DOI:
10.1007/s00440-017-0782-0

Hole probability for zeroes of Gaussian Taylor series with finite radii of convergence

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Abstract

We study a family of random Taylor series

$$F(z) = \sum_{n \geq 0} \zeta_n a_n z^n$$

with radius of convergence almost surely 1 and independent, identically distributed complex Gaussian coefficients (ζ_n); these Taylor series are distinguished by the invariance of their zero sets with respect to isometries of the unit disk. We find reasonably tight upper and lower bounds on the probability that F does not vanish in the disk $\{|z| \leqslant r\}$ as $r \uparrow 1$. Our bounds take different forms according to whether the non-random coefficients (a_n) grow, decay or remain of the same order. The results apply more generally to a class of Gaussian Taylor series whose coefficients (a_n) display power-law behavior.

1 Introduction

Random analytic functions are a classical topic of study, attracting the attention of analysts and probabilists [7]. One of the most natural instances of such functions is provided by Gaussian

*Supported by ISF Grants 1048/11 and 166/11, by ERC Grant 335141 and by the Raymond and Beverly Sackler Post-Doctoral Scholarship 2013–14.

†Supported by ISF Grants 1048/11 and 861/15 and by IRG Grant SPTRF.

‡Supported by ISF Grants 166/11 and 382/15.

analytic functions (GAFs, for short). The zero sets of Gaussian analytic functions are point processes exhibiting many interesting features [6, 11]. These features depend on the geometry of the domain of analyticity of the function and, sometimes, pique the interest of mathematical physicists, see, for instance, [1, 3, 4, 14].

In this work we consider Gaussian analytic functions in the unit disk represented by a Gaussian Taylor series

$$F(z) = \sum_{n\geq 0} \zeta_n a_n z^n \,, \tag{1}$$

where (ζ_n) is a sequence of independent, identically distributed standard complex normal random variables (i.e., the probability density of each ζ_n is $\frac{1}{\pi}e^{-|z|^2}$ with respect to Lebesgue measure on the complex plane), and (a_n) is a non-random sequence of non-negative numbers satisfying $\limsup \sqrt[n]{a_n} = 1$. Properties of zero sets of Gaussian Taylor series with infinite radius of convergence have been studied quite intensively in recent years, see, e.g., [6, 10, 13] and the references therein. When the radius of convergence is finite, the hyperbolic geometry often leads to some peculiarities and complications. In particular, this is the case in the study of the hole probability, that is, the probability of the event

$$\mathrm{Hole}(r) = \big\{F \neq 0 \text{ on } r\bar{\mathbb{D}}\big\},$$

an important characteristic both from the point of view of analytic function theory and the theory of point processes. For *arbitrary* Gaussian Taylor series with radius of convergence one, understanding the asymptotic behaviour of the hole probability as $r \uparrow 1$ seems difficult. Here, we focus on a natural family F_L of hyperbolic Gaussian analytic functions (hyperbolic GAFs, for short),

$$F_L(z) = \sum_{n\geq 0} \zeta_n \sqrt{\frac{L(L+1)\,\ldots\,(L+n-1)}{n!}}\, z^n \,, \qquad 0 < L < \infty \,, \tag{2}$$

distinguished by the invariance of the distribution of their zero set under Möbius transformations of the unit disk [6, Ch. 2], [18]. Note that the parameter L used to parameterize this family equals the mean number of zeros of F_L per unit hyperbolic area.

The main result of our work provides reasonably tight asymptotic bounds on the logarithm of the hole probability as r increases to 1. These bounds show a transition in the asymptotics of the hole probability according to whether $0 < L < 1$, $L = 1$ or $L > 1$. Curiously, a transition taking place at a different point, $L = \frac{1}{2}$, was observed previously in the study of the asymptotics of the variance of the number of zeroes of F_L in disks of increasing radii [2].

1.1 The main result

Theorem 1. *Suppose that F_L is a hyperbolic GAF, and that $r \uparrow 1$. Then,*

(i) *for $0 < L < 1$,*

$$\frac{1-L-o(1)}{2^{L+1}}\,\frac{1}{(1-r)^L}\,\log\frac{1}{1-r} \leqslant -\log \mathbb{P}[\mathrm{Hole}(r)] \leqslant \frac{1-L+o(1)}{2^L}\,\frac{1}{(1-r)^L}\,\log\frac{1}{1-r}\,;$$

(ii) *for $L = 1$,*

$$-\log \mathbb{P}[\mathrm{Hole}(r)] = \frac{\pi^2+o(1)}{12}\,\frac{1}{1-r}\,;$$

(iii) *for* $L > 1$,

$$-\log \mathbb{P}[\mathrm{Hole}(r)] = \frac{(L-1)^2 + o(1)}{4} \frac{1}{1-r} \log^2 \frac{1}{1-r} .$$

The case $L = 1$ in this theorem is due to Peres and Virág. We'll briefly discuss their work in Section 1.2.2. The rest is apparently new.

1.2 Previous work

1.2.1 Gaussian Taylor series with an infinite radius of convergence

For an *arbitrary* Gaussian Taylor series with an *infinite* radius of convergence, the logarithmic asymptotics of the hole probability was obtained in [12]. The main result [12, Theorem 1] says that when $r \to \infty$ outside an exceptional set of finite logarithmic length,

$$-\log \mathbb{P}[\mathrm{Hole}(r)] = S_F(r) + o(S_F(r)) , \tag{3}$$

where

$$S_F(r) = \sum_{n \geq 0} \log_+(a_n^2 r^{2n}) . \tag{4}$$

In this generality, the appearance of an exceptional set of values of r is unavoidable due to possible irregularities in the behaviour of the coefficients (a_n) (see [13, Section 17]).

For a Gaussian Taylor series with a finite radius of convergence the asymptotic rate of decay of the hole probability has been described only in several rather special cases.

1.2.2 The determinantal case: $L = 1$

Peres and Virág [15] (see also [6, Section 5.1]) discovered that the zero set of the Gaussian Taylor series

$$F(z) = \sum_{n \geq 0} \zeta_n z^n \tag{5}$$

(that corresponds to $L = 1$ in (2)) is a determinantal point process [15, Theorem 1] and, therefore, many of its characteristics can be explicitly computed. In particular, they found that [15, Corollary 3 (i)]

$$-\log \mathbb{P}[\mathrm{Hole}(r)] = \frac{\pi^2 + o(1)}{12} \frac{1}{1-r} \qquad \text{as } r \to 1. \tag{6}$$

For $L \neq 1$, the zero set of F_L is not a determinantal point process [6, p. 83], requiring the use of other techniques.

1.2.3 Fast growing coefficients

Skaskiv and Kuryliak [17] showed that the technique developed in [12] can be applied to Gaussian Taylor series on the disk, that grow very fast near the boundary. Put

$$\sigma_F(r)^2 = \mathbb{E}[|F(re^{\mathrm{i}\theta})|^2] = \sum_{n \geq 0} a_n^2 r^{2n} .$$

They proved [17, Theorem 4] that if

$$\lim_{r\to 1}(1-r)\log\log\sigma_F(r) = +\infty\,,$$

and the sequence (a_n) is logarithmically concave, then the same logarithmic asymptotic (3) holds when $r \to 1$ outside a small exceptional subset of $[\frac{1}{2}, 1)$. Note that in our case, $\sigma_{F_L}(r) = (1-r^2)^{-L/2}$.

1.2.4 The case when F is bounded on $\mathbb{D}$

At the opposite extreme, we may consider the case when, almost surely, the random series F is bounded on $\mathbb{D}$. Then $\mathbb{P}[\mathrm{Hole}(r)]$ has a positive limit as $r \to 1$. Indeed, put $F = F(0) + G$. If F is bounded on $\mathbb{D}$, then G is bounded on $\mathbb{D}$ as well. Take M so that $\mathbb{P}[\sup_{\mathbb{D}}|G| \leqslant M] \geq \frac{1}{2}$. Then

$$\big\{F \neq 0 \text{ on } \mathbb{D}\big\} \supset \big\{\sup_{\mathbb{D}}|G| \leqslant M,\ |F(0)| > M\big\}\,.$$

Since $F(0)$ and G are independent, we get

$$\mathbb{P}[\mathrm{Hole}(r)] \geq \mathbb{P}[\sup_{\mathbb{D}}|G| \leqslant M]\cdot\mathbb{P}[|F(0)| > M] \geq \frac{1}{2}e^{-M^2}\,.$$

In view of this observation, we recall a classical result that goes back to Paley and Zygmund and gives a sufficient condition for continuity (and hence, boundedness) of F on $\bar{\mathbb{D}}$. Introduce the sequence

$$s_j = \Big(\sum_{2^j\leqslant n<2^{j+1}} a_n^2\Big)^{1/2}\,.$$

If the sequence s_j decreases and $\sum_j s_j < \infty$, then, almost surely, F is a continuous function on $\bar{\mathbb{D}}$ [7, Section 7.1]. On the other hand, under a mild additional regularity condition, divergence of the series $\sum_j s_j = +\infty$ guarantees that, almost surely, F is unbounded in $\mathbb{D}$ [7, Section 8.4].

1.3 Several comments on Theorem 1

1.3.1

The proof of Theorem 1 combines the tools introduced in [19, 12] with several new ingredients. Unfortunately, in the case $0 < L < 1$, our techniques are insufficient for finding the main term in the logarithmic asymptotics of $\mathbb{P}[\mathrm{Hole}(r)]$. Also we cannot completely recover the aforementioned result of Peres and Virág. On the other hand, our arguments do not use the hyperbolic invariance of the zero distribution of F_L. We make use of the fact that

$$a_n^2 = \frac{\Gamma(n+L)}{\Gamma(L)\Gamma(n+1)} = (1+o(1))\frac{n^{L-1}}{\Gamma(L)}\,,\quad n\to\infty, \tag{7}$$

where Γ is Euler's Gamma function, and our techniques apply more generally to a class of Gaussian Taylor series whose coefficients (a_n) display power-law behavior. We will return to this in the concluding Section 8.

1.3.2

In the case $L > 1$, using (7), it is easy to see that

$$S_F(r) = \sum_{n\geq 0} \log_+(a_n^2 r^{2n}) = \frac{(L-1)^2 + o(1)}{4} \frac{1}{1-r} \log^2 \frac{1}{1-r}\,, \qquad r \to 1\,.$$

That is, in this case the hyperbolic geometry becomes less relevant and the main term in the logarithmic asymptotic of the hole probability is governed by the same function as in the planar case, discussed in Section 1.2.1.

1.3.3

For $0 < L < 1$, the gap between the upper and lower bounds in Theorem 1 remains unsettled. It would be also interesting to accurately explore the behaviour of the logarithm of the hole probability near the transition points $L = 0$ and $L = 1$; for instance, to consider the cases $a_n = n^{-1/2}\ell(n)$ and $a_n = \ell(n)$, where ℓ is a slowly varying function.

Of course, the ultimate goal would be to treat the case of *arbitrary* Gaussian Taylor series with finite radii of convergence.

Acknowledgements

The authors thank Alexander Borichev, Fedor Nazarov, and the referee for several useful suggestions.

Notation

- Gaussian analytic functions will be called GAFs. The Gaussian analytic functions F_L defined in (2) will be called hyperbolic GAFs.
- We suppress all of the dependence on L unless it is absolutely necessary. In particular, from here on,
  - $F = F_L$,
  - c and C denote positive constants that might only depend on L. The values of these constants are irrelevant for our purposes and may vary from line to line. By A, α, α_i, etc. we denote positive constants whose values we keep fixed throughout the proof in which they appear.
- The notation $X \simeq Y$ means that $cX \leqslant Y \leqslant CY$.
- We set
  $$\delta = 1 - r \quad \text{and} \quad r_0 = 1 - \kappa\delta \quad \text{with } 1 < \kappa \leqslant 2.$$
  Everywhere, except Section 6, we set $\kappa = 2$, that is, $r_0 = 1 - 2\delta$. The value of r is assumed to be sufficiently close to 1. Correspondingly, the value of δ is assumed to be sufficiently small.
- The variance $\sigma_F^2 = \sigma_F(r)^2$ is defined by
  $$\sigma_F^2 = \mathbb{E}\big[|F(re^{\mathrm{i}\theta})|^2\big] = (1-r^2)^{-L} = (1+o(1))(2\delta)^{-L}, \quad \text{as } \delta \to 0.$$
  Usually, we suppress the dependence on r and write σ_F instead of $\sigma_F(r)$. Notice that $\log \frac{1}{\delta} \simeq \log \sigma_F$.

- An event E depending on r will be called *negligible* if $-\log \mathbb{P}[\mathrm{Hole}(r)] = o(1)\big(-\log \mathbb{P}[E]]\big)$ as $r \to 1$. Notice that this may depend on the value of L.
- If f takes real values, then we define $f_+ = \max\{0, f\}$ and $f_- = \max\{0, -f\}$.
- $e(t) = e^{2\pi \mathrm{i} t}$.
- $[x]$ denotes the integer part of x.
- $n \equiv k\,(N)$ means that $n \equiv k$ modulo N.
- $\mathbb{D}$ denotes the open unit disk, $\mathbb{T}$ denotes the unit circle.
- The planar Lebesgue measure is denoted by m, and the (normalized) Lebesgue measure on $\mathbb{T}$ is denoted by μ.

2 Idea of the proof

We give a brief description of the proof of Theorem 1 in the cases $0 < L < 1$ and $L > 1$. In the case $L = 1$ our arguments suffice to estimate the logarithm of the hole probability up to a constant, as discussed in Section 8, and we briefly sketch the argument for this case as well. Our proof for the upper bound on the hole probability in the case $0 < L < 1$ is more involved than the other cases.

2.1 Upper bounds on the hole probability when $L > 1$ and $L = 1$

Our starting point for proving upper bounds is the mean-value property. On the hole event,

$$\int_{\mathbb{T}} \log |F(tr)| \, \mathrm{d}\mu(t) = \log |F(0)|.$$

Off an event of negligible probability, the integral may be discretized, yielding the inequality

$$\sum_{j=1}^{N} \log |F(\tau \omega^j r_0)| \leqslant N \log |F(0)| + 1 \tag{8}$$

in the slightly smaller radius $r_0 < r$, for a random $\tau \in \mathbb{T}$, taken out of a small set of possibilities, suitable N and $\omega = e(1/N)$ (see Lemma 8 and Lemma 9). Thus it suffices to bound from above, for each fixed $\tau \in \mathbb{T}$, the probability that (8) holds. We may further simplify by fixing a threshold $T > 0$, noting that $\mathbb{P}\left[|F(0)| \geq T\right] = \exp(-T^2)$ and writing

$$\mathbb{P}\Big[\sum_{j=1}^{N} \log |F(\tau \omega^j r_0)| \leqslant N \log |F(0)| + 1\Big] \leqslant \mathbb{P}\Big[\sum_{j=1}^{N} \log |F(\tau \omega^j r_0)| \leqslant N \log T + 1\Big] + e^{-T^2}. \tag{9}$$

We focus on the first summand, setting T sufficiently large so that the second summand is negligible. Taking $0 < \theta < 2$ and applying Chebyshev's inequality,

$$\begin{aligned} \mathbb{P}\Big[\sum_{j=1}^{N} \log |F(\tau \omega^j r_0)| \leqslant N \log T + 1\Big] &\leqslant \mathbb{P}\Big[\prod_{j=1}^{N} \big|F(\omega^j \tau r_0)\big|^{-\theta} \geq c T^{-\theta N}\Big] \\ &\leqslant C T^{\theta N} \, \mathbb{E}\Big[\prod_{j=1}^{N} \big|F(\omega^j \tau r_0)\big|^{-\theta}\Big]. \end{aligned} \tag{10}$$

It remains to estimate the expectation in the last expression. Our bounds for it make use of the fact that the covariance matrix Σ of the Gaussian vector $(F(\tau\omega^j r_0))$, $1 \leqslant j \leqslant N$, has a circulant structure, allowing it to be explicitly diagonalized. In particular, its eigenvalues are (see Lemma 10)

$$\lambda_m = N \sum_{n \equiv m\,(N)} a_n^2 r^{2n}\,, \qquad m = 0, \dots, N-1\,. \tag{11}$$

This is used together with the following, somewhat rough, bound (see Lemma 15)

$$\mathbb{E}\Big[\prod_{j=1}^{N} \big|F(\omega^j \tau r_0)\big|^{-\theta}\Big] \leqslant \frac{1}{\det \Sigma} \left(\Lambda^{\left(1-\frac{1}{2}\theta\right)} \cdot \Gamma\big(1-\tfrac{1}{2}\,\theta\big)\right)^N, \tag{12}$$

where Λ is the maximal eigenvalue of Σ and Γ is Euler's Gamma-function.

2.1.1 The case $L > 1$

We set the parameters to be $T = \delta^{-\frac{1}{2}} \exp\left(-\sqrt{\log \frac{1}{\delta}}\right)$, so that the factor e^{-T^2} in (9) is indeed negligible, $N = \left[\, \frac{L-1}{2\delta} \log \frac{1}{\delta} \,\right]$ and $\theta = 2 - \left(\log \frac{1}{\delta}\right)^{-1}$. With this choice, the dominant term in the combination of the bounds (10) and (12) is the factor $T^{\theta N} / \det \Sigma$. Its logarithmic asymptotics are calculated using (11) and yield the required upper bound.

We mention that choosing θ close to its maximal value of 2 corresponds, in some sense, to the fact that the event (8) constitutes a very large deviation for the random sum $\sum_{j=1}^{N} \log |F(\tau\omega^j r_0)|$.

