{P},1)$, where $\boldsymbol{\varepsilon}(\mathbf{P},1)$ is an estimation of the DOA. The polynomial fuzzy controller gains can be obtained by $\mathbf{G}_j(\mathbf{x})=\mathbf{D}_j(\mathbf{x})\mathbf{X}^{-1}$ and the auxiliary polynomial fuzzy controller gains can be obtained by $\mathbf{H}_j(\mathbf{x})=\mathbf{O}_j(\mathbf{x})\mathbf{X}^{-1}$, $\mathbf{X}=\mathbf{P}^{-1}$.

Corollary 3: For the PPFMB control system (6), if there exist positive definite matrix $\mathbf{X}\in\Re^{n\times n}$, polynomial matrices $\mathbf{D}_j(\mathbf{x})\in\Re^{m\times n}$ and $\hat{\mathbf{Y}}_{ij}(\mathbf{x})\in\Re^{n\times n}$, $\hat{\mathbf{\Omega}}_\varsigma(\mathbf{x})\in\Re^{n\times n}$ $\forall i\in\underline{p}$, $j\in\underline{c}$, $\varsigma\in\underline{\sigma}$ such that (20), (21), (29), (30), and the following SOS-based conditions are satisfied:

$$\nu^T(\hat{\mathbf{\Omega}}_\varsigma(\mathbf{x})-\varepsilon_5(\mathbf{x})\mathbf{I})\nu \text{ is SOS}, \ \forall\varsigma\in\underline{\sigma}; \quad (41)$$

$$-\nu^T\{\sum_{i=1}^{p}\sum_{j=1}^{c}[(\chi_{iji_1i_2\dots i_n\varsigma}(\mathbf{x})+\underline{\delta}_{ij\varsigma})\hat{\mathbf{\Xi}}_{ij}(\mathbf{x})+$$
$$(\bar{\delta}_{ij\varsigma}-\underline{\delta}_{ij\varsigma})\hat{\mathbf{Y}}_{ij}(\mathbf{x})]+\hat{\xi}_\varsigma(\mathbf{x})\hat{\mathbf{\Omega}}_\varsigma(\mathbf{x})+\varepsilon_6(\mathbf{x})\mathbf{I}\}\nu \text{ is SOS},$$
$$\forall i\in\underline{p}, j\in\underline{c}, \varsigma\in\underline{\sigma}, \mathbf{x}\in\mathbf{\Psi}_\varsigma, i_1,i_2,\dots,i_n\in\{1,2\}; \quad (42)$$

where ν is an arbitrary vector independent of $\mathbf{x}$; $\varepsilon_1>0$, $\varepsilon_2(\mathbf{x})>0$, …, $\varepsilon_6(\mathbf{x})>0$ are predefined scalar polynomials; $\hat{\mathbf{\Theta}}_{ij}^{(\alpha,\beta)}(\mathbf{x})$ and $\hat{\mathbf{\Xi}}_{ij}(\mathbf{x})$ are defined in (23) and (24), respectively, then system (6) is asymptotically stable and positive. The polynomial fuzzy controller gains can be obtained by $\mathbf{G}_j(\mathbf{x})=\mathbf{D}_j(\mathbf{x})\mathbf{X}^{-1}$, $\mathbf{X}=\mathbf{P}^{-1}$.

D. DOA Estimation Condition with Local Error and Boundary Information of Substate Space of Premise Variables

The condition $\boldsymbol{\varepsilon}(\mathbf{P},1)\subset\boldsymbol{L}(\mathbf{H}(\mathbf{x}))$ is hold if

$$\sum_{j=1}^{c}m_j(\mathbf{x})\mathbf{\Upsilon}_{j\iota}(\mathbf{x})\geq 0, \ \forall\iota\in\underline{m}. \quad (43)$$

