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DOI:

[10.1049/iet-cta.2016.0913](https://doi.org/10.1049/iet-cta.2016.0913)

Document Version

Publisher's PDF, also known as Version of record

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Citation for published version (APA):

Zhao, Y., Lam, H. K., Song, G., & Yin, X. (2017). Relaxed stability conditions for polynomial-fuzzy-model-based control system with membership function information. *Iet Control Theory And Applications*, 11(10), 1493-1503. <https://doi.org/10.1049/iet-cta.2016.0913>

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Relaxed stability conditions for polynomial-fuzzy-model-based control system with membership function information

ISSN 1751-8644
 Received on 16th July 2016
 Revised 17th January 2017
 Accepted on 20th February 2017
 E-First on 6th April 2017
 doi: 10.1049/iet-cta.2016.0913
 www.ietdl.org

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Abstract: This study presents a new approach for stability analysis of polynomial-fuzzy-model-based (PFMB) control system using membership function information. For the purpose of extracting the regional information of the membership functions, the operating domain is partitioned into several sub-domains. In each sub-domain, the boundaries of every single membership function overlap term and the numerical relation among all the membership function overlap terms are represented as a group of inequalities. Through the S-procedure, the regional membership function information is taken into account in the stability analysis to relax the stability conditions. The operating domain partition scheme naturally arises the motivation of constructing the PFMB control system with the sub-domain fuzzy controllers. Each polynomial fuzzy controller works in its corresponding sub-domain such that the compensation capability of controller is enhanced. The sum-of-squares (SOS) approach is proposed to obtain the stability conditions of the PFMB control system using the Lyapunov stability theory. The PFMB control system studied in this study has the feature that the number of fuzzy rules and the membership function shapes of the polynomial fuzzy controller can be designed independently from the polynomial fuzzy model. To verify the stability analysis result, a numerical example is given to demonstrate the validity of the proposed method.

1 Introduction

The extensive research of Takagi-Sugeno fuzzy-model-based (TSFMB) control [1] as an important nonlinear control technique has been taken over the last two decades [2–8]. Applying the linear matrix inequality (LMI) approach [1, 9] based on the Lyapunov stability theory, stability analysis and control design problem can be recasted as LMI problems. The stability conditions are represented as a group of LMIs which can be solved numerically by some convex optimisation techniques [10]. In recent years, the TSFMB control system has been extended to polynomial-fuzzy-model-based (PFMB) control system [11] and is widely applied to tackle various control problems [12–16]. For the reason that PFMB control system allows the polynomial terms to appear in the local models and the feedback gains, comparing with TSFMB control system, the capability of the fuzzy modelling and compensation capability of the controller can be enhanced extensively [11, 17]. Using the sum-of-squares (SOS) approach [11], the stability conditions for PFMB control system are represented as an SOS problem which takes the LMI problem as a special case. In the same time, the polynomial Lyapunov function candidate can be applied in the stability analysis instead of the quadratic Lyapunov candidate under the LMI approach. The above improvements make PFMB control system has a higher potential to achieve more effective control.

Although the stability analysis of PFMB control system using the SOS approach can make the stability conditions more relax, a large number of sources of conservativeness still exist in the fuzzy-model-based control system analysis and design scheme [18, 19]. It arises a strong interest in the research community that various techniques have been developed to tackle this problem for both of the TSFMB and PFMB control system analysis. In [20], a study of positivity of fuzzy summations using multi-dimensional summations was proposed. As the extension, the necessary and sufficient conditions for stability analysis were given in [21, 22] using Pólya theorem. Many works also focused on adopting more complicated Lyapunov function candidate such as fuzzy Lyapunov function [23–25], piecewise Lyapunov function [26–28], polynomial Lyapunov function [11, 19] and switching Lyapunov

function [29–31]. Recent researches also include the techniques of removing restrictions to choosing Lyapunov function candidates [32, 33] and improving the stability condition transformation process [34].

One of the important sources of the conservativeness is the partial use of the membership function information [18]. The stability analysis of both the LMI approach and the SOS approach in the early time [1, 11] did not involve the membership function information. It makes the stability conditions obtained work for a family of nonlinear systems but not the one considered at hands with specific membership function shapes [33, 35], thus the stability analysis result is very conservative. The early work to tackle this problem can be found in [36], in which the correlation between membership functions was taken into account in the stability analysis. The membership function information was further represented as a group of affine inequalities in [37] and a group of inequalities of the membership function overlap terms in [38], and the authors of [39] proposed the analysis approach to represent the membership function information in a more general form, i.e. a group of second-order polynomial inequalities. By constructing the inequalities, the membership function information was taken into the stability analysis through the S-procedure [10] in [37–40]. In [41], the approach proposed was to use a transformation matrix to bring the order relationship among the membership functions into the stability analysis. In the case of no order relations in the whole operating domain, the so-called induced relations were exploited through the operating domain partition and extracted the membership function information in each sub-domain.