2.1.2 The case $L = 1$

The same approach may be applied with the parameters $T = b\,\delta^{-1/2}$, for a small parameter $b > 0$, $N = \left[\,\delta^{-1}\,\right]$ and $\theta = 1$. For variety, Section 8.1 presents a slightly different alternative.

In the same sense as before, the choice $\theta = 1$ indicates that we are now considering a large deviation event.

2.2 Upper bound on the hole probability when $0 < L < 1$

Our goal here is to show that the intersection of the hole event with the event $\left\{|F(0)| \leqslant A\sigma_F \sqrt{\log \frac{1}{(1-r)\sigma_F^2}}\,\right\}$ is negligible when $A^2 < \frac{1}{2}$. The upper bound then follows from the estimate

$$\mathbb{P}\Big[|F(0)| > A\sigma_F \sqrt{\log \tfrac{1}{(1-r)\sigma_F^2}}\,\Big] = \exp\left(-A^2 \sigma_F^2 \log \tfrac{1}{(1-r)\sigma_F^2}\right).$$

The starting point is again the inequality (8) in which we choose the parameter

$$N = \left[\delta^{-\alpha}\right], \quad L < \alpha < 1,$$

with α eventually chosen close to 1. However, a more refined analysis is required here. First, we separate the constant term from the function F, writing

$$F(z) = F(0) + G(z).$$

Second, to have better control of the Gaussian vector

$$(G(\tau\omega^j r_0)), \quad 1 \leqslant j \leqslant N, \tag{13}$$

we couple G with two independent GAFs G_1 and G_2 so that $G = G_1 + G_2$ almost surely, the vector $(G_1(\tau\omega^j r_0))$, $1 \leqslant j \leqslant N$, is composed of independent, identically distributed variables, and G_2 is a polynomial of degree N with relatively small variance. In essence, we are treating the variables in (13) as independent, identically distributed up to a small error captured by G_2.

We proceed by conditioning on $F(0)$ and G_2, using the convenient notation

$$\mathbb{E}^{F(0),G_2}\big[\ .\ \big] = \mathbb{E}\big[\ .\ \big|\ F(0), G_2\big],$$
$$\mathbb{P}^{F(0),G_2}\big[\ .\ \big] = \mathbb{P}\big[\ .\ \big|\ F(0), G_2\big].$$

Applying Chebyshev's inequality we may use the above independence to exchange expectation and product, writing, for $0 < \theta < 2$,

$$\begin{aligned}\mathbb{P}^{F(0),G_2}\Big[\sum_{j=1}^{N} \log|F(\tau\omega^j r_0)| \leqslant N\log|F(0)| + 1\Big] &\leqslant C|F(0)|^{\theta N}\,\mathbb{E}^{F(0),G_2}\Big[\prod_{j=1}^{N}\big|F(\omega^j\tau r_0)\big|^{-\theta}\Big] \\ &= C|F(0)|^{\theta N}\prod_{j=1}^{N}\mathbb{E}^{F(0),G_2}\Big[\big|F(\omega^j\tau r_0)\big|^{-\theta}\Big].\end{aligned} \tag{14}$$

Thus we need to estimate expectations of the form

$$\mathbb{E}^{F(0),G_2}\Big[\Big|\frac{F(\omega^j\tau r_0)}{F(0)}\Big|^{-\theta}\Big] = \mathbb{E}^{F(0),G_2}\Big[\Big|1 + \frac{G_1(\omega^j\tau r_0) + G_2(\omega^j\tau r_0)}{F(0)}\Big|^{-\theta}\Big], \tag{15}$$

in which $F(0)$ and G_2 are given. Two bounds are used to this end. Given a standard complex Gaussian random variable ζ, real $t > 0$ and $0 < \theta \leqslant 1$ we have the simple estimate,

$$\sup_{w\in\mathbb{C}} \mathbb{E}\Big[\ \Big|w + \frac{\zeta}{t}\Big|^{-\theta}\ \Big] \leqslant t^{\theta}(1 + C\theta) \tag{16}$$

and, for $0 \leqslant \theta \leqslant \frac{1}{2}$, the more refined

$$\mathbb{E}\Big[\Big|1 + \frac{\zeta}{t}\Big|^{-\theta}\Big] \leqslant 1 - c\theta\,\frac{e^{-t^2}}{1+t^2} + C\theta^2\,, \tag{17}$$

see Lemma 11 and Lemma 14. The most important feature of the second bound is that it is less than 1 (though only slightly) when θ is very close to 0, satisfying $\theta \leqslant c(1+t^2)^{-1}e^{-t^2}$.

Combining (14) and (15) with the simple estimate (16) (with $\theta = 1$) already suffices to prove that the intersection of the hole event with the event $\{|F(0)| \leqslant a\sigma_F\}$ is negligible when a is a sufficiently small constant. However, on the event

$$\big\{a\sigma_F \leqslant |F(0)| \leqslant A\sigma_F\sqrt{\log\tfrac{1}{(1-r)\sigma_F^2}}\,\big\}$$

the error term G_2 becomes more relevant and we consider two cases according to its magnitude. Taking small constants $\varepsilon, \alpha_0 > 0$ and $\eta = \delta^{\alpha_0}$ we let

$$J = \Big\{1 \leqslant j \leqslant N\colon \Big|1 + \frac{G_2(\omega^j r_0)}{F(0)}\Big| \geq 1 + 2\varepsilon\Big\}.$$

2.2.1 The case $|J| > (1-2\eta)N$

Here, discarding (a priori, before conditioning on G_2) a negligible event to handle the rotation τ, for many values of j, the terms in (15) have $\left|1+\frac{G_2(\omega^j\tau r_0)}{F(0)}\right| \geq 1+\varepsilon$. This fact together with the bound (17) (in simplified form, with right-hand side $1+C\theta^2$), taking θ tending to 0 as a small power of δ, suffices to show that the probability in (14) is negligible.

2.2.2 The case $|J| \leqslant (1-2\eta)N$

In this case we change our starting point. By the mean-value inequality,

$$\int_{\mathbb{T}} \log\Big|\frac{F(tr)}{F(0)+G_2(tr)}\Big|\,\mathrm{d}\mu(t) \leqslant \log\Big|\frac{F(0)}{F(0)+G_2(0)}\Big| = 0.$$

Off an event of negligible probability, the integral may again be discretized, yielding

$$\sum_{j=1}^{N} \log\Big|\frac{F(\tau\omega^j r_0)}{F(0)+G_2(\tau\omega^j r_0)}\Big| \leqslant 1$$

in the slightly smaller radius $r_0 < r$, for a random τ, taken out of a small set of possibilities, see Lemma 8. As before, for each fixed τ, Chebyshev's inequality and independence show that,

$$\mathbb{P}^{F(0),G_2}\Big[\sum_{j=1}^{N} \log\Big|\frac{F(\tau\omega^j r_0)}{F(0)+G_2(\tau\omega^j r_0)}\Big| \leqslant 1\Big] \leqslant C\prod_{j=1}^{N} \mathbb{E}^{F(0),G_2}\Big[\Big|\frac{F(\tau\omega^j r_0)}{F(0)+G_2(\tau\omega^j r_0)}\Big|^{-\theta}\Big] \tag{18}$$

and we are left with the task of estimating terms of the form

$$\mathbb{E}^{F(0),G_2}\Big[\Big|\frac{F(\tau\omega^j r_0)}{F(0)+G_2(\tau\omega^j r_0)}\Big|^{-\theta}\Big] = \mathbb{E}^{F(0),G_2}\left[\left|1+\frac{G_1(\tau\omega^j r_0)}{F(0)\left(1+\frac{G_2(\tau\omega^j r_0)}{F(0)}\right)}\right|^{-\theta}\right]. \tag{19}$$

Again discarding (a priori) a negligible event to handle the rotation τ, for many values of j, the terms in (19) have $\left|1+\frac{G_2(\omega^j\tau r_0)}{F(0)}\right| \leqslant 1+\varepsilon$. These terms are estimated by using (17) with $t \leqslant (1+4\varepsilon)A\sqrt{\log\frac{1}{(1-r)\sigma_F^2}}$. Correspondingly we set $\theta = c\eta((1-r)\sigma_F^2)^{(1+10\varepsilon)A^2}$ and obtain that the probability in (18) satisfies

$$\mathbb{P}^{F(0),G_2}\Big[\sum_{j=1}^{N} \log\Big|\frac{F(\tau\omega^j r_0)}{F(0)+G_2(\tau\omega^j r_0)}\Big| \leqslant 1\Big] \leqslant \exp(-c\eta^2((1-r)\sigma_F^2)^{2(1+10\varepsilon)A^2}N).$$

Recalling that $\sigma_F^2 = (1-r^2)^{-L}$, $\eta = \delta^{\alpha_0}$ and $N = [\delta^{-\alpha}]$, and choosing ε and α_0 close to 0 and α close to 1 shows that this probability is negligible provided that $A^2 < \frac{1}{2}$.

In contrast to the cases $L > 1$ and $L = 1$, the fact that we take θ tending to 0 can be viewed as saying that we are now considering a moderate deviation event.

2.3 Lower bounds on the hole probability

The proofs of our lower bounds on the hole probability are less involved than the proofs of the upper bounds and the reader is referred to the relevant sections for details. We mention here that in all cases we rely on the same basic strategy: Fix a threshold $M > 0$ and observe that

$$\begin{aligned}\mathbb{P}\big[\mathrm{Hole}(r)\big] &\geq \mathbb{P}\Big[|F(0)| > M,\ \max_{r\bar{\mathbb{D}}} |F - F(0)| \leqslant M\Big] \\ &= e^{-M^2} \cdot \mathbb{P}\Big[\max_{r\mathbb{T}} |F - F(0)| \leqslant M\Big]. \end{aligned} \tag{20}$$

2.3.1 The case $0 < L < 1$

Here we take

$$M = \sqrt{1 - L + 2\varepsilon} \cdot \sigma_F \sqrt{\log \tfrac{1}{1-r}}\,.$$

To estimate the right-hand side of (20) we discretize the circle $r\mathbb{T}$ into $N = \big[(1-r)^{-(1+\varepsilon)}\big]$ equally-spaced points. We then use Hargé's version of the Gaussian correlation inequality to estimate, by bounding F', the probability that the maximum attained on the circle $r\mathbb{T}$ is not much bigger than the maximum attained on these points and that the value F attains at each of the points is not too large.

2.3.2 The case $L > 1$

Here we take $M = \frac{1}{\sqrt{\delta}} \left(\log \frac{1}{\delta}\right)^{\alpha}$ for some $\frac{1}{2} < \alpha < 1$. We also set $N = \left[\frac{2L}{\delta} \log \frac{1}{\delta}\right]$. We prove that the event on the right-hand side of (20) becomes typical after conditioning that the first N coefficients in the Taylor series of F are suitably small, and estimate the probability of the conditioning event.

2.3.3 The case $L = 1$

This case seems the most delicate of the lower bounds. We take $M = B\sqrt{\frac{1}{1-r}}$ for a large constant B. To estimate the probability on the right-hand side of (20) we write the Taylor series of F as an infinite sum of polynomials of degree $N = \left\lceil \frac{1}{1-r} \right\rceil$ and use an argument based on Bernstein's inequality and Hargé's version of the Gaussian correlation inequality.

3 Preliminaries

Here, we collect several lemmas, which will be used in the proof of Theorem 1.

3.1 GAFs

Lemma 1 ([6], Lemma 2.4.4)**.** *Let g be a GAF on $\mathbb{D}$, and let $\sup_{\mathbb{D}} \mathbb{E}[|g|^2] \leqslant \sigma^2$. Then, for every $\lambda > 0$,*

$$\mathbb{P}\Big[\max_{\frac{1}{2}\bar{\mathbb{D}}} |g| > \lambda\sigma\Big] \leqslant Ce^{-c\lambda^2}\,.$$

Lemma 2. *Let f be a GAF on $\mathbb{D}$, and $s \in (0, \delta)$. Put*

$$\sigma_f^2(r) = \max_{r\bar{\mathbb{D}}} \mathbb{E}[|f|^2]\,.$$

Then, for every $\lambda > 0$,

$$\mathbb{P}\Big[\max_{r\bar{\mathbb{D}}} |f| > \lambda\sigma_f\big(r+s\big)\Big] \leqslant \frac{C}{s}\cdot e^{-c\lambda^2}\,.$$

In particular, for every $\lambda > 0$,

$$\mathbb{P}\Big[\max_{r\bar{\mathbb{D}}} |f| > \lambda\sigma_f\big(\tfrac{1}{2}(1+r)\big)\Big] \leqslant C\delta^{-1}\,e^{-c\lambda^2}\,.$$

Proof of Lemma 2: Take an integer $N \simeq \frac{1}{s}$ and consider the scaled functions

$$g_j(w) = f(z_j + sw), \qquad z_j = re(j/N),\ j = 1\ldots N\,.$$

Since $\max_{r\bar{\mathbb{D}}}|f| = \max_{r\mathbb{T}}|f(z)|$, the the first statement follows by applying Lemma 1 to each g_j and using the union bound. The second statement follows from the first by taking $s = \frac{1}{2}\delta = \frac{1}{2}(1-r)$. □

3.2 A priori bounds for hyperbolic GAFs

Lemma 3. *Suppose that F is a hyperbolic GAF. Then, for $p > 1$,*

$$\mathbb{P}\Big[\max_{(1-\frac{1}{2}\delta)\bar{\mathbb{D}}} |F| \geq \sigma_F^p\Big] \leqslant C\delta^{-1}\,e^{-c\sigma_F^{2p-2}}\,.$$

Proof of Lemma 3: Since $\sigma_F\big(1-\frac{1}{4}\delta\big) \leqslant C\sigma_F$, this follows from Lemma 2. □

Lemma 4. *Suppose that F is a hyperbolic GAF. Then*

$$\mathbb{P}\Big[\max_{(1-2\delta)\bar{\mathbb{D}}} |F| \leqslant e^{-\log^2\sigma_F}\Big] \leqslant Ce^{-c\delta^{-1}\log^4\sigma_F}\,.$$

Proof of Lemma 4: Suppose that $\max_{(1-2\delta)\mathbb{D}}|F| \leqslant e^{-\log^2\sigma_F}$. Then, by Cauchy's inequalities,

$$|\zeta_n|a_n \leqslant (1-2\delta)^{-n}e^{-\log^2\sigma_F}\,,$$

whence, for $n > 0$, using the fact that $a_n \geq cn^{\frac{1}{2}(L-1)}$ (see (7)), we get

$$|\zeta_n| \leqslant Cn^{\frac{1}{2}(1-L)}(1-2\delta)^{-n}e^{-\log^2\sigma_F} < Cn^{\frac{1}{2}}e^{c\delta n-\log^2\sigma_F}\,.$$

In the range $n \simeq \frac{1}{\delta}\log^2\sigma_F$, we get

$$|\zeta_n| \leqslant C\big(\tfrac{1}{\delta}\log^2\sigma_F\big)^{\frac{1}{2}}e^{-c\log^2\sigma_F} \leqslant e^{-c\log^2\sigma_F} \tag{21}$$

provided that δ is sufficiently small. The probability that (21) holds simultaneously for all such n, does not exceed

$$\Big(e^{-c\log^2\sigma_F}\Big)^{c\delta^{-1}\log^2\sigma_F} = e^{-c\delta^{-1}\log^4\sigma_F}\,,$$

completing the proof. □

Next, we define "the good event" $\Omega_{\mathsf{g}} = \Omega_{\mathsf{g}}(r)$ by

$$\Omega_{\mathsf{g}} = \big\{\max_{(1-\frac{1}{2}\delta)\bar{\mathbb{D}}} |F| \leqslant \sigma_F^3\big\}\bigcap\big\{\max_{(1-2\delta)\bar{\mathbb{D}}} |F| \geq e^{-\log^2\sigma_F}\big\} \tag{22}$$

and note that by Lemma 3 and Lemma 4 the event Ω_{g}^c is negligible.