The membership functions $m_j(\mathbf{x})$ are approximated by TSMFs $\tilde{m}_j(\mathbf{x})$, $\tilde{m}_j(\mathbf{x})=\sum_{\varsigma=1}^{\sigma}\varphi_\varsigma(\mathbf{x})\tilde{m}_{j\varsigma}(\mathbf{x})$, where $\tilde{m}_{j\varsigma}(\mathbf{x})=\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n}\bar{\upsilon}_{ri_r\varsigma}(x_r)\bar{\chi}_{ji_1i_2\dots i_n\varsigma}(\mathbf{x})$. Denote $\Delta m_{j\varsigma}(\mathbf{x})=m_{j\varsigma}(\mathbf{x})-\tilde{m}_{j\varsigma}(\mathbf{x})$ as the approximation error in substate spaces $\mathbf{\Psi}_\varsigma$, where $m_{j\varsigma}(\mathbf{x})$ are the sub-membership functions of $m_j(\mathbf{x})$ in the substate space $\mathbf{\Psi}_\varsigma$, $\underline{\varrho}_{j\varsigma}$ and $\bar{\varrho}_{j\varsigma}$ are the minimal value and maximal value of $\Delta m_{j\varsigma}(\mathbf{x})$, respectively. Similar to Subsection IV-C, defining $0\leq\breve{\mathbf{Y}}_{j\iota}(\mathbf{x})=\breve{\mathbf{Y}}_{j\iota}^T(\mathbf{x})\in\Re^{n\times n}$ and $\breve{\mathbf{Y}}_{j\iota}\geq\mathbf{\Upsilon}_{j\iota}(\mathbf{x})$, the slack matrices $\breve{\mathbf{\Omega}}_{\iota\varsigma}(\mathbf{x})$ satisfy $\breve{\mathbf{\Omega}}_{\iota\varsigma}(\mathbf{x})=\breve{\mathbf{\Omega}}_{\iota\varsigma}^T(\mathbf{x})\geq 0$. Then, for $\forall\iota\in\underline{m}$, we have

$$\sum_{j=1}^{c}m_j(\mathbf{x})\mathbf{\Upsilon}_{j\iota}(\mathbf{x})$$
$$=\sum_{\varsigma=1}^{\sigma}\varphi_\varsigma(\mathbf{x})\sum_{j=1}^{c}[(\tilde{m}_{j\varsigma}(\mathbf{x})+\bar{\varrho}_{j\varsigma})+(m_{j\varsigma}(\mathbf{x})$$
$$-\tilde{m}_{j\varsigma}(\mathbf{x})-\bar{\varrho}_{j\varsigma})]\mathbf{\Upsilon}_{j\iota}(\mathbf{x})$$
$$\geq\sum_{\varsigma=1}^{\sigma}\varphi_\varsigma(\mathbf{x})\sum_{j=1}^{c}[(\tilde{m}_{j\varsigma}(\mathbf{x})+\bar{\varrho}_{j\varsigma})\mathbf{\Upsilon}_{j\iota}(\mathbf{x})+(\underline{\varrho}_{j\varsigma}-\bar{\varrho}_{j\varsigma})\breve{\mathbf{Y}}_{j\iota}].$$
$$\geq\sum_{\varsigma=1}^{\sigma}\varphi_\varsigma(\mathbf{x})\{\sum_{j=1}^{c}[(\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n}\bar{\upsilon}_{ri_r\varsigma}(x_r)\bar{\chi}_{ji_1i_2\dots i_n\varsigma}(\mathbf{x})$$
$$+\bar{\varrho}_{j\varsigma})\mathbf{\Upsilon}_{j\iota}(\mathbf{x})+(\underline{\varrho}_{j\varsigma}-\bar{\varrho}_{j\varsigma})\breve{\mathbf{Y}}_{j\iota}]-\hat{\xi}_\varsigma(\mathbf{x})\breve{\mathbf{\Omega}}_{\iota\varsigma}(\mathbf{x})\}. \quad (44)$$

According to the properties $\bar{\upsilon}_{ri_r\varsigma}(x_r)\geq 0$, $\bar{\upsilon}_{ri_r\varsigma}(x_r)$ are independent of rule j, ι and $\sum_{i_1}^{2}\sum_{i_2}^{2}\cdots\sum_{i_n}^{2}\prod_{r=1}^{n}\bar{\upsilon}_{ri_r\varsigma}(x_r)=1$ in substate space $\mathbf{\Psi}_\varsigma$, $\hat{\xi}_\varsigma(\mathbf{x})\breve{\mathbf{\Omega}}_{\iota\varsigma}(\mathbf{x})$ are independent of r and i_r in an certain substate space $\mathbf{\Psi}_\varsigma$, then $\sum_{j=1}^{c}m_j(\mathbf{x})\mathbf{\Upsilon}_{j\iota}(\mathbf{x})>0$ holds if $\sum_{j=1}^{c}[(\bar{\chi}_{ji_1i_2\dots i_n\varsigma}(\mathbf{x})+\bar{\varrho}_{j\varsigma})\mathbf{\Upsilon}_{j\iota}(\mathbf{x})+(\underline{\varrho}_{j\varsigma}-\bar{\varrho}_{j\varsigma})\breve{\mathbf{Y}}_{j\iota}]-\hat{\xi}_\varsigma(\mathbf{x})\breve{\mathbf{\Omega}}_{\iota\varsigma}(\mathbf{x})>0$, $\forall i_1,i_2,\dots,i_n\in\{1,2\},\varsigma\in\underline{\sigma}$, $\iota\in\underline{m}$.