With the study of the fuzzy-model-based control system with mismatched premise membership functions, the usage of numerical relationship between the model and the controller membership functions was taken into account to relax the conservativeness in the stability analysis and performance design [42]. For the purpose of extracting more membership function information, staircase membership functions [43] and piecewise linear membership functions [44] were used to approximate the original membership functions in the LMI approach. The ideas of approximated membership function were extended into the SOS approach. The

membership functions were approximated as piecewise membership functions [45] and polynomial membership function [46] using a systematic way, i.e. Taylor series method [35, 46] to extract more membership function information in the stability analysis of the PFMB control system. Membership functions were also handled as symbolic variables with the consideration of their properties and boundary information to relax the stability conditions [47].

For the approximated membership function methods mentioned, the regional information of memberships function can be extracted effectively. However, there is a drawback of heavy computational burden due to the large number of the variables and conditions. For the method proposed in [41], unfortunately, the operating domain partition heavily depends on the membership function shape and order relation is not a very accurate membership function shape information. In [45, 47], the information of membership functions is mainly the global/local boundary information of the membership functions, the numerical relation among all the membership functions was not considered. In this paper, an alternative stability analysis approach using membership function information is proposed that the operating domain of the PFMB control system is partitioned into several sub-domains uniformly which makes the operating domain partition to be more flexible comparing with [41]. Based on the operating domain partition, the regional information of the membership functions are extracted in the sub-domains. In each sub-domain, two types of membership function information are considered: the boundaries of the single membership function overlap term and the numerical relationship among all the membership function overlap terms. All the information are represented in the form of inequalities and are brought into the stability analysis through the S-procedure. More inequalities are considered with more membership function information being embedded, the conservativeness of the stability conditions can be progressively reduced.

The basic LMI and SOS approach introduced in the early works [11, 48] used to analysis the fuzzy-model-based system constructed under the parallel-distribution-compensation (PDC) scheme [48] that the fuzzy controller shares the same number of fuzzy rules and membership functions with the fuzzy model. The major drawback of the PDC design approach is that it limits the fuzzy controller design flexibility. It cannot avoid high implementation cost for stabilising a complicated fuzzy model which has a large number of fuzzy rules and/or the membership functions with complex structure. In this paper, the constraint of the PDC scheme in this aspect is dropped. The number of the fuzzy rules and the membership functions of the fuzzy controller are designed freely. For the purpose of enhancing the feedback compensation capability, according to the operating domain partitions, sub-domain fuzzy controller is proposed to work in every sub-domain.

This paper is organised as follows. In Section 2, the PFMB control system with sub-domain polynomial fuzzy controller is briefly presented. Section 3 presents the stability analysis without considering the membership function information. In Section 4, two types of regional membership function information are introduced and the relaxed stability conditions are given. Section 5 entails a simulation to verify the viability of the proposed approach. In Section 6, a conclusion is drawn.

2 Preliminaries

In this section, the preliminaries of polynomial fuzzy model and polynomial fuzzy controller are presented. A nonlinear plant is represented by a polynomial fuzzy model [11]. A set of sub-domain polynomial fuzzy controllers is employed to control the nonlinear plant.

2.1 Polynomial fuzzy model

Consider the following nonlinear plant to be controlled:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)) \quad (1)$$

where $f(\cdot)$ is a smooth nonlinear function. $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ is the state vector, and $\mathbf{u}(t) = [u_1(t), u_2(t), \dots, u_m(t)]^T$ is the input vector.

It is assumed that the nonlinear plant (1) can be represented by a p -rule polynomial fuzzy model where the i th rule is shown as

$$\begin{aligned} \text{Rule } i: & \text{ IF } f_i(\mathbf{x}(t)) \text{ is } M_i^i \text{ and } \dots \text{ and } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i \\ & \text{ THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \end{aligned} \quad (2)$$

where M_i^i is a fuzzy term of rule i corresponding to the function $f_i(\mathbf{x}(t))$, $\alpha = 1, 2, \dots, \Psi$, Ψ is a positive integer; $i = 1, 2, \dots, p$; $\mathbf{x}(t) \in \mathfrak{R}^n$ is the system state vector; $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \mathfrak{R}^N$ is a vector of monomials in $\mathbf{x}(t)$, in which a monomial in $\mathbf{x}(t)$ is in the form $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, $\alpha_1, \alpha_2, \dots, \alpha_n$ are non-negative integers; $\mathbf{u}(t) \in \mathfrak{R}^m$ is the input vector; $\mathbf{A}_i(\mathbf{x}(t)) \in \mathfrak{R}^{n \times n}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \mathfrak{R}^{n \times m}$ are the known polynomial system and input matrices, respectively. It is assumed that $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{0}$ iff $\mathbf{x}(t) = \mathbf{0}$. The system dynamics is described as

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^p w_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)) \quad (3)$$

where

$$w_i(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(\mathbf{x}(t)))}{\sum_{k=1}^p \prod_{l=1}^{\Psi} \mu_{M_l^k}(f_l(\mathbf{x}(t)))} \quad \forall i,$$

$$w_i(\mathbf{x}(t)) \geq 0 \quad \forall i, \quad \sum_{i=1}^p w_i(\mathbf{x}(t)) = 1$$

$w_i(\mathbf{x}(t))$, $i = 1, 2, \dots, p$, are the normalised grades of membership and $\mu_{M_i^i}(f_i(\mathbf{x}(t)))$, $\alpha = 1, 2, \dots, \Psi$, are the membership functions corresponding to the fuzzy term M_i^i .