Lemma 5. *Suppose that F is a hyperbolic GAF. If $\gamma > 1$ then*

$$\mathrm{Hole}(r) \bigcap \Omega_{\mathsf{g}}(r) \subset \Big\{ \min_{(1-\gamma\delta)\bar{\mathbb{D}}} |F| \geq \exp\big[-C(\gamma-1)^{-1}\delta^{-3}\big] \Big\}.$$

Proof of Lemma 5: Suppose that we are on the event $\mathrm{Hole}(r) \bigcap \Omega_{\mathsf{g}}(r)$. Since F does not vanish in $r\mathbb{D}$, the function $\log|F|$ is harmonic therein, and therefore,

$$\begin{aligned}
\max_{(1-\gamma\delta)\bar{\mathbb{D}}} \log \frac{1}{|F(z)|} &= \max_{(1-\gamma\delta)\bar{\mathbb{D}}} \int_{\mathbb{T}} \frac{r^2-|z|^2}{|rt-z|^2} \log \frac{1}{|F(rt)|} \, \mathrm{d}\mu(t) \\
&\leqslant \max_{(1-\gamma\delta)\bar{\mathbb{D}}} \Big(\frac{r+|z|}{r-|z|}\Big) \int_{\mathbb{T}} |\log|F(rt)|| \, \mathrm{d}\mu(t) \\
&\leqslant \frac{2}{(\gamma-1)\delta} \int_{\mathbb{T}} |\log|F(rt)|| \, \mathrm{d}\mu(t).
\end{aligned}$$

Furthermore, on Ω_{g}, let $w \in (1-2\delta)\mathbb{T}$ be a point where $|F(w)| \geq e^{-\log^2 \sigma_F}$. Then

$$\begin{aligned}
-\big(\log^2 \sigma_F\big) \leqslant \log|F(w)| &= \int_{\mathbb{T}} \frac{r^2-|w|^2}{|rt-w|^2} \log|F(rt)| \, \mathrm{d}\mu(t) \\
&= \int_{\mathbb{T}} \frac{r^2-|w|^2}{|rt-w|^2} \big[\log_+|F(rt)| - \log_-|F(rt)|\big] \, \mathrm{d}\mu(t),
\end{aligned}$$

whence,

$$\begin{aligned}
\int_{\mathbb{T}} \log_-|F(rt)| \, \mathrm{d}\mu(t) &\leqslant \frac{r+|w|}{r-|w|} \int_{\mathbb{T}} \frac{r^2-|w|^2}{|rt-w|^2} \log_-|F(rt)| \, \mathrm{d}\mu(t) \\
&\leqslant \frac{r+|w|}{r-|w|} \Big[\int_{\mathbb{T}} \frac{r^2-|w|^2}{|rt-w|^2} \log_+|F(rt)| \, \mathrm{d}\mu(t) + \log^2 \sigma_F\Big] \\
&\leqslant \frac{r+|w|}{r-|w|} \Big[\max_{r\mathbb{T}} \log|F| + \log^2 \sigma_F\Big] \\
&\leqslant \frac{2}{\delta}\big[3\log\sigma_F + \log^2\sigma_F\big] \leqslant \frac{C}{\delta} \cdot \log^2 \sigma_F.
\end{aligned}$$

Then,

$$\max_{(1-\gamma\delta)\bar{\mathbb{D}}} \log \frac{1}{|F|} \leqslant \frac{2}{(\gamma-1)\delta} \int_{\mathbb{T}} \big|\log|F(rt)|\big| \, \mathrm{d}\mu(t) \leqslant \frac{C}{(\gamma-1)\delta^2} \cdot \log^2 \sigma_F < \frac{C}{(\gamma-1)\delta^3},$$

proving the lemma. □

3.3 Averaging $\log|F|$ over roots of unity

We start with a polynomial version.

Lemma 6. *Let S be a polynomial of degree $n \geq 1$, let $k \geq 4$ be an integer, and let $\omega = e(1/k)$. Then there exists τ with $\tau^{k^2 n} = 1$ so that*

$$\frac{1}{k}\sum_{j=1}^{k} \log|S(\tau\omega^j)| \geq \log|S(0)| - \frac{C}{k^2}\,.$$

Proof of Lemma 6: Assume that $S(0) = 1$ (otherwise, replace S by $S/S(0)$). Then $S(z) = \prod_{\ell=1}^{n}(1 - s_\ell z)$ and

$$\prod_{j=1}^{k} S(\omega^j z) = \prod_{\ell=1}^{n}(1 - s_\ell^k z^k) = S_1(z^k)\,,$$

where $S_1(w) \stackrel{\text{def}}{=} \prod_{\ell=1}^{n}(1 - s_\ell^k w)$ is a polynomial of degree n.

Let $M = \max_{\mathbb{T}}|S_1|$. By the maximum principle, $M \geq |S_1(0)| = 1$. By Bernstein's inequality, $\max_{\mathbb{T}}|S_1'| \leqslant nM$. Now, let $t_0 = e^{\mathrm{i}\varphi_0} \in \mathbb{T}$ be a point where $|S_1(t_0)| = M$. Then, for any $t = e^{\mathrm{i}\varphi} \in \mathbb{T}$ with $|\varphi - \varphi_0| < \frac{\varepsilon}{n}$ (with $0 \leqslant \varepsilon < 1$), we have

$$|S_1(t)| \geq M - \frac{\varepsilon}{n}\cdot nM = (1-\varepsilon)M \geq 1 - \varepsilon\,.$$

The roots of unity of order kn form a $\frac{\pi}{kn}$-net on $\mathbb{T}$. Thus, there exists t such that $t^{kn} = 1$ and $\log|S_1(t)| \geq -\frac{C}{k}$. Taking τ so that $\tau^k = t$, we get

$$\frac{1}{k}\sum_{j=1}^{k}\log|S(\tau\omega^j)| = \frac{1}{k}\,\log|S_1(\tau^k)| \geq -\frac{C}{k^2}\,,$$

while $\tau^{k^2 n} = 1$. □

Lemma 7. *Let f be an analytic function on $\mathbb{D}$ such that $M \stackrel{\text{def}}{=} \sup_{\mathbb{D}}|f| < +\infty$. Let $0 < \rho < 1$, and denote by q the Taylor polynomial of f around 0 of degree N. If*

$$N \geq \frac{1}{1-\rho}\,\log\frac{M}{1-\rho}\,, \tag{23}$$

then $\max_{\rho\bar{\mathbb{D}}}|f - q| < 1$.

Proof of Lemma 7: Let

$$f(z) = \sum_{n\geq 0} c_n z^n\,.$$

Then, by Cauchy's inequalities, $|c_n| \leqslant M$, whence

$$\max_{\rho\bar{\mathbb{D}}}|f-q| \leqslant \sum_{n\geq N+1}|c_n|\rho^n \leqslant M\,\frac{\rho^{N+1}}{1-\rho} = \frac{M}{1-\rho}\,(1-(1-\rho))^{N+1} < \frac{M}{1-\rho}\,e^{-N(1-\rho)} \leqslant 1. \quad \square$$

Lemma 8. *Let $0 < m < 1$, $0 < \rho < 1$, $k \geq 4$ be an integer, and $\omega = e(1/k)$. Suppose that F is an analytic function on $\mathbb{D}$ with $m \leqslant \inf_{\mathbb{D}} |F| \leqslant \sup_{\mathbb{D}} |F| \leqslant m^{-1}$ and that P is a polynomial with $P(0) \neq 0$. There exist a positive integer*

$$K \leqslant k^2 \Big[\deg P + \frac{C}{1-\rho} \log \frac{k}{m(1-\rho)} \Big] ,$$

and $\tau \in \mathbb{T}$ satisfying $\tau^K = 1$ such that

$$\frac{1}{k} \sum_{j=1}^{k} \log \Big| \frac{F}{P}(\tau \omega^j \rho) \Big| \leqslant \log \Big| \frac{F}{P}(0) \Big| + \frac{C}{k^2} .$$

Proof of Lemma 8: Let $0 < \varepsilon < m$ be a small parameter. Applying Lemma 7 to the function $f = (\varepsilon F)^{-1}$, we get a polynomial Q with

$$\deg Q \leqslant \frac{C}{1-\rho} \log \frac{1}{\varepsilon m (1-\rho)} ,$$

such that

$$\max_{\rho \bar{\mathbb{D}}} \Big| \frac{1}{F} - Q \Big| < \varepsilon .$$

Then, assuming that $\varepsilon < \frac{1}{2} m$, we get

$$\max_{\rho \bar{\mathbb{D}}} \big| \log |Q| + \log |F| \big| = \max_{\rho \bar{\mathbb{D}}} \big| \log |1 + F(Q - \tfrac{1}{F})| \, \big| \leqslant C \varepsilon m^{-1} . \tag{24}$$

Applying Lemma 6 to the polynomial $S = P \cdot Q$ and taking into account (24), we see that there exists τ so that $\tau^{k^2 \deg S} = 1$ and

$$\begin{aligned} \frac{1}{k} \sum_{j=1}^{k} \log \Big| \frac{F}{P}(\tau \omega^j \rho) \Big| &\leqslant -\frac{1}{k} \sum_{j=1}^{k} \log \Big| S(\tau \omega^j \rho) \Big| + C \varepsilon m^{-1} \\ &\leqslant -\log |S(0)| + C \big[k^{-2} + \varepsilon m^{-1} \big] \\ &\leqslant \log \Big| \frac{F}{P}(0) \Big| + C \big[k^{-2} + \varepsilon m^{-1} \big] . \end{aligned}$$

It remains to let $\varepsilon = \frac{1}{2} m k^{-2}$ and $K = k^2 \deg S = k^2 (\deg P + \deg Q)$. □

We will only need the full strength of this lemma in Section 4.3.3. Elsewhere, the following lemma will be sufficient.

Lemma 9. *Let F be a hyperbolic GAF. Let*

$$1 < \kappa \leqslant 2, \quad r_0 = 1 - \kappa \delta .$$

Let $k \geq 4$ be an integer and $\omega = e(1/k)$. Then

$$\mathbb{P}\big[\mathrm{Hole}(r)\big] \leqslant \frac{c k^2 \log k}{(\kappa - 1)^2 \delta^4} \cdot \sup_{\tau \in \mathbb{T}} \mathbb{P}\Big[\sum_{j=1}^{k} \log |F(\tau \omega^j r_0)| \leqslant k \log |F(0)| + C \Big] + \langle \text{negligible terms} \rangle .$$

Proof of Lemma 9: Outside the negligible event $\mathrm{Hole}(r) \setminus \Omega_{\mathsf{g}}$ we have the bound

$$\max_{(1-\frac{1}{2}\delta)\bar{\mathbb{D}}} |F| \leqslant \sigma_F^3 \leqslant \frac{C}{\delta^{3L/2}}.$$

According to Lemma 5 applied with $\gamma = 1 + \frac{1}{2}(\kappa - 1)$, we have

$$\mathrm{Hole}(r) \setminus \Omega_{\mathsf{g}} \subset \Big\{ \min_{(1-\gamma\delta)\bar{\mathbb{D}}} |F| \geq \exp\big[-C(\kappa-1)^{-1}\delta^{-3}\big] \Big\}.$$

Therefore, letting $m = \exp\big[-C(\kappa-1)^{-1}\delta^{-3}\big]$, we get

$$\mathbb{P}\big[\mathrm{Hole}(r)\big] \leqslant \mathbb{P}\Big[m \leqslant \min_{(1-\gamma\delta)\bar{\mathbb{D}}} |F| \leqslant \max_{(1-\gamma\delta)\bar{\mathbb{D}}} |F| \leqslant m^{-1}\Big] + \langle \text{negligible terms} \rangle\,.$$

Applying Lemma 8 to the function $F\big((1-\gamma\delta)z\big)$ with $P \equiv 1$ and

$$\rho = \frac{r_0}{(1-\gamma\delta)} = 1 - \frac{(\kappa-\gamma)\delta}{1-\gamma\delta}\,,$$

we find that

$$\Big\{ m \leqslant \min_{(1-\gamma\delta)\bar{\mathbb{D}}} |F| \leqslant \max_{(1-\gamma\delta)\bar{\mathbb{D}}} |F| \leqslant m^{-1} \Big\} \subset \bigcup_{\tau\colon \tau^K=1} \Big\{ \sum_{j=1}^{k} \log\Big|F(\tau\omega^j r_0)\Big| \leqslant k \log|F(0)| + \frac{C}{k} \Big\}$$

with some

$$K \leqslant \frac{Ck^2}{(\kappa-\gamma)\delta} \log \frac{k}{m(\kappa-\gamma)\delta} \leqslant \frac{Ck^2 \log k}{(\kappa-1)^2\delta^4}\,,$$

completing the proof. □

Note that everywhere except for Section 6 we use this lemma with $\kappa = 2$.

3.4 The covariance matrix

We will constantly exploit the fact that the covariance matrix of the random variables $F(\omega^j r)$, $\omega = e(1/N)$, $1 \leqslant j \leqslant N$, has a simple structure. This is valid for general Gaussian Taylor series, as the following lemma details.

Lemma 10. *Let F be any Gaussian Taylor series of the form* (1) *with radius of convergence R and let $z_j = re(j/N)$ for $r < R$ and $j = 0, \dots, N-1$. Consider the covariance matrix $\Sigma = \Sigma(r, N)$ of the random variables $F(z_0)$, …, $F(z_{N-1})$, that is,*

$$\Sigma_{jk} = \mathbb{E}\Big[F(z_j)\overline{F(z_k)}\Big] = \sum_{n\geq 0} a_n^2 r^{2n} e((j-k)n/N)\,.$$

Then, the eigenvalues of Σ are

$$\lambda_m = N \sum_{n \equiv m\,(N)} a_n^2 r^{2n}\,, \qquad m = 0, \dots, N-1\,,$$

where $n \equiv m\,(N)$ denotes that n is equivalent to m modulo N.

Proof of Lemma 10: Observe that

$$F(z_j) = \sum_{m=0}^{N-1} \sum_{n \equiv m\,(N)} \zeta_n a_n r^n e(jn/N) = \sum_{m=0}^{N-1} e(jm/N) \sum_{n \equiv m\,(N)} \zeta_n a_n r^n \,.$$

Define the $N \times N$ matrix U by

$$U_{jm} = \frac{1}{\sqrt{N}}\, e(jm/N)\,, \qquad 0 \leqslant j, m \leqslant N-1\,,$$

and the vector Υ by

$$\Upsilon_m = \sqrt{N} \sum_{n \equiv m\,(N)} \zeta_n a_n r^n \,.$$

Then U is a unitary matrix (it is the discrete Fourier transform matrix) and the components of Υ are independent complex Gaussian random variables with

$$\mathbb{E}\big[\,|\Upsilon_m|^2\,\big] = N \sum_{n \equiv m\,(N)} a_n^2 r^{2n} \,.$$

Finally, note that

$$\begin{pmatrix} F(z_0) \\ F(z_1) \\ \vdots \\ F(z_{N-1}) \end{pmatrix} = U\Upsilon \,.$$

Hence, the covariance matrices of $\big(F(z_0), \ldots, F(z_{N-1})\big)$ and of Υ have the same set of eigenvalues. □

3.5 Negative moments of Gaussian random variables

In the following lemmas ζ is a standard complex Gaussian random variable. Recall that, for $\theta < 2$,

$$\mathbb{E}\big[\,|\zeta|^{-\theta}\,\big] = \Gamma\big(1 - \tfrac{1}{2}\,\theta\big)\,, \tag{25}$$

where Γ is Euler's Gamma-function.

Lemma 11. *There exists a numerical constant $C > 0$ such that, for every $t > 0$ and $0 < \theta \leqslant 1$,*

$$\sup_{w \in \mathbb{C}} \mathbb{E}\Big[\,\Big|w + \frac{\zeta}{t}\Big|^{-\theta}\,\Big] \leqslant t^{\theta}(1 + C\theta)\,.$$

Proof of Lemma 11: Write

$$\mathbb{E}\Big[\,\Big|w + \frac{\zeta}{t}\Big|^{-\theta}\,\Big] = \frac{1}{\pi} \int_{\mathbb{C}} \Big|w + \frac{z}{t}\Big|^{-\theta} e^{-|z|^2}\, \mathrm{d}m(z)\,,$$

where m is the planar Lebesgue measure, and use the Hardy-Littlewood rearrangement inequality, noting that the symmetric decreasing rearrangement of $\big|w + \frac{z}{t}\big|^{-\theta}$ is $\big|\frac{z}{t}\big|^{-\theta}$, and that $e^{-|z|^2}$ is already symmetric and decreasing. □

Lemma 12. *For each $\tau > 0$ there exists a $C(\tau) > 0$ such that for every integer $n \geq 1$,*

$$\sup_{|t|\geq\tau} \mathbb{E}\Big[\Big|\log\big|1+\frac{\zeta}{t}\big|\Big|^n\Big] \leqslant C(\tau)n!\,.$$

Proof of Lemma 12: By the symmetry of ζ we may assume that $t > 0$. Put $X = \big|1+\frac{\zeta}{t}\big|$ and write

$$\mathbb{E}\big[\big|\log X\big|^n\big] = \mathbb{E}\big[\big(\log\tfrac{1}{X}\big)^n \mathbb{1}_{\{X\leqslant\frac{1}{2}\}}\big] + \mathbb{E}\big[\big(\log\tfrac{1}{X}\big)^n \mathbb{1}_{\{\frac{1}{2}<X\leqslant 1\}}\big] + \mathbb{E}\big[\big(\log X\big)^n \mathbb{1}_{\{X>1\}}\big]\,.$$

We have

$$\begin{aligned}
\mathbb{E}\big[\big(\log\tfrac{1}{X}\big)^n \mathbb{1}_{\{X\leqslant\frac{1}{2}\}}\big] &= \frac{1}{\pi}\int_{\{|1+\frac{z}{t}|\leqslant\frac{1}{2}\}} \Big(\log\big|1+\frac{z}{t}\big|^{-1}\Big)^n e^{-|z|^2}\,\mathrm{d}m(z) \\
&\leqslant \frac{e^{-t^2/4}}{\pi}\int_{\{|1+\frac{z}{t}|\leqslant\frac{1}{2}\}} \Big(\log\big|1+\frac{z}{t}\big|^{-1}\Big)^n \,\mathrm{d}m(z) \\
&= \frac{e^{-t^2/4}t^2}{\pi}\int_{\{|w|\leqslant\frac{1}{2}\}} \Big(\log\frac{1}{|w|}\Big)^n \,\mathrm{d}m(w) \\
&\leqslant C\int_0^{1/2}\Big(\log\frac{1}{s}\Big)^n s\,\mathrm{d}s \\
&= C\int_{\log 2}^{\infty} x^n e^{-2x}\,\mathrm{d}x \leqslant Cn!\,.
\end{aligned}$$

In addition,

$$\mathbb{E}\big[\big(\log\tfrac{1}{X}\big)^n \mathbb{1}_{\{\frac{1}{2}<X\leqslant 1\}}\big] \leqslant \big(\log 2\big)^n < 1\,.$$