Theorem 4: For the PPFMB control system (5), if there exist positive definite matrix $\mathbf{X}\in\Re^{n\times n}$, polynomial matrices $\mathbf{D}_j(\mathbf{x})\in\Re^{m\times n}$ and $\mathbf{O}_j(\mathbf{x})\in\Re^{m\times n}$, polynomial symmetric matrices $\mathbf{Y}_{ijk}(\mathbf{x})\in\Re^{n\times n}$ and $\mathbf{\Omega}_{k\varsigma}(\mathbf{x})\in\Re^{n\times n}$, $\breve{\mathbf{Y}}_{j\iota}(\mathbf{x})\in$

$\Re^{n\times n}$, $\breve{\mathbf{\Omega}}_{\iota\varsigma}(\mathbf{x}) \in \Re^{n\times n}$ $\forall i \in \underline{p}$, $j \in \underline{c}$, $k \in \underline{2^m}$, $\varsigma \in \underline{\sigma}$, $\iota \in \underline{m}$, and scalar $\gamma > 0$ such that the following SOS-based conditions of optimization problem are satisfied:

$$
\begin{aligned}
&\min \gamma \\
&s.t. 1) \text{ Sub-equations } 1) \text{ to } 3) \text{ of } (40); \\
&2)\ \nu^T(\breve{\mathbf{Y}}_{j\iota}(\mathbf{x}) - \varepsilon_7(\mathbf{x})\mathbf{I})\nu \text{ is SOS}, \ \forall j \in \underline{c}, \iota \in \underline{m}; \\
&3)\ \nu^T(\breve{\mathbf{Y}}_{j\iota}(\mathbf{x}) - \mathbf{\Upsilon}_{j\iota}(\mathbf{x}) - \varepsilon_8(\mathbf{x})\mathbf{I})\nu \text{ is SOS}, \forall j \in \underline{c}, \iota \in \underline{m}; \\
&4)\ \nu^T(\breve{\mathbf{\Omega}}_{\iota\varsigma}(\mathbf{x}) - \varepsilon_9(\mathbf{x})\mathbf{I})\nu \text{ is SOS}, \ \forall \iota \in \underline{m}, \varsigma \in \underline{\sigma}; \\
&5)\ \nu^T\{\sum_{j=1}^{c}[(\bar{\chi}_{ji_1i_2\ldots i_n\varsigma}(\mathbf{x}) + \overline{\varrho}_{j\varsigma})\mathbf{\Upsilon}_{j\iota}(\mathbf{x}) + (\underline{\varrho}_{j\varsigma} - \overline{\varrho}_{j\varsigma})\breve{\mathbf{Y}}_{j\iota}] \\
&\quad - \hat{\xi}_\varsigma(\mathbf{x})\breve{\mathbf{\Omega}}_{\iota\varsigma}(\mathbf{x})\}\nu \text{ is SOS}, \ \forall j \in \underline{c}, \iota \in \underline{m}, \varsigma \in \underline{\sigma}, \mathbf{x} \in \mathbf{\Psi}_\varsigma, \\
&\quad i_1, i_2, \ldots, i_n \in \{1, 2\}; \qquad (45)
\end{aligned}
$$

where ν is an arbitrary vector independent of $\mathbf{x}$; $\varepsilon_1 > 0$, $\varepsilon_2(\mathbf{x}) > 0, \ldots, \varepsilon_9(\mathbf{x}) > 0$ are predefined scalar polynomials; $\tilde{\mathbf{\Theta}}_{ijk}^{(\alpha,\beta)}(\mathbf{x})$, $\tilde{\mathbf{\Xi}}_{ijk}(\mathbf{x})$ and $\mathbf{\Upsilon}_{j\iota}(\mathbf{x})$ are defined in (17), (18) and (19), then the system (5) is asymptotically stable and positive with initial conditions contained in $\boldsymbol{\varepsilon}(\mathbf{P}, 1)$, where $\boldsymbol{\varepsilon}(\mathbf{P}, 1)$ is an estimation of the DOA. The polynomial fuzzy controller gains can be obtained by $\mathbf{G}_j(\mathbf{x}) = \mathbf{D}_j(\mathbf{x})\mathbf{X}^{-1}$ and the auxiliary polynomial fuzzy controller gains can be obtained by $\mathbf{H}_j(\mathbf{x}) = \mathbf{O}_j(\mathbf{x})\mathbf{X}^{-1}$, $\mathbf{X} = \mathbf{P}^{-1}$.