Finally, for $t \geq \tau$,

$$\mathbb{E}\big[\big(\log X\big)^n \mathbb{1}_{\{X>1\}}\big] \leqslant \mathbb{E}\big[(X-1)_+^n\big] \leqslant \mathbb{E}\big[|X-1|^n\big] \leqslant \mathbb{E}\Big[\frac{|\zeta|^n}{t^n}\Big] \overset{(25)}{=} \frac{\Gamma(\frac{1}{2}n+1)}{t^n} \leqslant \tau^{-n}n!\,,$$

completing the proof. □

Lemma 13. *For $t > 0$,*

$$\mathbb{E}\Big[\log\Big|1+\frac{\zeta}{t}\Big|\Big] > \frac{e^{-t^2}}{2(t^2+1)}\,.$$

Proof of Lemma 13: We have

$$\mathbb{E}\Big[\log\Big|1+\frac{\zeta}{t}\Big|\Big] = \frac{1}{\pi}\int_{\mathbb{C}} \log\Big|1+\frac{z}{t}\Big|\, e^{-|z|^2}\,\mathrm{d}m(z)$$

$$= 2\int_0^\infty e^{-r^2} r \int_{-\pi}^{\pi} \log\Big|1+\frac{re^{\mathrm{i}\theta}}{t}\Big|\,\frac{\mathrm{d}\theta}{2\pi}\,\mathrm{d}r$$

$$= 2\int_0^\infty \log_+\Big(\frac{r}{t}\Big)\, e^{-r^2} r\,\mathrm{d}r$$

$$= \frac{1}{2}\int_{t^2}^\infty \frac{e^{-u}}{u}\,\mathrm{d}u = I\,.$$

Integrating by parts once again, we see that

$$I = \frac{e^{-t^2}}{2t^2} - \int_{t^2}^\infty \frac{e^{-u}}{2u^2}\,\mathrm{d}u > \frac{e^{-t^2}}{2t^2} - \frac{1}{t^2}\,I\,,$$

whence,

$$I > \frac{e^{-t^2}}{2(t^2+1)}\,,$$

completing the proof. □

Lemma 14. *There exist numerical constants $c, C > 0$ such that, for every $t > 0$ and $0 \leqslant \theta \leqslant \frac{1}{2}$,*

$$\mathbb{E}\Big[\Big|1+\frac{\zeta}{t}\Big|^{-\theta}\Big] \leqslant 1 - c\theta\,\frac{e^{-t^2}}{1+t^2} + C\theta^2\,.$$

Proof of Lemma 14: Lemma 11 yields that there exist $c, \tau > 0$ such that, for every $0 < t \leqslant \tau$ and $0 \leqslant \theta \leqslant \frac{1}{2}$, we have

$$\mathbb{E}\Big[\Big|1+\frac{\zeta}{t}\Big|^{-\theta}\Big] \leqslant 1 - c\theta\,.$$

Thus, we need to consider the case $t \geq \tau$. Write

$$\Big|1+\frac{\zeta}{t}\Big|^{-\theta} = \exp\Big(-\theta\log\Big|1+\frac{\zeta}{t}\Big|\Big) = 1 + \sum_{n\geq 1}\frac{\big(-\theta\log|1+\frac{\zeta}{t}|\big)^n}{n!}\,.$$

Using Lemma 12, we get

$$\mathbb{E}\Big[\Big|1+\frac{\zeta}{t}\Big|^{-\theta}\Big] \leqslant 1 - \theta\,\mathbb{E}\Big[\log\Big|1+\frac{\zeta}{t}\Big|\Big] + \sum_{n\geq 2}\frac{\theta^n}{n!}\,\mathbb{E}\Big[\Big|\log\Big|1+\frac{\zeta}{t}\Big|\Big|^n\Big]$$

$$\overset{0\leqslant\theta\leqslant\frac{1}{2}}{\leqslant} 1 - \theta\,\mathbb{E}\Big[\log\Big|1+\frac{\zeta}{t}\Big|\Big] + C(\tau)\,\frac{\theta^2}{1-\theta} \leqslant 1 - \theta\,\mathbb{E}\Big[\log\Big|1+\frac{\zeta}{t}\Big|\Big] + 2C(\tau)\theta^2\,.$$

Applying Lemma 13, we get the result. □

Lemma 15. *Suppose that $(\eta_j)_{1\leqslant j\leqslant N}$ are complex Gaussian random variables with covariance matrix Σ. Then, for $0 \leqslant \theta < 2$,*

$$\mathbb{E}\left[\prod_{j=1}^{N} \frac{1}{|\eta_j|^\theta}\right] \leqslant \frac{1}{\det \Sigma}\left(\Lambda^{\left(1-\frac{1}{2}\theta\right)} \cdot \Gamma\left(1-\tfrac{1}{2}\,\theta\right)\right)^N,$$

where Λ is the maximal eigenvalue of Σ.

Proof of Lemma 15: We have

$$\mathbb{E}\Big[\prod_{j=1}^{N} \frac{1}{|\eta_j|^\theta}\Big] = \frac{1}{\pi^N \det \Sigma} \int_{\mathbb{C}^N} \prod_{j=1}^{N} \frac{1}{|Z_j|^\theta}\, e^{-\langle \Sigma^{-1} Z, Z\rangle}\, \mathrm{d}m(Z_1)\ldots \mathrm{d}m(Z_N)$$

$$\leqslant \frac{1}{\pi^N \det \Sigma} \int_{\mathbb{C}^N} \prod_{j=1}^{N} \frac{1}{|Z_j|^\theta}\, e^{-\Lambda^{-1}|Z|^2}\, \mathrm{d}m(Z_1)\ldots \mathrm{d}m(Z_N)$$

$$= \frac{1}{\det \Sigma}\left(\frac{1}{\pi}\int_{\mathbb{C}} |z|^{-\theta}\, e^{-\Lambda^{-1}|z|^2}\, \mathrm{d}m(z)\right)^N$$

$$\overset{(25)}{=} \frac{1}{\det \Sigma}\left(\Lambda^{\left(1-\frac{1}{2}\theta\right)} \cdot \Gamma\left(1-\tfrac{1}{2}\,\theta\right)\right)^N,$$

proving the lemma. □

4 Upper bound on the hole probability for $0 < L < 1$

Now we are ready to prove the lower bound part of Theorem 1, in the case $0 < L < 1$. Throughout the proof, we use the parameters

$$L < \alpha < 1, \quad r_0 = 1 - 2\delta, \quad N = \left[\delta^{-\alpha}\right], \quad \omega = e(1/N),$$

where

$$\frac{1}{2} \leqslant r < 1, \quad \delta = 1 - r.$$

In many instances, we assume that r is sufficiently close to 1, that is, that δ is sufficiently small.

It will be convenient to separate the constant term from the function F, letting

$$F = F(0) + G.$$

4.1 Splitting the function G

We define two independent GAFs G_1 and G_2 so that $G = G_1 + G_2$ and

- $G_1(\omega^j r_0)$, $1 \leqslant j \leqslant N$, are independent, identically distributed Gaussian random variables with variance close to $\sigma_F^2(r_0)$;
- G_2 is a polynomial of degree $N-1$ and, for $|z| = r_0$, the variance of $G_2(z)$ is much smaller than $\sigma_F^2(r_0)$.

Let $(\zeta_n')_{n\geq 1}$ and $(\zeta_n'')_{1\leqslant n\leqslant N-1}$ be two independent sequences of independent standard complex Gaussian random variables, and let

$$G_1(z) = \sum_{n=1}^{\infty} \zeta_n' b_n z^n \,,$$
$$G_2(z) = \sum_{n=1}^{N-1} \zeta_n'' d_n z^n \,,$$

where the non-negative coefficients b_n are defined by

$$\begin{cases} b_n^2 r_0^{2n} = \sum_{k\geq 1} \Big[a_{kN}^2 r_0^{2kN} - a_{kN+n}^2 r_0^{2(kN+n)} \Big], & 1 \leqslant n \leqslant N-1, \\ b_n = a_n, & n \geq N. \end{cases}$$

Since the sequence (a_n) does not increase, the expression in the brackets is positive. For the same reason, for $1 \leqslant n \leqslant N-1$, we have

$$a_n^2 r_0^{2n} + \sum_{k\geq 1} a_{kN+n}^2 r_0^{2(kN+n)} = \sum_{k\geq 0} a_{kN+n}^2 r_0^{2(kN+n)} \geq \sum_{k\geq 1} a_{kN}^2 r_0^{2kN} \,,$$

whence, for these values of n we have $b_n \leqslant a_n$. The coefficients $d_n \geq 0$ are defined by $a_n^2 = b_n^2 + d_n^2$. This definition implies that the random Gaussian functions G and $G_1 + G_2$ have the same distribution, and we couple G, G_1 and G_2 (that is, we couple the sequences (ζ_n), (ζ_n') and (ζ_n'')) so that $G = G_1 + G_2$ almost surely.

Lemma 16. *For any $\tau \in \mathbb{T}$, the random variables $\big(G_1(\omega^j \tau r_0)\big)$, $1 \leqslant j \leqslant N$, are independent, identically distributed $\mathcal{N}_{\mathbb{C}}(0, \sigma_{G_1}^2(r_0))$ with*

$$\sigma_{G_1}^2(r_0) = N \sum_{k\geq 1} a_{kN}^2 r_0^{2kN} \,. \tag{26}$$

In addition, we have

$$0 \leqslant \sigma_F^2(r_0) - \sigma_{G_1}^2(r_0) \leqslant C \sigma_F^{2-c}(r_0) \,. \tag{27}$$

Proof of Lemma 16: Applying Lemma 10 to the function G_1 evaluated at τz we see that the eigenvalues of the covariance matrix of the random variables $(G_1(\omega^j \tau r_0))_{1\leqslant j\leqslant N}$ are all equal to $N \sum_{k\geq 1} a_{kN}^2 r_0^{2kN}$. Hence, the covariance matrix of these Gaussian random variables is diagonal, that is, they are independent, and the relation (26) holds.

To prove estimate (27), observe that

$$\sigma_{G_1}^2(r_0) \leqslant \sum_{n\geq 0} a_n^2 r_0^{2n} = \sigma_F^2(r_0),$$

and since the sequence (a_n) does not increase, we have

$$\sigma_{G_1}^2(r_0) \geq \sum_{n\geq N} a_n^2 r_0^{2n} = \sigma_F^2(r_0) - \sum_{n=0}^{N-1} a_n^2 r_0^{2n} \,.$$

Recalling (see (7)) that

$$a_n^2 = \frac{\Gamma(n+L)}{\Gamma(L)\Gamma(n+1)} \sim \frac{n^{L-1}}{\Gamma(L)}, \qquad n \to \infty \,,$$

we see that

$$\sum_{n=0}^{N-1} a_n^2 r_0^{2n} \leqslant C \sum_{n=0}^{N-1} (1+n)^{L-1} \leqslant CN^L \leqslant C\sigma_F^{2-c}(r_0)\,,$$

proving the lemma. □

We henceforth condition on $F(0)$ and G_2 (that is, on ζ_0 and $(\zeta_n'')_{1\leqslant n\leqslant N-1}$), and write

$$\mathbb{E}^{F(0),G_2}\big[\ .\ \big] = \mathbb{E}\big[\ .\ \big|\, F(0), G_2\big],$$
$$\mathbb{P}^{F(0),G_2}\big[\ .\ \big] = \mathbb{P}\big[\ .\ \big|\, F(0), G_2\big].$$

In the following section, we consider the case when $|F(0)|/\sigma_F$ is sufficiently small.

4.2 $|F(0)| \leqslant a\sigma_F$

We show that the intersection of the hole event with the event $\{|F(0)| \leqslant a\sigma_F\}$ is negligible for a sufficiently small.

By Lemma 9 (with $\kappa = 2$ and $k = N$), it suffices to estimate the probability

$$\mathbb{P}^{F(0),G_2}\Big[\sum_{j=1}^{N} \log\Big|1 + \frac{G(\omega^j \tau r_0)}{F(0)}\Big| \leqslant C\Big] = \mathbb{P}^{F(0),G_2}\Big[\prod_{j=1}^{N}\Big|1 + \frac{G(\omega^j \tau r_0)}{F(0)}\Big|^{-1} \geq e^{-C}\Big]$$

with some fixed $\tau \in \mathbb{T}$. By Chebyshev's inequality, the right-hand side is bounded by

$$C\,\mathbb{E}^{F(0),G_2}\Big[\prod_{j=1}^{N}\Big|1 + \frac{G(\omega^j \tau r_0)}{F(0)}\Big|^{-1}\Big] = C\prod_{j=1}^{N}\mathbb{E}^{F(0),G_2}\Big[\Big|1 + \frac{G(\omega^j \tau r_0)}{F(0)}\Big|^{-1}\Big]$$

where in the last equality we used the independence of $\big(G_1(\omega^j \tau r_0)\big)_{1\leqslant j\leqslant N}$ proven in Lemma 16, and the independence of G_1 and $F(0)$, G_2. Then, applying Lemma 11 with $t = |F(0)|/\sigma_{G_1}(r_0)$, we get using (27), for each $1 \leqslant j \leqslant N$,

$$\begin{aligned}\mathbb{E}^{F(0),G_2}&\Big[\Big|1 + \frac{G(\omega^j \tau r_0)}{F(0)}\Big|^{-1}\Big] \\ &= \mathbb{E}^{F(0),G_2}\Big[\Big|1 + \frac{G_2(\omega^j \tau r_0)}{F(0)} + \frac{G_1(\omega^j \tau r_0)}{F(0)}\Big|^{-1}\Big] \leqslant C\,\frac{|F(0)|}{\sigma_{G_1}} \leqslant C\cdot a < \frac{1}{2}\,,\end{aligned}$$

provided that the constant a is sufficiently small.

Thus,

$$\mathbb{P}^{F(0),G_2}\Big[\prod_{j=1}^{N}\Big|1 + \frac{G(\omega^j \tau r_0)}{F(0)}\Big|^{-1} \geq e^{-C}\Big] \leqslant C2^{-N} \leqslant Ce^{-c\delta^{-\alpha}}\,.$$

Since $\alpha > L$, this case gives a negligible contribution to the probability of the hole event.

4.3 $a\sigma_F \leqslant |F(0)| \leqslant A\sigma_F\sqrt{\log\frac{1}{(1-r)\sigma_F^2}}$

Our strategy is to make the constant A as large as possible, while keeping the hole event negligible. Then, up to negligible terms, we will bound $\mathbb{P}[\mathrm{Hole}(r)]$ by

$$\mathbb{P}\Big[|F(0)| > A\sigma_F\sqrt{\log\tfrac{1}{(1-r)\sigma_F^2}}\Big] = \exp\Big[-A^2\sigma_F^2\log\tfrac{1}{(1-r)\sigma_F^2}\Big]\,.$$

We fix a small positive parameter ε, put

$$J = \left\{1 \leqslant j \leqslant N \colon \left|1 + \frac{G_2(\omega^j r_0)}{F(0)}\right| \geq 1 + 2\varepsilon\right\},$$

and introduce the event $\big\{|J| \leqslant (1-2\eta)N\big\}$, where $\eta = \delta^{\alpha_0}$, α_0 is a sufficiently small positive constant, and $|J|$ denotes the size of the set J. We will estimate separately the probabilities of the events

$$\mathrm{Hole}(r) \bigcap \big\{a\sigma_F \leqslant |F(0)| \leqslant A\sigma_F \sqrt{\log \tfrac{1}{(1-r)\sigma_F^2}}\big\} \bigcap \big\{|J| \leqslant (1-2\eta)N\big\}$$

and

$$\mathrm{Hole}(r) \bigcap \big\{a\sigma_F \leqslant |F(0)| \leqslant A\sigma_F \sqrt{\log \tfrac{1}{(1-r)\sigma_F^2}}\big\} \bigcap \big\{|J| > (1-2\eta)N\big\}.$$

Given $\tau \in \mathbb{T}$, put

$$J_-(\tau) = \left\{1 \leqslant j \leqslant N \colon \left|1 + \frac{G_2(\omega^j \tau r_0)}{F(0)}\right| \geq 1 + \varepsilon\right\},$$
$$J_+(\tau) = \left\{1 \leqslant j \leqslant N \colon \left|1 + \frac{G_2(\omega^j \tau r_0)}{F(0)}\right| \geq 1 + 3\varepsilon\right\}.$$

Our first goal is to show that, outside a negligible event, the random sets $J_\pm(\tau)$ are similar in size to the set J.

4.3.1 Controlling G_2 at the points $\big(\omega^j \tau r_0\big)_{1 \leqslant j \leqslant N}$

Lemma 17. *Given $\alpha_0 > 0$, there is an event $\widetilde{\Omega} = \widetilde{\Omega}(r)$ with $\mathbb{P}^{F(0)}\big[\widetilde{\Omega}\big] \leqslant e^{-\delta^{-(L+c)}}$ on $\{|F(0)| \geq a\sigma_F\}$, for r sufficiently close to 1, such that, for any $\tau \in \mathbb{T}$, on the complement $\widetilde{\Omega}^c$ we have*

$$\begin{aligned} |J_-(\tau)| &\geq |J| - \eta N, \\ |J_+(\tau)| &\leqslant |J| + \eta N \end{aligned} \tag{28}$$

with $\eta = \delta^{\alpha_0}$.