Remark 2: Theorems 1 to 4 are used to estimate DOA for PPFMB control system with input saturation while Corollaries 1 to 3 are used to obtain the stability region for PPFMB control system without input saturation. A larger DOA means that larger initial conditions can be driven to the origin. A larger stability region means that a wider range of systems can be stabilized. In order to relax the estimation of DOA and stability region, MFD method is applied to the stability analysis and positivity analysis. Theorem 2 and Corollary 2 are obtained by using TSMFD analysis method [41] which introduces the TSMFs interpolation point information, global approximation error and minimal value of the TSMFs into the basic results (Theorem 1 and Corollary 1). In order to further relax the results of Theorem 2 and Corollary 2, the TSMFD analysis method is improved by employing local approximation error and boundary information of substate space of premise variables, then the ITSMFD analysis method is proposed and used to obtain stability conditions. As a result, more relaxed results shown in Theorem 3 and Corollary 3 are obtained. In addition, in order to further relax the estimation of DOA, more relaxed results in Theorem 4 are obtained by applying the ITSMFD analysis method to both the stability conditions and the DOA estimation conditions.

V. Simulation Examples

In this section, two examples are provided to demonstrate the effectiveness and applicability of the analysis results. In the first example, the effect of the ITSMFD analysis method on the stability region is verified by the PPFMB control system without input saturation; In the second example, the proposed ITSMFD method will be applied to the example with input saturation to verify its effect on the estimation of DOA.

A. Simulation Example without Input Saturation

A three-rules PPFMB system is considered. The system and input matrices are as follows:

$$
\begin{aligned}
\mathbf{A}_1(x_1) &= \begin{bmatrix} -0.039 & 28.82 \\ 1 & -2 - x_1^2 - x_1 \end{bmatrix}, \\
\mathbf{A}_2(x_1) &= \begin{bmatrix} -0.037 & 26.71 \\ 0.80 & -4 - 1.20x_1^2 \end{bmatrix}, \\
\mathbf{A}_3(x_1) &= \begin{bmatrix} -0.033 & 22.07 \\ a & -2 - x_1^2 - x_1 - b \end{bmatrix}, \\
\mathbf{B}_1(x_1) &= \begin{bmatrix} 3.27 + 0.05x_1^2 \\ 0.05 \end{bmatrix}, \mathbf{B}_2(x_1) = \begin{bmatrix} 2.90 + 0.02x_1^2 \\ 0.05 \end{bmatrix}, \\
\mathbf{B}_3(x_1) &= \begin{bmatrix} 2.09 + 0.10x_1^2 \\ b \end{bmatrix}, \qquad (46)
\end{aligned}
$$

where a and b are constant scalars to be specified.

The membership functions of the PPFMB system are chosen as $w_1(x_1) = 1 - \frac{1}{1+e^{-5(x_1-1.6)}}$, $w_3(x_1) = \frac{1}{1+e^{-5(x_1-2.4)}}$, $w_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$. In this paper, we adopt IPC concept [35]–[41] to design the polynomial fuzzy controller, which means that the membership functions and/or number of rules between the polynomial fuzzy model and controller are allowed to be different. In order to reduce the complexity of the designed controller, the number of rules for the fuzzy controller is chosen as 2 and the membership functions are chosen as $m_1(x_1) = 1 - \frac{1}{1+e^{-5(x_1-2.0)}}$ and $m_2(x_1) = 1 - m_1(x_1)$.

When the membership functions are approximated by TSMFs, the function $\hat{\upsilon}_{11\varsigma}(x_1)$ and $\hat{\upsilon}_{12\varsigma}(x_1)$ are chosen as: $\hat{\upsilon}_{11\varsigma}(x_1) = (x_{1max\varsigma} - x_1)/(x_{1max\varsigma} - x_{1min\varsigma})$, $\hat{\upsilon}_{12\varsigma}(x_1) = 1 - \hat{\upsilon}_{11\varsigma}(x_1)$, where $x_{1max\varsigma}$ and $x_{1min\varsigma}$ are the maximum value and minimum value of x_1 in substate space $\mathbf{\Psi}_\varsigma$, respectively.