Proof of Lemma 17: We take small positive constants α_1 and α_2 satisfying

$$\begin{aligned} &\alpha_1 < \max\left\{\frac{(1-\alpha)L}{2}, \frac{(1-L)\alpha_2}{2}\right\} \quad \text{and} \\ &\alpha_1 + 2\alpha_2 < \alpha, \end{aligned} \tag{29}$$

let $M = [\delta^{-\alpha_2}]$, write $G_2 = G_3 + G_4$ as follows

$$G_3(z) = \sum_{n=1}^{M} \zeta_n'' d_n z^n,$$
$$G_4(z) = \sum_{n=M+1}^{N-1} \zeta_n'' d_n z^n,$$

and define the events

$$\Omega_3 = \bigcup_{n=1}^{M} \big\{ |\zeta_n''| \geq \delta^{-\alpha_1} |F(0)| \big\},$$
$$\Omega_4 = \Big\{ \sum_{n=M+1}^{N-1} |\zeta_n''|^2 d_n^2 \geq \delta^{2\alpha_1} |F(0)|^2 \Big\}.$$

First, we show that (under our running assumption $|F(0)| \geq a\sigma_F$) the events Ω_3 and Ω_4 are negligible. Then, outside these events, we estimate the functions G_3 and G_4 at the points $\big(\omega^j \tau r_0\big)_{1 \leqslant j \leqslant N}$. We may assume without loss of generality that $|\arg(\tau)| \leqslant \pi/N$, as rotating τ by $2\pi k/N$ leaves $|J_\pm(\tau)|$ unchanged.

Lemma 18. *For r sufficiently close to 1, we have $\mathbb{P}^{F(0)}\big[\Omega_3\big] \leqslant e^{-\delta^{-(L+c)}}$ on $\{|F(0)| \geq a\sigma_F\}$.*

Proof of Lemma 18: Using the union bound, we get

$$\mathbb{P}^{F(0)}\big[\Omega_3\big] \leqslant \sum_{n=1}^{M} \mathbb{P}^{F(0)}\big[|\zeta_n''| \geq \delta^{-\alpha_1}|F(0)|\big] \leqslant \sum_{n=1}^{M} \mathbb{P}^{F(0)}\big[|\zeta_n''| \geq \delta^{-\alpha_1} a\sigma_F\big] \leqslant M e^{-ca^2\delta^{-(L+2\alpha_1)}}. \quad \square$$

Lemma 19. *For r sufficiently close to 1, we have $\mathbb{P}^{F(0)}\big[\Omega_4\big] \leqslant e^{-\delta^{-(L+c)}}$ on $\{|F(0)| \geq a\sigma_F\}$.*

Proof of Lemma 19: Put

$$X = \sum_{n=M+1}^{N-1} |\zeta_n''|^2 d_n^2 .$$

Let $\lambda > 0$ satisfy $\lambda d_n^2 < 1$ for $M+1 \leqslant n \leqslant N-1$. Then,

$$\begin{aligned}\mathbb{P}\big[X \geq t\big] = \mathbb{P}\big[e^{\lambda X} \geq e^{\lambda t}\big] &\leqslant e^{-\lambda t}\mathbb{E}\Big[\prod_{n=M+1}^{N-1} e^{\lambda|\zeta_n''|^2 d_n^2}\Big] \\ &= e^{-\lambda t}\prod_{n=M+1}^{N-1}\frac{1}{1-\lambda d_n^2} = \exp\Big[-\lambda t + \sum_{n=M+1}^{N-1}\log\frac{1}{1-\lambda d_n^2}\Big].\end{aligned}$$

We take $\lambda = 1/(2a_M^2)$, recalling that $a_n^2 = b_n^2 + d_n^2$ with (a_n) non-increasing. Then,

$$\log\frac{1}{1-\lambda d_n^2} \leqslant 2\lambda d_n^2 .$$

Thus,

$$\mathbb{P}\big[X \geq t\big] \leqslant \exp\Big[-\lambda\Big(t - 2\sum_{n=M+1}^{N-1} d_n^2\Big)\Big] \leqslant \exp\big[-\tfrac{1}{2}\lambda t\big], \tag{30}$$

provided that

$$t \geq 4\sum_{n=M+1}^{N-1} d_n^2 . \tag{31}$$

We use estimate (30) with $t = \delta^{2\alpha_1}|F(0)|^2$. Note that $\delta^{2\alpha_1}|F(0)|^2 \geq c\delta^{-(L-2\alpha_1)}$ and that

$$4\sum_{n=M+1}^{N-1} d_n^2 \leqslant 4\sum_{n=M+1}^{N-1} a_n^2 \leqslant CN^L < C\delta^{-\alpha L},$$

which satisfies (31) by (29). Finally, using again (29), we get

$$\mathbb{P}\big[X \geq \delta^{2\alpha_1}|F(0)|^2\big] \leqslant \exp\big[-\tfrac{1}{(2a_M)^2}\cdot\delta^{2\alpha_1}|F(0)|^2\big]$$

$$\leqslant \exp\big[-cM^{1-L}\delta^{2\alpha_1}|F(0)|^2\big] \leqslant \exp\big[-ca^2\delta^{-L-(1-L)\alpha_2+2\alpha_1}\big] \leqslant \exp\big[-\delta^{-(L+c)}\big]. \quad \square$$

Lemma 20. *Suppose that r is sufficiently close to 1. Then on the event Ω_3^c we have*

$$\sup_{|\arg\tau|<\pi N^{-1}}\ \sup_{1\leqslant j\leqslant N}\Big|G_3(\omega^j\tau r_0) - G_3(\omega^j r_0)\Big| < \frac{1}{2}\,\varepsilon\,|F(0)|, \qquad 1 \leqslant j \leqslant N.$$

Proof of Lemma 20: For each $1 \leqslant j \leqslant N$, since $0 \leqslant d_n \leqslant 1$, we have

$$\big|G_3(\omega^j\tau r_0) - G_3(\omega^j r_0)\big| \leqslant \max_{\bar{\mathbb{D}}}|G_3'|\cdot\frac{\pi}{N} \leqslant M\sum_{n=1}^{M}|\zeta_n''|\cdot|d_n|\cdot\frac{\pi}{N}$$
$$\leqslant \frac{CM^2}{N}\cdot\delta^{-\alpha_1}|F(0)| \leqslant C\delta^{-2\alpha_2+\alpha-\alpha_1}|F(0)| < \tfrac{1}{2}\varepsilon|F(0)|\,,$$

provided that $2\alpha_2 + \alpha_1 < \alpha$, and that r is sufficiently close to 1. $\square$

Lemma 21. *Suppose that r is sufficiently close to 1. Then, on the event Ω_4^c, for any $\tau \in \mathbb{T}$, the cardinality of the set*

$$\Big\{1 \leqslant j \leqslant N\colon\ \max\big(|G_4(\omega^j r_0)|, |G_4(\omega^j\tau r_0)|\big) \geq \tfrac{1}{4}\,\varepsilon|F(0)|\Big\}$$

does not exceed ηN.

Proof of Lemma 21: We have

$$\frac{1}{N}\sum_{j=1}^{N}|G_4(\omega^j r_0)|^2 = \frac{1}{N}\sum_{j=1}^{N}\Big|\sum_{n=M+1}^{N-1}\zeta_n'' d_n(\omega^j r_0)^n\Big|^2$$
$$= \sum_{n_1,n_2=M+1}^{N-1}\zeta_{n_1}''\overline{\zeta_{n_2}''}d_{n_1}d_{n_2}r_0^{n_1+n_2}\,\frac{1}{N}\sum_{j=1}^{N}\omega^{j(n_1-n_2)}$$
$$= \sum_{n=M+1}^{N-1}|\zeta_n''|^2 d_n^2 r_0^{2n}$$
$$\leqslant \sum_{n=M+1}^{N-1}|\zeta_n''|^2 d_n^2 \leqslant \delta^{2\alpha_1}|F(0)|^2 \qquad \text{on } \Omega_4^c\,,$$

and similarly,

$$\frac{1}{N}\sum_{j=1}^{N}|G_4(\omega^j\tau r_0)|^2 \leqslant \delta^{2\alpha_1}|F(0)|^2 \qquad \text{on } \Omega_4^c\,.$$

Hence, on Ω_4^c, the cardinality of the set we are interested in does not exceed

$$\frac{32\,\delta^{2\alpha_1}N}{\varepsilon^2} < \delta^{\alpha_0}N = \eta N\,,$$

provided that $2\alpha_1 < \alpha_0$ and that r is sufficiently close to 1. $\square$

Now, Lemma 17 is a straightforward consequence of Lemmas 18, 19, 20, and 21. $\square$

4.3.2 $|J| > (1-2\eta)N$, $\eta = \delta^{\alpha_0}$

In this section we show that the intersection of the hole event with the event

$$\Big\{a\sigma_F \leqslant |F(0)| \leqslant A\sigma_F\sqrt{\log\tfrac{1}{(1-r)\sigma_F^2}}\Big\} \cap \Big\{|J| > (1-2\eta)N\Big\} \tag{32}$$

is negligible. Taking into account the fact that $J, J_-(\tau)$ and $J_+(\tau)$ are measurable with respect to $F(0)$ and G_2, Lemma 9 (with $\kappa = 2$ and $k = N$) and Lemma 17 show that it suffices to estimate uniformly in $\tau \in \mathbb{T}$, on the intersection of the events (32) and (28), the probability

$$\mathbb{P}^{F(0),G_2}\Big[\sum_{j=1}^{N}\log\Big|1+\frac{G(\omega^j\tau r_0)}{F(0)}\Big| \leqslant C\Big].$$

Taking some positive $\theta_1 = \theta_1(r)$ tending to 0 as $r \to 1$ (the function $\theta_1(r)$ will be chosen later) and applying Chebyshev's inequality, the last probability does not exceed

$$\begin{aligned} C\,\mathbb{E}^{F(0),G_2}\Big[\prod_{j=1}^{N}\Big|1+\frac{G(\omega^j\tau r_0)}{F(0)}\Big|^{-\theta_1}\Big] &= C\,\mathbb{E}^{F(0),G_2}\Big[\prod_{j=1}^{N}\Big|1+\frac{G_2(\omega^j\tau r_0)}{F(0)}+\frac{G_1(\omega^j\tau r_0)}{F(0)}\Big|^{-\theta_1}\Big] \\ &= C\prod_{j=1}^{N}\mathbb{E}^{F(0),G_2}\Big[\Big|1+\frac{G_2(\omega^j\tau r_0)}{F(0)}+\frac{G_1(\omega^j\tau r_0)}{F(0)}\Big|^{-\theta_1}\Big]. \end{aligned} \tag{33}$$

Once again, we used the independence of $\big(G_1(\omega^j\tau r_0)\big)_{1\leqslant j\leqslant N}$ and the independence of G_1 and $F(0)$, G_2.

For $j \in J_-(\tau)$, we have

$$\begin{aligned} \Big|1+\frac{G_2(\omega^j\tau r_0)}{F(0)}+\frac{G_1(\omega^j\tau r_0)}{F(0)}\Big| &= \Big|1+\frac{G_2(\omega^j\tau r_0)}{F(0)}\Big|\cdot\Big|1+\frac{G_1(\omega^j\tau r_0)}{F(0)+G_2(\omega^j\tau r_0)}\Big| \\ &\geq (1+\varepsilon)\Big|1+\frac{G_1(\omega^j\tau r_0)}{F(0)+G_2(\omega^j\tau r_0)}\Big|. \end{aligned}$$

Hence, by Lemma 14, for such j we obtain

$$\mathbb{E}^{F(0),G_2}\Big[\Big|1+\frac{G_2(\omega^j\tau r_0)}{F(0)}+\frac{G_1(\omega^j\tau r_0)}{F(0)}\Big|^{-\theta_1}\Big] \leqslant (1+\varepsilon)^{-\theta_1}(1+C\theta_1^2) \leqslant e^{-c\varepsilon\theta_1+C\theta_1^2} < e^{-c\varepsilon\theta_1},$$

provided that r is so close to 1 that $\theta_1(r)$ is much smaller than ε (recall that ε is small but fixed).

For $j \notin J_-(\tau)$, using Lemma 11 (with $t = \frac{|F(0)|}{\sigma_G} \leqslant CA\sqrt{\log\frac{1}{(1-r)\sigma_F^2}}$), we get

$$\mathbb{E}^{F(0),G_2}\Big[\Big|1+\frac{G(\omega^j\tau r_0)}{F(0)}\Big|^{-\theta_1}\Big] \leqslant (CA\sqrt{\log\tfrac{1}{(1-r)\sigma_F^2}})^{\theta_1}(1+C\theta_1) < C(\log\tfrac{1}{(1-r)\sigma_F^2})^{\frac{1}{2}\theta_1}.$$

Thus, (33) does not exceed

$$\exp\Big[-c\varepsilon\theta_1|J_-(\tau)| + \Big(C+\frac{1}{2}\theta_1\log\log\tfrac{1}{(1-r)\sigma_F^2}\Big)(N-|J_-(\tau)|)\Big].$$

As we are on the intersection of the events (32) and (28) we have $|J_-(\tau)| \geq (1-3\eta)N$. Therefore, the expression in the last displayed formula does not exceed

$$\exp\big[-c\varepsilon\theta_1 N + C\eta N(1+\theta_1 \log\log \tfrac{1}{(1-r)\sigma_F^2})\big] \stackrel{\text{def}}{=} E.$$

Then, letting $\theta_1 = \delta^c$ with $c < \min\{\alpha_0, \alpha - L\}$, and using that

$$\theta_1 \log\log \tfrac{1}{(1-r)\sigma_F^2} \leqslant \delta^c\big(\log\log \frac{1}{\delta} + C\big) \to 0 \quad \text{as } \delta \to 0\,,$$

we see that $E \leqslant \exp\big[-c\varepsilon\theta_1 N\big]$ (recall that $\eta = \delta^{\alpha_0}$) and conclude that the event

$$\mathrm{Hole}(r) \bigcap \big\{a\sigma_F \leqslant |F(0)| \leqslant A\sigma_F \sqrt{\log \tfrac{1}{(1-r)\sigma_F^2}}\big\} \bigcap \big\{|J| > (1-2\eta)N\big\}$$

is negligible.

4.3.3 $|J| \leqslant (1-2\eta)N$, $\eta = \delta^{\alpha_0}$

Here we show that the intersection of the hole event with the event

$$\Big\{a\sigma_F \leqslant |F(0)| \leqslant A\sigma_F \sqrt{\log \tfrac{1}{(1-r)\sigma_F^2}}\Big\} \cap \Big\{|J| \leqslant (1-2\eta)N\Big\} \tag{34}$$

is negligible, provided that $A^2 < \frac{1}{2}$.

In this case, our starting point is Lemma 8 which we apply with the polynomial $P = F(0) + G_2$. Combined with Lemma 5 and Lemma 17, it tells us that it suffices to estimate uniformly in $\tau \in \mathbb{T}$, on the intersection of the events (34) and (28), the probability

$$\mathbb{P}^{F(0),G_2}\Big[\sum_{j=1}^N \log\Big|\frac{F}{P}(\omega^j \tau r_0)\Big| \leqslant C\Big]\,.$$

Noting that

$$\frac{F}{P} = 1 + \frac{G_1}{F(0)+G_2},$$

we rewrite this expression as

$$\begin{aligned}
&\mathbb{P}^{F(0),G_2}\Big[\prod_{j=1}^N \Big|1 + \frac{G_1(\omega^j\tau r_0)}{F(0)+G_2(\omega^j\tau r_0)}\Big| \leqslant e^C\Big] \\
&\qquad = \mathbb{P}^{F(0),G_2}\Big[\prod_{j=1}^N \Big|1 + \frac{G_1(\omega^j\tau r_0)}{F(0)+G_2(\omega^j\tau r_0)}\Big|^{-\theta_2} \geq e^{-C\theta_2}\Big] \\
&\qquad \leqslant e^{C\theta_2}\, \mathbb{E}^{F(0),G_2}\Big[\prod_{j=1}^N \Big|1 + \frac{G_1(\omega^j\tau r_0)}{F(0)+G_2(\omega^j\tau r_0)}\Big|^{-\theta_2}\Big]\,.
\end{aligned}$$

The positive $\theta_2 = \theta_2(r)$ (again tending to 0 as $r \to 1$) will be chosen later. By the independence of $(G_1(\omega^j\tau r_0)_{1\leqslant j\leqslant N}$ proven in Lemma 16, and the independence of G_1 and $F(0)$, G_2, it suffices to estimate the product

$$\prod_{j=1}^N \mathbb{E}^{F(0),G_2}\Big[\Big|1 + \frac{G_1(\omega^j\tau r_0)}{F(0)+G_2(\omega^j\tau r_0)}\Big|^{-\theta_2}\Big]\,.$$

First we consider the terms with $j \in J_+(\tau)$. In this case, by Lemma 14, we have

$$\mathbb{E}^{F(0),G_2}\Big[\Big|1+\frac{G_1(\omega^j\tau r_0)}{F(0)+G_2(\omega^j\tau r_0)}\Big|^{-\theta_2}\Big] \leqslant 1 + C\theta_2^2 < e^{C\theta_2^2}\,. \tag{35}$$

For the terms with $j \notin J_+(\tau)$, we apply Lemma 14 with

$$\begin{aligned} t = \frac{|F(0)+G_2(\omega^j\tau r_0)|}{\sigma_{G_1}} &\leqslant (1+3\varepsilon)\frac{|F(0)|}{\sigma_{G_1}} \\ &= (1+3\varepsilon)\frac{\sigma_F}{\sigma_{G_1}}\,\frac{|F(0)|}{\sigma_F} \overset{(27)}{\leqslant} (1+4\varepsilon)A\sqrt{\log\tfrac{1}{(1-r)\sigma_F^2}}\,, \end{aligned}$$

provided that r is sufficiently close to 1. Then, by Lemma 14 (using that $(1-r)\sigma_F^2$ is sufficiently small)

$$\mathbb{E}^{F(0),G_2}\Big[\Big|1+\frac{G_1(\omega^j\tau r_0)}{F(0)+G_2(\omega^j\tau r_0)}\Big|^{-\theta_2}\Big] \leqslant 1 - c\theta_2\big((1-r)\sigma_F^2\big)^{(1+10\varepsilon)A^2} + C\theta_2^2\,.$$

Assuming that

$$\theta_2 = o(1)\big((1-r)\sigma_F^2\big)^{(1+10\varepsilon)A^2}\,, \tag{36}$$

we continue our estimate as follows

$$\leqslant 1 - c\theta_2\big((1-r)\sigma_F^2\big)^{(1+10\varepsilon)A^2} \leqslant \exp\big[-c\theta_2\big((1-r)\sigma_F^2\big)^{(1+10\varepsilon)A^2}\big]\,. \tag{37}$$

Multiplying the bounds (35) and (37), we get

$$\begin{aligned} &\prod_{j=1}^N \mathbb{E}^{F(0),G_2}\Big[\Big|1+\frac{G_1(\omega^j\tau r_0)}{F(0)+G_2(\omega^j\tau r_0)}\Big|^{-\theta_2}\Big] \\ &\qquad\qquad \leqslant \exp\big[C\theta_2^2|J_+(\tau)| - c\theta_2\big((1-r)\sigma_F^2\big)^{(1+10\varepsilon)A^2}(N-|J_+(\tau)|)\big]\,. \end{aligned}$$

As we are on the intersection of the events (34) and (28), we have $|J_+(\tau)| \leqslant |J| + \eta N \leqslant (1-\eta)N$. Then, the expression in the exponent on the right-hand side of the previous displayed equation does not exceed

$$\big(C\theta_2^2 - c\eta\theta_2\big((1-r)\sigma_F^2\big)^{(1+10\varepsilon)A^2}\big)N\,.$$

Letting $\theta_2 = c\eta\big((1-r)\sigma_F^2\big)^{(1+10\varepsilon)A^2}$ (which satisfies our previous requirement (36) since $\eta \to 0$ as $r \to 1$), we estimate the previous expression by

$$-c\eta^2\big((1-r)\sigma_F^2\big)^{2(1+10\varepsilon)A^2}N = -c\delta^{2\alpha_0+2(1+10\varepsilon)A^2(1-L)-\alpha}\,.$$

To make the event with probability bounded by $\exp\big[-c\delta^{2\alpha_0+2(1+10\varepsilon)A^2(1-L)-\alpha}\big]$ negligible, we need to be sure that $\alpha - 2(1+10\varepsilon)A^2(1-L) - 2\alpha_0 > L$. Since the constants ε and α_0 can be made arbitrarily small, while α can be made arbitrarily close to 1, we conclude that the event

$$\mathrm{Hole}(r)\bigcap\big\{a\sigma_F \leqslant |F(0)| \leqslant A\sigma_F\sqrt{\log\tfrac{1}{(1-r)\sigma_F^2}}\big\}\bigcap\{|J| \leqslant (1-2\eta)N\}$$

is negligible, provided that $A^2 < \frac{1}{2}$.

We conclude that the event

$$\mathrm{Hole}(r) \bigcap \big\{ a\sigma_F < |F(0)| < A\sigma_F \sqrt{\log \tfrac{1}{(1-r)\sigma_F^2}} \big\}$$

is negligible whenever $A^2 < \frac{1}{2}$, and therefore, combined with the bound of Section 4.2, for any such A,

$$\begin{aligned} \mathbb{P}\big[\mathrm{Hole}(r)\big] &\leqslant \mathbb{P}\Big[|F(0)| \geq A\sigma_F \sqrt{\log \tfrac{1}{(1-r)\sigma_F^2}} \Big] + \langle \text{negligible terms} \rangle \\ &= \exp\big(-A^2 \sigma_F^2 \log \tfrac{1}{(1-r)\sigma_F^2} \big) + \langle \text{negligible terms} \rangle \,, \end{aligned}$$

whence, letting $r \uparrow 1$,

$$\begin{aligned} \mathbb{P}\big[\mathrm{Hole}(r)\big] &\leqslant \exp\Big[-\frac{1-o(1)}{2}\, \sigma_F^2 \log \tfrac{1}{(1-r)\sigma_F^2} \Big] \\ &= \exp\Big[-\frac{1-o(1)}{2} \cdot \frac{1}{(2\delta)^L} \cdot (1-L) \log \frac{1}{\delta} \Big] \\ &= \exp\Big[-\frac{1-L-o(1)}{2^{L+1}} \cdot \frac{1}{\delta^L} \log \frac{1}{\delta} \Big] \end{aligned}$$

completing the proof of the upper bound in the case $0 < L < 1$. □

5 Lower bound on the hole probability for $0 < L < 1$

As before, let $F = F(0) + G$. Fix $\varepsilon > 0$, set

$$M = \frac{\sqrt{1-L+2\varepsilon}}{(1-r^2)^{L/2}} \cdot \sqrt{\log \frac{1}{1-r}} \,,$$

define the events

$$\Omega_1 = \big\{ |F(0)| > M \big\}, \qquad \Omega_2 = \big\{ \max_{r\mathbb{D}} |G| \leqslant M \big\} = \big\{ \max_{r\mathbb{T}} |G| \leqslant M \big\} \,,$$

and observe that $\mathrm{Hole}(r) \supset \Omega_1 \bigcap \Omega_2$ and that Ω_1 and Ω_2 are independent.

Put

$$N = \big[(1-r)^{-1-\varepsilon} \big] \,, \quad \omega = e(1/N) \,, \quad z_j = r\omega^j, \ 1 \leqslant j \leqslant N \,,$$

$$M' = \frac{\sqrt{1-L+\varepsilon}}{(1-r^2)^{L/2}} \sqrt{\log \frac{1}{1-r}} \,, \quad M'' = \frac{A}{(1-r^2)^{(L+2)/2}} \sqrt{\log \frac{1}{1-r}} \,,$$

with a sufficiently large positive constant A that will be chosen later, and define the events

$$\Omega_3 = \Big\{ \max_{1 \leqslant j \leqslant N} |G(z_j)| \leqslant M' \Big\} \,, \quad \Omega_4 = \Big\{ \max_{r\mathbb{T}} |G'| \leqslant M'' \Big\} \,.$$

Then, $\Omega_2 \supset \Omega_3 \bigcap \Omega_4$ when r is sufficiently close to 1 as a function of ε. Hence,

$$\mathbb{P}\big[\mathrm{Hole}(r)\big] \geq \mathbb{P}\big[\Omega_1\big] \cdot \mathbb{P}\big[\Omega_3 \bigcap \Omega_4\big] \,.$$

For $K \in \mathbb{N}$, we put $\lambda = e\big(1/2^K\big)$, $w_k = \lambda^k r$, $1 \leqslant k \leqslant 2^K$ and define

$$\Omega_4^K = \Big\{ \max_{1 \leqslant k \leqslant 2^K} |G'(w_k)| \leqslant M'' \Big\}.$$

Notice that $\Omega_4^K \downarrow \Omega_4$ as $K \to \infty$. In order to bound $\mathbb{P}\big[\,\Omega_3 \bigcap \Omega_4\,\big]$ from below we use Hargé's version [5] of the Gaussian correlation inequality[1]:

Theorem (G. Hargé) *Let γ be a Gaussian measure on $\mathbb{R}^n$, let $A \subset \mathbb{R}^n$ be a convex symmetric set, and let $B \subset \mathbb{R}^n$ be an ellipsoid (that is, a set of the form $\{X \in \mathbb{R}^n \colon \langle CX, X\rangle \leqslant 1\}$, where C is a non-negative symmetric matrix). Then $\gamma(A \bigcap B) \geq \gamma(A)\gamma(B)$.*

We apply this inequality N times to the Gaussian measure on $\mathbb{R}^{2\left(N-j+1+2^K\right)}$, $1 \leqslant j \leqslant N$, generated by

$$X_j^r = \operatorname{Re} G\left(z_j\right), \ldots, X_N^r = \operatorname{Re} G\left(z_N\right),$$
$$X_j^i = \operatorname{Im} G\left(z_j\right), \ldots, X_N^i = \operatorname{Im} G\left(z_N\right),$$

and

$$X_{N+1}^r = \operatorname{Re} G'(w_1), \ldots, X_{N+2^K}^r = \operatorname{Re} G'\left(w_{2^K}\right),$$
$$X_{N+1}^i = \operatorname{Im} G'(w_1), \ldots, X_{N+2^K}^i = \operatorname{Im} G'\left(w_{2^K}\right),$$

and the sets

$$A_j = \left\{\max_{j+1 \leqslant k \leqslant N} |G(z_j)| \leqslant M'\right\} \cap \left\{\max_{1 \leqslant k \leqslant 2^K} |G'(w_k)| \leqslant M''\right\},$$

$$B_j = \left\{|G(z_j)|^2 = (X_j^r)^2 + (X_j^i)^2 \leqslant (M')^2\right\}.$$

Thus, we get

$$\mathbb{P}[\Omega_3 \cap \Omega_4^K] \geq \prod_{j=1}^N \mathbb{P}\big[|G(z_j)| \leqslant M'\,\big] \cdot \mathbb{P}[\Omega_4^K].$$

Thus, by the monotone convergence of Ω_4^K to Ω_4,

$$\mathbb{P}\big[\Omega_3 \bigcap \Omega_4\big] \geq \mathbb{P}\big[\max_{r\bar{\mathbb{D}}} |G'| \leqslant M''\big] \cdot \prod_{j=1}^N \mathbb{P}\big[|G(z_j)| \leqslant M'\big].$$

For each $1 \leqslant j \leqslant N$, $G(z_j)$ is a complex Gaussian random variable with variance at most $(1-r^2)^{-L}$, so that

$$\prod_{j=1}^N \mathbb{P}\big[|G(z_j)| \leqslant M'\big] \geq \left(1 - e^{-M'^2(1-r^2)^L}\right)^N = \left(1 - e^{(1-L+\varepsilon)\,\log(1-r)}\right)^N = \left(1 - (1-r)^{1-L+\varepsilon}\right)^N$$
$$\geq \exp\big[-cN(1-r)^{1-L+\varepsilon}\big] = \exp\big[-c(1-r)^{-1-\varepsilon+1-L+\varepsilon}\big] = \exp\big[-c(1-r)^{-L}\big]$$

with r sufficiently close to 1.

Next note that

$$G'(z) = \sum_{n \geq 0} \zeta_{n+1}(n+1)a_{n+1}z^n,$$

[1] Recently, Royen [16] proved the full version of the Gaussian correlation inequality, see also the paper by Latała and Matlak [8].

and therefore we have

$$\max_{s\bar{\mathbb{D}}} \mathbb{E}\big[|G'|^2\big] \leqslant C(1-s^2)^{-(L+2)}\,, \qquad 0 < s < 1\,.$$

Thus, noting that

$$M'' \geq cA\sqrt{\log(1-r)^{-1}} \cdot \sigma_{G'}(\tfrac{1}{2}(1+r))$$

and applying Lemma 2, we see that

$$\mathbb{P}\big[\max_{r\bar{\mathbb{D}}} |G'| > M''\big] \leqslant \frac{C}{1-r} \exp\big[cA^2 \log(1-r)\big] < \tfrac{1}{2}\,,$$

provided that the constant A in the definition of M'' is chosen sufficiently large. Thus, $\mathbb{P}\big[\max_{r\bar{\mathbb{D}}} |G'| \leqslant M''\big] \geq \frac{1}{2}$. Then, piecing everything together, we get

$$\begin{aligned}
\mathbb{P}\big[\mathrm{Hole}(r)\big] &\geq \mathbb{P}\big[\Omega_1\big] \cdot \mathbb{P}\big[\Omega_3 \bigcap \Omega_4\big] \\
&\geq \mathbb{P}\big[\Omega_1\big] \cdot \mathbb{P}\big[\max_{r\bar{\mathbb{D}}} |G'| \leqslant M''\big] \cdot \prod_{j=1}^{N} \mathbb{P}\big[|G(z_j)| \leqslant M'\big] \\
&\geq \mathbb{P}\big[\Omega_1\big] \cdot \tfrac{1}{2} \exp\big[-c(1-r)^{-L}\big]\,.
\end{aligned}$$

Finally, the theorem follows from the fact that $\mathbb{P}\big[\Omega_1\big] = e^{-M^2}$. □

5.1 Remark

Having in mind the gap between the upper and lower bounds on the hole probability for $0 < L < 1$, as given in Theorem 1, we note here that a different method would be required in order to improve the lower bound. Precisely, setting $F = F(0) + G$ as before, we will show that

$$\begin{aligned}
&\sup_M \mathbb{P}\Big[|F(0)| > M,\ \max_{r\bar{\mathbb{D}}} |G| \leqslant M\Big] \\
&\qquad = \exp\left(-\frac{1-L+o(1)}{2^L} \frac{1}{(1-r)^L} \log\frac{1}{1-r}\right), \quad \text{as } r \uparrow 1. \qquad (38)
\end{aligned}$$

Our proof above shows that this holds with a greater or equal sign instead of the equality sign and it remains to establish the opposite inequality. It is clear that

$$\mathbb{P}\Big[|F(0)| > M,\ \max_{r\bar{\mathbb{D}}} |G| \leqslant M\Big] \leqslant \mathbb{P}\Big[|F(0)| > M\Big] = e^{-M^2}$$

and hence the opposite inequality need only be verified in the regime where

$$M \leqslant \frac{\sqrt{1-L-o(1)}}{(1-r^2)^{L/2}} \cdot \sqrt{\log\frac{1}{1-r}}, \quad \text{as } r \uparrow 1.$$

Now let $0 < \varepsilon < \frac{1}{2}(1-L)$ and suppose that

$$M \leqslant \frac{\sqrt{1-L-2\varepsilon}}{(1-r^2)^{L/2}} \cdot \sqrt{\log\frac{1}{1-r}}. \qquad (39)$$

Set also $N = \big[(1-r)^{-1+\epsilon}\big]$ and $\omega = e(1/N)$. Let G_1 and G_2 be as in Section 4.1, so that $G = G_1 + G_2$ and the random variables $(G_1(\omega^j r))$, $1 \leqslant j \leqslant N$, are independent and identically

distributed by Lemma 16. (The decomposition in Section 4.1 yields independent random variables at radius $r_0 = 1 - 2(1-r)$ but may easily be modified to radius r (or indeed any radius).)

$$\mathbb{P}\Big[|F(0)| > M,\ \max_{r\bar{\mathbb{D}}} |G| \leqslant M\Big] \leqslant \mathbb{P}\Big[\max_{1\leqslant j\leqslant N} |G_1(\omega^j r) + G_2(\omega^j r)| \leqslant M\Big].$$

We condition on G_2 and write $\mathbb{P}^{G_2}\big[\ .\ \big] = \mathbb{P}\big[\ .\ \big|\, G_2\big]$. Recalling that $(G_1(\omega^j r))$ are independent and applying the Hardy-Littlewood rearrangement inequality, we get

$$\begin{aligned}\mathbb{P}^{G_2}\Big[\max_{1\leqslant j\leqslant N} |G_1(\omega^j r) + G_2(\omega^j r)| \leqslant M\Big] &= \prod_{j=1}^{N} \mathbb{P}^{G_2}\Big[|G_1(\omega^j r) + G_2(\omega^j r)| \leqslant M\Big] \\ &\leqslant \prod_{j=1}^{N} \mathbb{P}^{G_2}\Big[|G_1(\omega^j r)| \leqslant M\Big] = \Big(1 - e^{-M^2/\sigma^2_{G_1}(r)}\Big)^N \leqslant e^{-N\exp(-M^2/\sigma^2_{G_1}(r))}.\end{aligned}$$

Since the right-hand side does not depend on G_2, we can drop the conditioning on the left-hand side and finally get

$$\mathbb{P}\Big[|F(0)| > M,\ \max_{r\bar{\mathbb{D}}} |G| \leqslant M\Big] \leqslant e^{-N\exp(-M^2/\sigma^2_{G_1}(r))}.$$

It remains to note that with our choice of N and using the upper bound (39) on M and the relation of $\sigma^2_F(r)$ and $\sigma^2_{G_1}(r)$ given in Lemma 16 we have

$$N\exp(-M^2/\sigma^2_{G_1}(r)) \geq \Big(\frac{1}{1-r}\Big)^{L+\varepsilon-o(1)}, \quad \text{as } r \uparrow 1.$$

We conclude that

$$\mathbb{P}\Big[|F(0)| > M,\ \max_{r\bar{\mathbb{D}}} |G| \leqslant M\Big] \leqslant \exp\Big[-\Big(\frac{1}{1-r}\Big)^{L+\varepsilon-o(1)}\Big], \quad \text{as } r \uparrow 1.$$

for all M satisfying the upper bound (39). As ε can be taken arbitrarily small, this completes the proof of (38).