In order to verify that the proposed ITSMFD method can further relax the stability region, Corollaries 1 to 3 are applied. The constant parameters a and b are chosen in the range of $8 \leq a \leq 38$ at the interval of 2 and $0 \leq b \leq 1$ at the interval of 0.1. For Corollary 2, the expansion points are chosen as $x_1 = \{0, 0.16, \ldots, 3.84, 4\}$, λ is chosen as 3; we choose $\varepsilon_1 = \ldots = \varepsilon_6 = 1 \times 10^{-3}$; $\mathbf{D}_j(x_1)$, $\mathbf{Y}_{ij}(x_1)$ and $\mathbf{W}_{ij}(x_1)$ are all of degrees from 0 to 2 in x_1. For Corollary 3, the expansion points are chosen as $x_1 = \{0, 0.4, \ldots, 3.6, 4\}$; λ is chosen as 1 and 3, respectively, where $\lambda = 1$ means the Taylor series membership functions reduce to piecewise linear membership functions; $\mathbf{W}_{ij}(x_1)$ is removed, and $\mathbf{\Omega}_\varsigma(x_1)$ of degrees 0, 1 and 2 are introduced, the other parameters and settings are kept the same as Corollary 2. The stability regions obtained by Corollaries 1 to 3 are shown in Fig. 2.

Discussion and Explanation

In Fig. 2, a and b describe the floating range of system parameters, the larger stability region means that a wider range

TABLE I
Comparison of Corollary 2 and Corollary 3

	number of decision matrices	number of SOS conditions	times of obtaining results
Corollary 2	15	57	700s
Corollary 3	19	55	346s

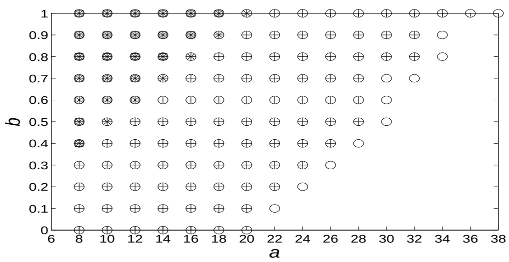


Fig. 2. Stability regions for PPFMB fuzzy system without limits on control signal in Corollary 1 (" × ") and Corollary 2 ("□"), Stability regions given by Corollary 3 with $\lambda = 1$ and $\lambda = 3$ are indicated by " + " and " ○ ", respectively.

of systems can be stabilized. "□" and " ○ " represent the stability regions which are given by Corollary 2 and Corollary 3, respectively. It can be seen that the stability region given by Corollary 3 is larger than the stability region given by Corollary 2 under the same λ. The number of decision matrices, the number of SOS conditions and the running time of Corollary 2 and Corollary 3 with the same λ are shown in Table I. From Fig. 2 and Table I, it can be concluded that Corollary 3 is more relaxed than Corollary 2 under roughly the same computation burden. It demonstrates that the local approximation error information and boundary information of substate space of premise variables lead to a larger stability region than global error information, because the local approximation error is generally smaller than the global approximation error, and every sub-condition used to ensure $\dot{V} < 0$ is allowed to be satisfied only in the corresponding substate space $\mathbf{\Psi}_\varsigma$ instead of the whole state space $\mathbf{\Psi}$.

In Fig. 2, the stability regions given by Corollary 3 with $\lambda = 1$ and $\lambda = 3$ are indicated by "+" and "○", respectively. The stability region indicated by "○" is larger than the stability region indicated by " + ", which demonstrates that the higher order of polynomial for the approximation of membership functions generally lead to a larger stability region. However, it should be noted that higher order of polynomial does not necessarily lead to larger stability region. Higher order of polynomial gives a larger stability region only when a higher order of polynomial can achieve smaller approximation error.

Remark 3: Corollaries 1 and 2 are obtained based on the membership-functions-independent (MFI) analysis method and MFD analysis method, respectively. Generally, MFD analysis method can lead to more relaxed result than MFI analysis method. However, as shown in Fig. 2, the stability region indicated by "□" (MFD) is smaller than the stability region indicated by " × " (MFI). Because the boundary information of substate space of premise variables is not considered in Corollary 2, which means that every sub-condition used to ensure $\dot{V} < 0$ needs to be satisfied in the whole state space $\mathbf{\Psi}$. However, when the local approximate membership functions corresponding to every sub-condition are extended to the whole state space $\mathbf{\Psi}$, the extended part of the membership functions may be very different from the original membership functions or even become negative, which may lead to very conservative results. In Corollary 3, the terms $\xi_\varsigma(\mathbf{x})$ which carry boundary information of the substate space of the premise variables are included in the stability conditions, which leads to larger stability regions indicated by " + " and " ○ " in Fig. 2.

B. Simulation Example with Input Saturation

In this example, we consider the system (46) with input saturation where $a = 1$ and $b = 0.05$, then, the auxiliary polynomial fuzzy controller is designed to help deal with saturation nonlinearity. For this example, the saturation value u_{lim} is 10. The settings of the membership functions $w_1(x_1)$, $w_2(x_1)$, $w_3(x_1)$, $m_1(x_1)$ and $m_2(x_1)$ are the same as the settings of them in Subsection V-A. When the membership functions are approximated by TSMFs, λ is chosen as 3, the settings of the functions $\bar{\upsilon}_{11\varsigma}(x_1)$ and $\bar{\upsilon}_{12\varsigma}(x_1)$ also are the same as the settings of $\hat{\upsilon}_{11\varsigma}(x_1)$ and $\hat{\upsilon}_{12\varsigma}(x_1)$ in Subsection V-A. Inspired by [50], the reference set is chosen as $\boldsymbol{\chi}_R := co\{(\sin(\theta), \cos(\theta))\}$, $\theta \in [0, \frac{\pi}{2}]$. In Theorems 1 to 4, we choose $\varepsilon_1 = \ldots = \varepsilon_9 = 1 \times 10^{-3}$.

Referring to Theorem 2, the expansion points are chosen as $x_1 = \{0, 0.16, \ldots, 3.84, 4\}$; $\mathbf{D}_j(x_1)$, $\mathbf{O}_j(x_1)$, $\mathbf{Y}_{ijk}(x_1)$, $\mathbf{W}_{ijk}(x_1)$ are all of degrees from 0 to 2 in x_1.

Referring to Theorem 3, the expansion points are chosen as $x_1 = \{0, 0.4, \ldots, 3.6, 4\}$; $\mathbf{D}_j(x_1)$, $\mathbf{O}_j(x_1)$, $\mathbf{Y}_{ijk}(x_1)$, $\mathbf{\Omega}_{k\varsigma}(x_1)$ are all of degrees from 0 to 2 in x_1.

Referring to Theorem 4, the expansion points are chosen as $x_1 = \{0, 0.4, \ldots, 3.6, 4\}$; $\mathbf{D}_j(x_1)$, $\mathbf{O}_j(x_1)$, $\mathbf{Y}_{ijk}(x_1)$, $\mathbf{\Omega}_{k\varsigma}(x_1)$, $\breve{\mathbf{Y}}_{j\iota}(x_1)$, $\breve{\mathbf{\Omega}}_{\iota\varsigma}$ are all of degrees from 0 to 2 in x_1.

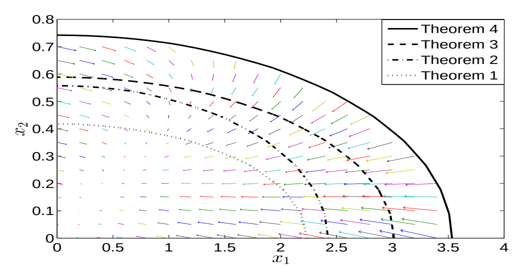


Fig. 3. Estimated DOAs indicated by elliptical area and quiver plot for polynomial fuzzy system with $a = 1$ and $b = 0.05$.