6 Upper bound on the hole probability for $L > 1$

6.1 Beginning the proof

Put $\delta = 1 - r$, $1 < \kappa \leqslant 2$ and $r_0 = 1 - \kappa\delta$. We take

$$N = \Big[\frac{L-1}{2\delta}\log\frac{1}{\delta}\Big] \tag{40}$$

and put $\omega = e(1/N)$. By Lemma 9, it suffices to estimate the probability

$$\sup_{\tau\in\mathbb{T}} \mathbb{P}\Big[\sum_{j=1}^{N} \log|F(\omega^j \tau r_0)| \leqslant N\log|F(0)| + C\Big].$$

Set

$$a = \Big(\log\frac{1}{\delta}\Big)^{-\frac{1}{2}}. \tag{41}$$

Discarding a negligible event, we assume that $|F(0)| \leqslant \delta^{-(\frac{1}{2}+a)}$. Let $0 < \theta < 2$ be a parameter depending on δ which we will choose later. Then,

$$\mathbb{P}\Big[\sum_{j=1}^{N} \log|F(\omega^j \tau r_0)| \leqslant N \log|F(0)| + C\Big] \leqslant \mathbb{P}\Big[\prod_{j=1}^{N} \big|F(\omega^j \tau r_0)\big|^{-\theta} \geq C\delta^{N\theta(\frac{1}{2}+a)}\Big]$$
$$\leqslant C\delta^{-N\theta(\frac{1}{2}+a)}\, \mathbb{E}\Big[\prod_{j=1}^{N} \big|F(\omega^j \tau r_0)\big|^{-\theta}\Big].$$

Using Lemma 15, we can bound the expectation on the right by

$$\frac{1}{\det \Sigma}\Big(\Lambda^{\left(1-\frac{1}{2}\theta\right)} \Gamma\big(1-\tfrac{1}{2}\theta\big)\Big)^N,$$

where Σ is the covariance matrix of $\big(F(\omega^j \tau r_0)\big)_{1\leqslant j\leqslant N}$ and Λ is the maximal eigenvalue of Σ. Note that, since the distribution of $F(z)$ is rotation invariant, the covariance matrix Σ does not depend on τ.

6.2 Estimating the eigenvalues of Σ

By Lemma 10, the eigenvalues of Σ are

$$\lambda_m(\Sigma) = N \sum_{n\equiv m\,(N)} a_n^2 r_0^{2n} = N\Big(a_m^2 r_0^{2m} + \sum_{j\geq 1} a_{m+jN}^2 r_0^{2(m+jN)}\Big), \qquad m = 0, 1, \ldots, N-1\,.$$

Now, for small δ and $j \geq 1$, we have

$$\frac{a_{m+(j+1)N}^2 r_0^{2(m+(j+1)N)}}{a_{m+jN}^2 r_0^{2(m+jN)}} \leqslant C\Big(\frac{m+(j+1)N}{m+jN}\Big)^{L-1} r_0^{2N}$$
$$= C\Big(1+\frac{N}{m+jN}\Big)^{L-1}\big(1-\kappa\delta\big)^{2N} \leqslant Ce^{-2\kappa\delta N}\,.$$

Take δ sufficiently small so that

$$\frac{a_{m+(j+1)N}^2 r_0^{2(m+(j+1)N)}}{a_{m+jN}^2 r_0^{2(m+jN)}} \leqslant \frac{1}{2}\,,$$

which yields

$$\lambda_m(\Sigma) \leqslant N\Big(a_m^2(1-\kappa\delta)^{2m} + a_{m+N}^2(1-\kappa\delta)^{2(m+N)} \sum_{j\geq 1} 2^{-j}\Big)$$
$$= N\big(a_m^2(1-\kappa\delta)^{2m} + a_{m+N}^2(1-\kappa\delta)^{2(m+N)}\big)\,.$$

Put

$$M \stackrel{\text{def}}{=} \frac{Lr_0^2-1}{1-r_0^2} = (1+o(1))\,\frac{L-1}{2\kappa}\,\frac{1}{\delta} \leqslant \frac{C}{\delta} \qquad \text{as } \delta \to 0\,,$$

and note that the sequence

$$m \mapsto a_m^2 r_0^{2m} = \frac{L(L+1)\,\ldots\,(L+m-1)}{m!}\, r_0^{2m}$$

increases for $m \leqslant [M]$ and decreases for $m \geq [M]+1$. Thus, the maximal term of this sequence does not exceed

$$CM^{L-1}(1-\kappa\delta)^{2M} \leqslant \frac{C}{\delta^{L-1}}\,,$$

whence

$$\lambda_m(\Sigma) \leqslant \frac{CN}{\delta^{L-1}}\,, \qquad m = 0, 1, \ldots, N-1\,.$$

Also

$$\begin{aligned}\det(\Sigma) = \prod_{m=0}^{N-1} \lambda_m(\Sigma) &\geq \prod_{m=0}^{N-1} N a_m^2 (1-\kappa\delta)^{2m} \geq N^N (1-\kappa\delta)^{N(N-1)} \prod_{m=0}^{N-1} c(m+1)^{L-1} \\ &\geq (cN)^N (N!)^{L-1} (1-\kappa\delta)^{N(N-1)} = \exp\big[(1+o(1))(LN\log N - \kappa\delta N^2)\big]\end{aligned}$$

as $\delta \to 0$.

6.3 Completing the proof

Finally,

$$\begin{aligned}\log \mathbb{E}\Big[\prod_{j=1}^{N} |F(\omega^j \tau r_0)|^{-\theta}\Big] &\leqslant -\log\det\Sigma + N\big(1-\tfrac{1}{2}\theta\big)\log\Lambda + N\log\Gamma\big(1-\tfrac{1}{2}\theta\big) \\ &\leqslant -(1+o(1))\big(LN\log N - \kappa\delta N^2\big) \\ &\qquad + N\big(1-\tfrac{1}{2}\theta\big)\big(\log N + (L-1)\log\tfrac{1}{\delta} + O(1)\big) + N\log\Gamma\big(1-\tfrac{1}{2}\theta\big)\,,\end{aligned}$$

and then,

$$\begin{aligned}&\log \mathbb{P}\Big[\sum_{j=1}^{N} \log|F(\omega^j \tau r_0)| \leqslant N\log|F(0)| + C\Big] \\ &\quad\leqslant N\theta\big(\tfrac{1}{2}+a\big)\log\tfrac{1}{\delta} + O(1) + \log\mathbb{E}\Big[\prod_{j=1}^{N}|F(\omega^j\tau r_0)|^{-\theta}\Big] \\ &\quad\leqslant N\theta\big(\tfrac{1}{2}+a\big)\log\tfrac{1}{\delta} + O(1) - (1+o(1))\big(LN\log N - \kappa\delta N^2\big) \\ &\qquad + N\big(1-\tfrac{1}{2}\theta\big)\big(\log N + (L-1)\log\tfrac{1}{\delta} + O(1)\big) + N\log\Gamma\big(1-\tfrac{1}{2}\theta\big) \\ &\quad\stackrel{\text{def}}{=} (1+o(1))N\cdot P_N.\end{aligned}$$

We set $\theta = 2-a^2$, $\kappa = 1+\delta$, and continue to bound P_N (using the choices (40) and (41)):

$$\begin{aligned}P_N &= (2-a^2)(\tfrac{1}{2}+a)\log\tfrac{1}{\delta} - L\log N + (1+\delta)\delta N + \tfrac{a^2}{2}(\log N + (L-1)\log\tfrac{1}{\delta}) + \log\Gamma(\tfrac{a^2}{2}) \\ &= \log\tfrac{1}{\delta} + O(\sqrt{\log\tfrac{1}{\delta}}) - L\log\tfrac{1}{\delta} + \tfrac{L-1}{2}\log\tfrac{1}{\delta} + O(\delta\log\tfrac{1}{\delta}) + O(\log\log\tfrac{1}{\delta}) + O(1) + O(\log\log\tfrac{1}{\delta}) \\ &= -\tfrac{L-1}{2}\log\tfrac{1}{\delta}(1+o(1)), \quad \delta\to 0.\end{aligned}$$

Thus,

$$\log \mathbb{P}\Big[\sum_{j=1}^{N} \log |F(\omega^j \tau r_0)| \leqslant N \log |F(0)| + C\Big] \leqslant -(1+o(1))\, \frac{(L-1)^2}{4}\, \frac{1}{\delta} \Big(\log \frac{1}{\delta}\Big)^2 .$$

From Lemma 9 we have

$$\begin{aligned} \log \mathbb{P}\big[\mathrm{Hole}(r)\big] &\leqslant C \log \frac{1}{\delta} + 2 \log \frac{1}{\kappa - 1} - (1+o(1))\, \frac{(L-1)^2}{4}\, \frac{1}{\delta} \Big(\log \frac{1}{\delta}\Big)^2 \\ &= -(1+o(1))\, \frac{(L-1)^2}{4}\, \frac{1}{\delta} \Big(\log \frac{1}{\delta}\Big)^2 , \end{aligned}$$

by our choice of κ, completing the proof of the upper bound in the case $L > 1$. □

7 Lower bound on the hole probability for $L > 1$

As before, let $\delta = 1 - r$ and assume that r is sufficiently close to 1. Introduce the parameter

$$\tfrac{1}{2} < \alpha < 1 ,$$

and put

$$N = \Big[\, \frac{2L}{\delta} \log \frac{1}{\delta} \Big], \quad M = \frac{1}{\sqrt{\delta}} \Big(\log \frac{1}{\delta}\Big)^{\alpha} .$$

We now introduce the events

$$\mathcal{E}_1 = \big\{ |\zeta_0| > M \big\} , \ \mathcal{E}_2 = \Big\{ \max_{r\mathbb{T}} \Big| \sum_{1 \leqslant n \leqslant N} \zeta_n a_n z^n \Big| \leqslant \frac{M}{2} \Big\} , \ \mathcal{E}_3 = \Big\{ \max_{r\mathbb{T}} \Big| \sum_{n > N} \zeta_n a_n z^n \Big| \leqslant \frac{M}{2} \Big\}.$$

Then

$$\mathbb{P}\big[\mathrm{Hole}(r)\big] \geq \mathbb{P}\big[\mathcal{E}_1\big] \cdot \mathbb{P}\big[\mathcal{E}_2\big] \cdot \mathbb{P}\big[\mathcal{E}_3\big] .$$

We have

$$\mathbb{P}\big[\mathcal{E}_1\big] = \exp\big[-M^2\big] = \exp\Big[-\frac{1}{\delta} \Big(\log \frac{1}{\delta}\Big)^{2\alpha}\Big].$$

In order to give a lower bound for $\mathbb{P}\big[\mathcal{E}_2\big]$, we rely on a comparison principle between Gaussian analytic functions which might be of use in other contexts. To introduce this principle let us say that a random analytic function G has the $\mathrm{GAF}(b_n)$ distribution, for some sequence of complex numbers $(b_n)_{n \geq 0}$, if G has the same distribution as

$$z \mapsto \sum_{n \geq 0} \zeta_n b_n z^n, \quad z \in \mathbb{C},$$

where, as usual, (ζ_n) is a sequence of independent standard complex Gaussian random variables.

Lemma 22. *Let (b_n), (c_n), $n \geq 0$, be two sequences of complex numbers, such that $|c_n| \leqslant |b_n|$ for all n, and put*

$$Q = \prod_{n \geq 0} \Big| \frac{c_n}{b_n} \Big| ,$$

where we take the ratio c_n/b_n to be 1 if both b_n and c_n are zero. If $Q > 0$, then there exists a probability space supporting a random analytic function G with the $\mathrm{GAF}(b_n)$ distribution, and an event E satisfying $\mathbb{P}[E] = Q^2$, such that, conditioned on the event E, the function G has the $\mathrm{GAF}(c_n)$ distribution.

The proof of Lemma 22 uses the following simple property of Gaussian random variables.

Lemma 23. *Let $0 < \sigma \leqslant 1$. There exists a probability space supporting a standard complex Gaussian random variable ζ, and an event E satisfying $\mathbb{P}[E] = \sigma^2$, such that, conditioned on the event E, the random variable ζ has the complex Gaussian distribution with variance $\mathbb{E}[|\zeta|^2 \mid E] = \sigma^2$.*

Proof of Lemma 23: We may assume that $\sigma < 1$. Write

$$f(z) = \frac{1}{\pi}\exp(-|z|^2), \quad f_\sigma(z) = \frac{1}{\pi\sigma^2}\exp\big(-\tfrac{1}{\sigma^2}|z|^2\big), \quad z \in \mathbb{C},$$

for the density of a standard complex Gaussian, and the density of a complex Gaussian with variance σ^2, respectively. Observe that, since $\sigma < 1$,

$$f = \sigma^2 \cdot f_\sigma + (1-\sigma^2) g_\sigma \tag{42}$$

for some non-negative function g_σ with integral 1. Now, suppose that our probability space supports a complex Gaussian ζ_σ with $\mathbb{E}|\zeta_\sigma|^2 = \sigma^2$, a random variable Y_σ with density function g_σ, and a Bernoulli random variable I_σ, satisfying $\mathbb{P}[I_\sigma = 1] = 1 - \mathbb{P}[I_\sigma = 0] = \sigma^2$, which is independent of both ζ_σ and Y_σ. Then (42) implies that the random variable

$$\zeta = I_\sigma \cdot \zeta_\sigma + (1 - I_\sigma) Y_\sigma$$

has the standard complex Gaussian distribution. In this probability space, after conditioning that $I_\sigma = 1$, the distribution of ζ is that of a complex Gaussian with variance $\mathbb{E}[|\zeta|^2 | I_\sigma = 1] = \sigma^2$, as required. □

Proof of Lemma 22: Let $\sigma_n = |c_n/b_n|$ (where again, the ratio is defined to be 1 if b_n and c_n are both zero). Lemma 23 yields, for each n, a probability space supporting a standard Gaussian random variable ζ_n and an event E_n with $\mathbb{P}[E_n] = \sigma_n^2$. Clearly we may assume that the sequence ζ_n is mutually independent and we take the probability space to be the product of these probability spaces, extend each E_n to this space in the obvious way and define

$$G(z) = \sum_{n \geq 0} \zeta_n b_n z^n.$$

The claim follows with the event $E = \bigcap_{n \geq 0} E_n$. □

7.1 Estimating $\mathbb{P}\big[\mathcal{E}_2\big]$

Put

$$G(z) = \sum_{1 \leqslant n \leqslant N} \zeta_n a_n z^n,$$

and let $(q_n)_{n=1}^N$ be a sequence of numbers in $[0,1]$ to be specified below. According to Lemma 22, there is an event E with probability $Q^2 = \prod_{n=1}^N q_n^2$, such that on the event E, the function G has the same distribution as

$$G_Q(z) \stackrel{\text{def}}{=} \sum_{1 \leqslant n \leqslant N} \zeta_n q_n a_n z^n.$$

Notice also that

$$\sigma_Q(\rho)^2 \stackrel{\text{def}}{=} \mathbb{E}\Big[\Big|\sum_{n=1}^N \zeta_n q_n a_n (\rho e^{\mathrm{i}\theta})^n\Big|^2\Big] = \sum_{n=1}^N q_n^2 a_n^2 \rho^{2n}\,.$$

If we now set

$$r_2 = r + \delta^2, \quad \lambda = \Big(\log\frac{1}{\delta}\Big)^\alpha,$$

then applying first Lemma 22, and then Lemma 2 to the function G_Q, with λ as above, we obtain that for δ sufficiently small

$$\begin{aligned}
\mathbb{P}\Big[\max_{r\bar{\mathbb{D}}}\Big|\sum_{n=1}^N \zeta_n a_n z^n\Big| \leqslant \lambda\,\sigma_Q(r_2)\Big] &\geq Q^2\cdot\mathbb{P}\Big[\max_{r\bar{\mathbb{D}}}\Big|\sum_{n=1}^N \zeta_n a_n z^n\Big| \leqslant \lambda\,\sigma_Q(r_2)\Big|E\Big] \\
&= Q^2\cdot\mathbb{P}\Big[\max_{r\bar{\mathbb{D}}}\Big|\sum_{n=1}^N \zeta_n q_n a_n z^n\Big| \leqslant \lambda\,\sigma_Q(r_2)\Big] \\
&\geq Q^2\Big(1 - C\delta^{-1}\exp\big(-c(\log\tfrac{1}{\delta})^{2\alpha}\big)\Big) \\
&\stackrel{\alpha>\frac12}{\geq} \tfrac12 Q^2.
\end{aligned}$$

For this estimate to be useful in our context we have to ensure, by choosing the sequence q_n appropriately, that

$$\sigma_Q(r_2) \leqslant \frac{M}{2\lambda} = \frac{1}{2\sqrt{\delta}}. \tag{43}$$

First, it is straightforward to verify that

$$\exp(-\delta) \leqslant r_2 \leqslant \exp(-\delta+\delta^2), \quad \forall r\in(0,1).$$

Since $cn^{L-1} \leqslant a_n^2 \leqslant Cn^{L-1}$, this implies that for $n\in\{1,\dots,N\}$

$$a_n^2 r_2^{2n} = \exp\big((L-1)\log n - 2n\delta + O(1)\big).$$

Putting

$$N_1 = \Big[\,\frac{L-1}{2\delta}\,\log\frac{1}{\delta}\,\Big]$$

and noting that the function $x\mapsto (L-1)\log x - 2x\delta$ attains its maximum at $x = \frac{L-1}{2\delta}$ we see that $a_n^2 r_2^{2n} \geq c$ for $n\leqslant N_1$, while for $n\in\{N_1,\dots,N\}$ we have $a_n^2 r_2^{2n} \leqslant C\left(\log\frac{1}{\delta}\right)^{L-1}$.