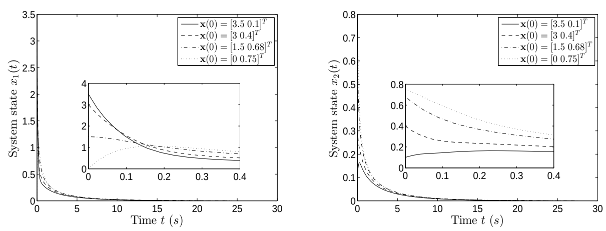
Discussion and Explanation

Theorems 1 to 4 are applied to system (46) with input saturation to estimate DOA where the minimum value of γ is obtained when $\theta = \frac{\pi}{2}$. The obtained minimum value of γ, Lyapunov matrix $\mathbf{P}$, polynomial fuzzy controller gains $\mathbf{G}_j(x_1)$ and auxiliary polynomial fuzzy controller gains $\mathbf{H}_j(x_1)$ are shown in Table II. The contractive invariant sets $\boldsymbol{\varepsilon}(\mathbf{P}, 1)$ (DOAs) obtained by applying Theorems 1 to 4 are shown in Fig. 3. The quiver plot demonstrates that the system state variables in estimated DOAs can be steered by the designed controller and auxiliary controller to the origin. The DOA bounded by dotted line is smaller than other DOAs, which means that the MFD analysis method has a stronger ability to relax the estimation of DOA than MFI analysis method. The DOAs bounded by dashed and solid lines are larger than the DOA bounded by dash-dotted line, which means that the

TABLE II

MINIMUM VALUE OF γ, LYAPUNOV MATRIX $\mathbf{P}$, POLYNOMIAL FUZZY CONTROLLER GAINS $\mathbf{G}_j(x_1)$ AND AUXILIARY POLYNOMIAL FUZZY CONTROLLER GAINS $\mathbf{H}_j(x_1)$ OF THEOREM 1 TO 4

Theorem	γ	Lyapunov matrix $\mathbf{P}$	controller gain $\mathbf{G}_j(x_1)$ and auxiliary controller gain $\mathbf{H}_j(x_1)$
1	0.20	diag$\{0.2000, 5.6883\}$	$\mathbf{G}_1(x_1) = [-6.2911\times10^{-8}x_1^2-3.1709\times10^{-7}x_1-6.3314, 2.0275\times10^{-6}x_1^2-1.7750\times10^{-5}x_1+2.3639]$ $\mathbf{G}_2(x_1) = [-6.2907\times10^{-8}x_1^2-3.1708\times10^{-7}x_1-6.3314, 3.0275\times10^{-6}x_1^2-1.7750\times10^{-5}x_1+2.3639]$ $\mathbf{H}_1(x_1) = [-6.0299\times10^{-10}x_1^2-1.0629\times10^{-8}x_1-4.4715, 9.6311\times10^{-8}x_1^2-2.1681\times10^{-7}x_1-0.0465]$ $\mathbf{H}_2(x_1) = [-5.8940\times10^{-10}x_1^2-1.0629\times10^{-8}x_1-4.4715, 9.5866\times10^{-8}x_1^2-2.1682\times10^{-7}x_1-0.0465]$
2	0.17	diag$\{0.1700, 3.2041\}$	$\mathbf{G}_1(x_1) = [-2.2351\times10^{-4}x_1^2-5.0541\times10^{-4}x_1-4.6895, 1.2236\times10^{-3}x_1^2+4.2167\times10^{-4}x_1-1.6846]$ $\mathbf{G}_2(x_1) = [-1.0984\times10^{-4}x_1^2-5.1236\times10^{-4}x_1-5.1441, 4.4902\times10^{-4}x_1^2-2.1446\times10^{-4}x_1-1.4441]$ $\mathbf{H}_1(x_1) = [2.4378\times10^{-7}x_1^2+1.6431\times10^{-4}x_1-4.0891, 1.8896\times10^{-4}x_1^2+3.3059\times10^{-4}x_1-1.4639]$ $\mathbf{H}_2(x_1) = [-1.2085\times10^{-7}x_1^2-5.5256\times10^{-5}x_1-4.1029, 3.1075\times10^{-4}x_1^2+9.7772\times10^{-6}x_1-1.6247]$
3	0.11	diag$\{0.1100, 2.8711\}$	$\mathbf{G}_1(x_1) = [-4.0530\times10^{-5}x_1^2-5.0645\times10^{-4}x_1-3.9154, 3.0551\times10^{-4}x_1^2-8.2127\times10^{-4}x_1-0.5052]$ $\mathbf{G}_2(x_1) = [-1.9243\times10^{-5}x_1^2-6.0030\times10^{-4}x_1-3.8156, 1.1253\times10^{-4}x_1^2-1.2107\times10^{-3}x_1+1.0490]$ $\mathbf{H}_1(x_1) = [-5.5271\times10^{-7}x_1^2-7.1655\times10^{-6}x_1-3.3105, 8.0600\times10^{-5}x_1^2-5.0126\times10^{-8}x_1-0.8455]$ $\mathbf{H}_2(x_1) = [-2.1418\times10^{-7}x_1^2-2.6459\times10^{-5}x_1-3.1560, 6.8087\times10^{-6}x_1^2-9.0189\times10^{-4}x_1+0.6557]$
4	0.08	diag$\{0.0801, 1.8109\}$	$\mathbf{G}_1(x_1) = [-1.1025\times10^{-4}x_1^2+0.0158x_1-3.3265, 1.4422\times10^{-3}x_1^2-1.2365\times10^{-3}x_1-1.4960]$ $\mathbf{G}_2(x_1) = [-5.9452\times10^{-5}x_1^2+4.4087\times10^{-3}x_1-3.4139, 3.6064\times10^{-4}x_1^2-0.0227x_1+0.1257]$ $\mathbf{H}_1(x_1) = [-8.4141\times10^{-5}x_1^2+0.0665x_1-2.7904, 3.4783\times10^{-3}x_1^2-1.9309\times10^{-5}x_1-2.0858]$ $\mathbf{H}_2(x_1) = [-5.9247\times10^{-5}x_1^2+0.0128x_1-1.9970, 1.0024\times10^{-3}x_1^2-0.0145x_1-0.8727]$



(a) Time response of $x_1(t)$ (b) Time response of $x_2(t)$

Fig. 4. Time response of system states with four different initial conditions.

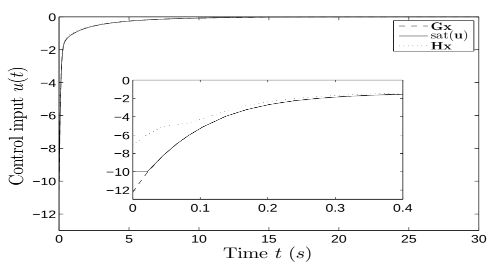


Fig. 5. Time response of control input signal $\mathbf{G}(x_1)\mathbf{x}$, auxiliary control input signal $\mathbf{H}(x_1)\mathbf{x}$ and saturation control signal sat$(\mathbf{u})$ with initial condition $\mathbf{x}(0) = [3.5\ 0.1]^T$.

proposed ITSMFD analysis method can result in a larger DOA than the existing TSMFD analysis method. The DOA bounded by solid line is larger than other DOAs, which means that applying the proposed ITSMFD analysis method to the DOA estimation conditions can further relax the estimation of DOA.

When Theorem 4 is used, saturation control signal sat$(\mathbf{u})$ is combined in the form sat$(\mathbf{u}) = \eta_1\mathbf{G}(x_1)\mathbf{x} + \eta_2\mathbf{H}(x_1)\mathbf{x}$ by control signals generated by the obtained polynomial fuzzy controller and auxiliary polynomial fuzzy controller. Then, sat$(\mathbf{u})$ is applied to the polynomial fuzzy system (46) with $a = 1$, $b = 0.05$, and four different initial conditions are chosen, the time response of the system states are shown in Fig. 4. For the initial condition $\mathbf{x}(0) = [3.5\ 0.1]$ which is on the boundary of the DOA obtained by Theorem 4, as shown in Fig. 5, the control input signal $\mathbf{G}(x_1)\mathbf{x}$ exceeds the limit value 10 for a period of time at the beginning, $\mathbf{H}(x_1)\mathbf{x}$ is always less than 10. sat$(\mathbf{u})$ satisfying (1) can be obtained by combining $\mathbf{G}(x_1)\mathbf{x}$ and $\mathbf{H}(x_1)\mathbf{x}$ in the form of convex combination, where $0 \le \eta_1, \eta_2 \le 1$ are time-varying. The above simulation result means that the system states in the estimated DOA can be driven to the origin by the designed controller and auxiliary controller even if the input signal is saturated. In Fig. 4 and 5, in order to provide a clearer result, the time response curves for a certain period of time are enlarged in the figure insets.

VI. CONCLUSION

The stability and positivity of PPFMB fuzzy system have been investigated in presence or absence of input saturation. The IPC concept has been employed in both two situations to increase the flexibility of controller design. For the case that there no any limits on the control signal, the ITSMFD method has been proposed to give more relaxed stability region. For the case that the input saturation is considered, the ITSMFD method not only has been adopted on the stability conditions but has been extended to DOA estimation conditions to relax the estimation of DOA. Two simulation examples have been presented to verify the feasibility of the proposed methods. In the future, more diverse MFD methods can be used to excavate more information of saturated input signals, so that more relaxed estimation of DOA can be obtained. Also, the effective one-step convexification method of the non-convex conditions derived by linear copositive Lyapunov function will be explored to relax the research results for positive nonlinear systems.

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