We therefore choose

$$q_n^2 = \begin{cases} \alpha_1\big(a_n^2 r_2^{2n}\log\frac{1}{\delta}\big)^{-1}, & n\in\{1,\dots,N_1\}, \\[2ex] \alpha_1\big(\log\frac{1}{\delta}\big)^{-L}, & n\in\{N_1+1,\dots,N\}, \end{cases}$$

where we choose the constant $\alpha_1 > 0$ sufficiently small to ensure that $q_n \leqslant 1$ for all $n\in\{1,\dots N\}$. With this choice we have

$$\sigma_Q^2(r_2) = \sum_{n=1}^N q_n^2 a_n^2 r_2^{2n} \leqslant \alpha_1\Big(\log\frac{1}{\delta}\Big)^{-1} N_1 + \alpha_1\cdot C\Big(\log\frac{1}{\delta}\Big)^{-1}(N-N_1) \leqslant C\cdot\frac{\alpha_1}{\delta}$$

and further choosing α_1 small if necessary, Condition (43) is satisfied.

It remains to estimate the probability of the event E. Notice that

$$\begin{aligned}\mathbb{P}[E] = Q^2 = \prod_{n=1}^{N} q_n^2 &= \alpha_1^N \big(\log \tfrac{1}{\delta}\big)^{-L(N-N_1)-N_1} \prod_{n=1}^{N_1} \big(a_n^2 r_2^{2n}\big)^{-1} \\ &\geq \exp\big(-CN\log\log\tfrac{1}{\delta}\big) \cdot \prod_{n=1}^{N_1} \big(Cn^{L-1}\exp[-2n(\delta-\delta^2)]\big)^{-1} \\ &\geq \exp\big(-CN\log\log\tfrac{1}{\delta}\big) \cdot \frac{e^{(\delta-\delta^2)N_1(N_1+1)}}{(N_1!)^{L-1}}.\end{aligned}$$

Recalling that $N \leqslant CN_1$, and using that $N_1 = \big[\,\frac{L-1}{2\delta}\,\log\frac{1}{\delta}\,\big]$, we obtain

$$\begin{aligned}\mathbb{P}\big[\mathcal{E}_2\big] \geq \tfrac{1}{2}Q^2 &\geq \exp\big(-N_1\big[\,(L-1)\log N_1 - \delta N_1 + C\log\log\tfrac{1}{\delta}\,\big]\big) \\ &= \exp\big(-\tfrac{1}{4}[(L-1)^2+o(1)]\,\tfrac{1}{\delta}\,\log^2\tfrac{1}{\delta}\big).\end{aligned}$$

7.2 Estimating $\mathbb{P}\big[\mathcal{E}_3\big]$

Put

$$H(z) = \sum_{n>N} \zeta_n a_n z^n\,.$$

Then

$$\sigma_H(\rho)^2 = \mathbb{E}\Big[\Big|\sum_{n>N}\zeta_n a_n(\rho e^{\mathrm{i}\theta})^n\Big|^2\Big] = \sum_{n>N} a_n^2\rho^{2n}\,.$$

The choice of the parameter N guarantees that the 'tail' H will be small.

Lemma 24. *Put $r_1 = \frac{1}{2}(1+r)$. There exists a constant $C > 0$ such that, for δ sufficiently small,*

$$\sigma_H(r_1)^2 = \sum_{n>N} a_n^2 r_1^{2n} \leqslant C.$$

Proof of Lemma 24: Since the function $r \mapsto 2\log(1+r) - r$, $0 \leqslant r \leqslant 1$ attains its maximum at $r = 1$ we have $r_1^2 \leqslant \exp(-\delta)$ for $\delta = 1 - r \in [0,1]$. Recalling that $N = \big[\,\frac{2L}{\delta}\,\log\frac{1}{\delta}\,\big]$, we observe that for $n > N$ we have

$$\frac{n}{\log n} \geq \frac{2(L-1)}{\delta},$$

and therefore $n^{L-1} \leqslant \exp(\frac{1}{2}\delta n)$. Then, using that $a_n^2 \leqslant Cn^{L-1}$, we have

$$\sigma_H(r_1)^2 \leqslant C\sum_{n>N} n^{L-1}\exp(-\delta n) \leqslant C\sum_{n>N}\exp(-\tfrac{1}{2}\delta n) \leqslant C\delta^{-1}\exp(-\tfrac{1}{2}\delta N) \leqslant C\delta^{L-1}.$$

Since $L > 1$ this is a stronger result than we claimed. □

By Lemma 24,

$$\mathbb{P}\big[\mathcal{E}_3^c\big] = \mathbb{P}\big[\max_{r\bar{\mathbb{D}}}|H| > \tfrac{1}{2}M\big] \leqslant \mathbb{P}\big[\max_{r\bar{\mathbb{D}}}|H| > cM\cdot\sigma_H\big(\tfrac{1}{2}(1+r)\big)\big]\,,$$

if $c > 0$ is sufficiently small. By Lemma 2, the right-hand side is at most $C\delta^{-1}\,\exp\big[-cM^2\big] \to 0$ as $\delta \to 0$. Thus, $\mathbb{P}\big[\mathcal{E}_3\big] \geq \frac{1}{2}$ for δ sufficiently small.

7.3 Putting the estimates together

Finally,

$$\begin{aligned}\mathbb{P}\big[\mathrm{Hole}(r)\big] &\geq \mathbb{P}\big[\mathcal{E}_1\big]\cdot\mathbb{P}\big[\mathcal{E}_2\big]\cdot\mathbb{P}\big[\mathcal{E}_3\big]\\ &\geq \frac{1}{2}\exp\Big[-\frac{1}{\delta}\big(\log\frac{1}{\delta}\big)^{2\alpha}-\frac{(L-1)^2+o(1)}{4}\,\frac{1}{\delta}\,\log^2\frac{1}{\delta}\Big]\\ &\overset{\alpha<1}{\geq}\exp\Big[-\frac{(L-1)^2+o(1)}{4}\,\frac{1}{\delta}\,\log^2\frac{1}{\delta}\Big],\end{aligned}$$

completing the proof of the lower bound in the case $L>1$, and hence of Theorem 1. □

8 The hole probability for non-invariant GAFs with regularly distributed coefficients

The proofs we gave do not use the hyperbolic invariance of the zero distribution of $F=F_L$. In the case $0<L<1$ we could assume that the sequence of coefficients (a_n) in (1) does not increase, that $a_0=1$ and that

$$a_n \simeq n^{\frac{1}{2}(L-1)}\,,\quad n\geq 1. \tag{44}$$

Then, setting as before

$$\sigma_F(r)^2=\mathbb{E}\big[|F(re^{\mathrm{i}\theta})|^2\big]=\sum_{n\geq 0}a_n^2r^{2n}$$

the same proof yields the bounds

$$\begin{aligned}\frac{1-L+o(1)}{2}\,\sigma_F(r)^2\log\frac{1}{1-r} &\leqslant -\log\mathbb{P}\big[\mathrm{Hole}(r)\big]\\ &\leqslant (1-L+o(1))\,\sigma_F(r)^2\log\frac{1}{1-r}\,,\qquad r\to 1\,.\end{aligned}$$

In the case $L>1$, assuming that $a_0=1$ and (44), we also get the same answer as in Theorem 1:

$$-\log\mathbb{P}\big[\mathrm{Hole}(r)\big]=\frac{(L-1)^2+o(1)}{4}\,\frac{1}{1-r}\,\log^2\frac{1}{1-r}\,,\qquad r\to 1\,.$$

In the case $L=1$ (that is, $a_n=1$, $n\geq 0$) the result of Peres and Virág relies on their proof that the zero set of F_1 is a determinantal point process. This ceases to hold under a slight perturbation of the coefficients a_n, while our techniques still work, though yielding less precise bounds:

Theorem 2. *Suppose that F is a Gaussian Taylor series of the form* (1) *with $a_n\simeq 1$ for $n\geq 0$. Then*

$$-\log\mathbb{P}\big[\mathrm{Hole}(r)\big]\simeq\frac{1}{1-r}\,,\qquad \frac{1}{2}\leqslant r<1\,.$$

For the reader's convenience, we supply the proof of Theorem 2, which is based on arguments similar to those we have used above. As before, we put $\delta=1-r$, $\sigma_F=\sigma_F(r)$, and note that under the assumptions of Theorem 2, $\sigma_F(r)^2\simeq\delta^{-1}$. Also put

$$N=\left\lceil\frac{1}{1-r}\right\rceil$$

and $\omega=e(1/N)$.

8.1 Upper bound on the hole probability in Theorem 2

8.1.1 Beginning the proof

The starting point is the same as in the proofs of the upper bound in the cases $L \neq 1$. Put $r_0 = 1 - 2\delta$. Then, by Lemma 9 (with $\kappa = 2$ and $k = N$),

$$\mathbb{P}\big[\mathrm{Hole}(r)\big] \leqslant \delta^{-c} \sup_{\tau \in \mathbb{T}} \mathbb{P}\Big[\sum_{j=1}^{N} \log|F(\omega^j \tau r_0)| \leqslant N \log|F(0)| + C\Big] + \langle \text{negligible terms} \rangle \,.$$

For fixed $\tau \in \mathbb{T}$ and for a small positive parameter b, we have

$$\begin{aligned}
&\mathbb{P}\Big[\sum_{j=1}^{N} \log|F(\omega^j \tau r_0)| \leqslant N \log|F(0)| + C\Big] \\
&\quad \leqslant \mathbb{P}\Big[\sum_{j=1}^{N} \log|F(\omega^j \tau r_0)| \leqslant N \log\big(b\sigma_F\big) + C\Big] + \mathbb{P}\big[|F(0)| > b\sigma_F\big] \\
&\quad \leqslant \mathbb{P}\Big[\sum_{j=1}^{N} \log|w_j| \leqslant N \log\big(Cb\big)\Big] + \mathbb{P}\big[|F(0)| > b\sigma_F\big] \,,
\end{aligned}$$

where $w_j = F(\omega^j \tau r_0)/\sqrt{N}$ are complex Gaussian random variables. Next,

$$\mathbb{P}\Big[\sum_{j=1}^{N} \log|w_j| \leqslant N \log\big(Cb\big)\Big] = \mathbb{P}\Big[\prod_{j=1}^{N} |w_j|^{-1} \geq (Cb)^{-N}\Big] \leqslant (Cb)^N \, \mathbb{E}\Big[\prod_{j=1}^{N} |w_j|^{-1}\Big] \,.$$

Thus, up to negligible terms, the hole probability is bounded from above by

$$(Cb)^N \, \mathbb{E}\Big[\prod_{j=1}^{N} |w_j|^{-1}\Big] + e^{-b^2\sigma_F^2} \,.$$

What remains is to show that the expectation of the product of $|w_j|^{-1}$ grows at most exponentially with N. Then, choosing the constant b so small that the prefactor $(Cb)^N$ overcomes this growth of the expectation, we will get the result.

8.1.2 Estimating $\mathbb{E}\Big[\prod_{j=1}^{N} |w_j|^{-1}\Big]$

One can use Lemma 15 in order to bound the expectation above, below we give an alternative argument. Put $z_j = \omega^j \tau r_0$, $1 \leqslant j \leqslant N$, and consider the covariance matrix

$$\Gamma_{ij} = \mathbb{E}\big[w_i \bar{w}_j\big] = N^{-1} \mathbb{E}\big[F(z_i)\bar{F}(z_j)\big] \,, \qquad 1 \leqslant i, j \leqslant N \,.$$

For each non-empty subset $I \subset \{1, 2, \ldots, N\}$, we put $\Gamma_I = \big(\Gamma_{ij}\big)_{i,j \in I}$.

Lemma 25. *For each $I \subset \{1, 2, \ldots, N\}$, we have* $\det \Gamma_I \geq c^{|I|}$.

Proof of Lemma 25: By Lemma 10, the eigenvalues of the matrix Γ are

$$\lambda_m = \sum_{n \equiv m\,(N)} a_n^2 r_0^{2n} \geq c \sum_{n \equiv m\,(N)} (1 - 2\delta)^{2n} \geq c \sum_{k \geq 0} e^{-CNk\delta} \geq c \sum_{k \geq 0} e^{-Ck} \geq c > 0 \,,$$

that is, the minimal eigenvalue of Γ is separated from zero. It remains to recall that the $N-1$ eigenvalues of any minor of order $N-1$ of an Hermitian matrix of order N interlace with the N eigenvalues of the original matrix. Applying this principle several times, we conclude that the minimal eigenvalues of the matrix Γ_I cannot be less than the minimal eigenvalue of the full matrix Γ. □

Now, we write

$$\mathbb{E}\Big[\prod_{j=1}^{N}|w_j|^{-1}\Big] \leqslant \sum_{I\subset\{1,2,\dots,N\}} \mathbb{E}\Big[\prod_{i\in I}|w_i|^{-1}\mathbb{1}_{\{|w_i|\leqslant 1\}}\Big]\,.$$

By Lemma 25, the expectations on the right-hand side do not exceed

$$\frac{1}{\pi^{|I|}\det\Gamma_I}\left(\int_{|w|\leqslant 1}\frac{1}{|w|}\,e^{-\lambda_I^{-1}|w|^2}\,\mathrm{d}m(w)\right)^{|I|} \leqslant C^{|I|}$$

(here, λ_I is the maximal eigenvalue of Γ_I). Since the number of subsets I of the set $\{1,2,\dots,N\}$ is 2^N, we finally get

$$\mathbb{E}\Big[\prod_{j=1}^{N}|w_j|^{-1}\Big] \leqslant (2C)^N\,.$$

This completes the proof of the upper bound on the hole probability in Theorem 2. □

8.2 Lower bound on the hole probability in Theorem 2

We write

$$F(z) = F(0) + G(z) = F(0) + \sum_{k\geq 0} z^{kN} S_k(z)\,,$$

where the $S_k(z)$ are independent random Gaussian polynomials,

$$S_k(z) = \sum_{j=1}^{N}\zeta_{j+kN}a_{j+kN}z^j\,.$$

We have

$$\max_{r\mathbb{T}}|G| \leqslant \sum_{k\geq 0} r^{kN}\max_{r\mathbb{T}}|S_k| \overset{N\log r<-1}{\leqslant} \sum_{k\geq 0} e^{-k}\max_{r\mathbb{T}}|S_k|\,.$$

We fix $1<A<e$ and consider the independent events $\mathcal{E}_k = \big\{\max_{r\mathbb{T}}|S_k| < A^k\sqrt{N}\big\}$. If these events occur together, then

$$\max_{r\mathbb{T}}|G| < \sqrt{N}\sum_{k\geq 0}\big(Ae^{-1}\big)^k < B\sqrt{N}$$

with some positive numerical constant B. Then, $F(z)\neq 0$ on $r\bar{\mathbb{D}}$, provided that $|F(0)|>B\sqrt{N}$, and that all the events $\mathcal{E}_k$ occur together, that is,

$$\mathbb{P}\big[\mathrm{Hole}(r)\big] \geq e^{-B^2N}\prod_{k\geq 0}\mathbb{P}\big[\mathcal{E}_k\big]\,.$$

It remains to estimate from below the probability of each event $\mathcal{E}_k$ and to multiply the estimates.

8.2.1 Estimating $\mathbb{P}[\mathcal{E}_k]$

Take $4N$ points $z_j = re\big(\frac{j}{4N}\big)$, $1 \leqslant j \leqslant 4N$. By the Bernstein inequality,

$$\max_{\theta\in[0,2\pi]} \Big|\frac{\partial S_k}{\partial\theta}\big(re^{\mathrm{i}\theta}\big)\Big| \leqslant N \max_{r\mathbb{T}} |S_k| \,.$$

Therefore,

$$\max_{r\mathbb{T}} |S_k| \leqslant \max_{1\leqslant j\leqslant 4N} |S_k(z_j)| + \frac{\pi}{4N}\cdot N \max_{r\mathbb{T}} |S_k| \,,$$

whence,

$$\max_{r\mathbb{T}} |S_k| \leqslant C \max_{1\leqslant j\leqslant 4N} |S_k(z_j)| \,,$$

and

$$\mathbb{P}\big[\mathcal{E}_k\big] \geq \mathbb{P}\Big[\max_{1\leqslant j\leqslant 4N} |S_k(z_j)| < CA^k\sqrt{N}\Big] \,.$$

Then, applying as in Section 5, Hargé's version of the Gaussian correlation inequality, we get

$$\mathbb{P}\big[\mathcal{E}_k\big] \geq \prod_{j=1}^{4N} \mathbb{P}\big[|S_k(z_j)| < CA^k\sqrt{N}\big] \,.$$

(In fact, passing to real and imaginary parts we could use here the simpler Khatri-Sidak version of the Gaussian correlation inequality [9, Section 2.4].) Each value $S_k(z_j)$ is a complex Gaussian random variable with variance

$$\sigma^2_{S_k} = \sum_{j=1}^{N} a^2_{j+kN} r^{2j} \simeq \sum_{j=1}^{N} r^{2j} \simeq \frac{1}{\delta} \simeq N \,.$$

Therefore,

$$\mathbb{P}\big[\,|S_k(z_j)| < CA^k\sqrt{N}\,\big] = 1 - \mathbb{P}\big[\,|S_k(z_j)| \geq CA^k\sqrt{N}\,\big] \geq 1 - e^{-cA^{2k}} \,,$$

whence,

$$\mathbb{P}\big[\mathcal{E}_k\big] \geq \Big(1 - e^{-cA^{2k}}\Big)^{4N} \,,$$

and then,

$$\prod_{k\geq 0} \mathbb{P}[\mathcal{E}_k] \geq \prod_{k\geq 0}\Big(1 - e^{-cA^{2k}}\Big)^{4N} \geq e^{-cN} \,,$$

completing the proof. □

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