2.2 Sub-domain polynomial fuzzy controllers

To enhance the feedback compensation capability of the polynomial fuzzy controller, the system operating domain is partitioned into D connected sub-domains, D is a positive integer. Recalling that the system state vector is denoted as $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$, the sub-domain denoted as s_d , $d = 1, 2, \dots, D$, is characterised by the state variables $\underline{x}_{\zeta d} \leq x_{\zeta}(t) \leq \bar{x}_{\zeta d}$, $\zeta = 1, 2, \dots, n$, where $\underline{x}_{\zeta d}$ and $\bar{x}_{\zeta d}$ are predefined constants used to denote the lower and upper bounds of the state variable x_{ζ} in the sub-domain s_d , respectively. A set of sub-domain polynomial fuzzy controllers is proposed according to above operating domain partition. As an extension to the controller proposed in [49] which has constant feedback gains working in each sub-domain, the controllers proposed here have polynomial feedback gains to stabilise the nonlinear plant (1) represented by the polynomial fuzzy model (3).

Corresponding to the sub-domain s_d , the sub-domain polynomial fuzzy controller is described by the following c -rules.

$$\begin{aligned} \text{Rule } j: & \text{ IF } g_1(\mathbf{x}(t)) \text{ is } N_1^j \text{ and } \dots \text{ and } g_{\Omega}(\mathbf{x}(t)) \text{ is } N_{\Omega}^j \\ & \text{ THEN } \mathbf{u}(t) = \mathbf{G}_{jd}(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) \end{aligned} \quad (4)$$

where N_i^j is a fuzzy term of rule j corresponding to the function $g_i(\mathbf{x}(t))$, $\beta = 1, 2, \dots, \Omega$, Ω is a positive integer, $j = 1, 2, \dots, c$ and $\mathbf{G}_{jd}(\mathbf{x}) \in \mathfrak{R}^{m \times N}$, $j = 1, 2, \dots, c$, $d = 1, 2, \dots, D$, are the polynomial feedback gains to be determined.

The polynomial fuzzy controller in sub-domain s_d is defined as

$$\mathbf{u}(t) = \sum_{j=1}^c m_j(\mathbf{x}(t)) \mathbf{G}_{jd}(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t)) \quad (5)$$

where

$$m_j(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Omega} \mu_{N_l^j}(g_l(\mathbf{x}(t)))}{\sum_{k=1}^c \prod_{l=1}^{\Omega} \mu_{N_l^k}(g_l(\mathbf{x}(t)))} \quad \forall j,$$

$$m_j(\mathbf{x}(t)) \geq 0 \quad \forall j, \quad \sum_{j=1}^c m_j(\mathbf{x}(t)) = 1$$

$m_j(\mathbf{x}(t))$, $j = 1, 2, \dots, c$, are the normalised grades of membership and $\mu_{N_\beta^j}(f_\beta(\mathbf{x}(t)))$, $\beta = 1, 2, \dots, \Omega$, are the membership functions corresponding to the fuzzy term N_β^j .

In the control operating operation, the d th sub-domain polynomial fuzzy controller is employed when the system state falls into the sub-domain s_d . In the boundary of the connected sub-domains, either one of the sub-domain fuzzy controllers can be applied for the control process.

Remark 1: In practice, if the switching control scheme is not suitable, when physical constraints are considered such as the issues of actuator, performance, controller structure, implementation costs etc., by choosing all sets of feedback gains from all sub-domains to be the same, i.e. $\mathbf{G}_j(\mathbf{x}(t)) = \mathbf{G}_{jd}(\mathbf{x}(t))$ for all d , the polynomial fuzzy controller (5) will be reduced to the traditional fuzzy controller without switching at the boundary of sub-domains.

Remark 2: The number of rules c and membership function $m_j(\mathbf{x}(t))$ can be different from the fuzzy model in each sub-domain which can enhance the design flexibility of the polynomial fuzzy controller.

2.3 PFMB control system

The PFMB control system is formed by connecting the polynomial fuzzy model (3) and the sub-domain polynomial fuzzy controller (5) in a closed loop where the dynamics is described as follows:

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) = & \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(t)) m_j(\mathbf{x}(t)) (\mathbf{A}_i(\mathbf{x}(t)) \\ & + \mathbf{B}_i(\mathbf{x}(t)) \mathbf{G}_{jd}(\mathbf{x}(t))) \hat{\mathbf{x}}(\mathbf{x}(t)) \quad \forall d \end{aligned} \quad (6)$$

The control objective is to determine the polynomial feedback gains $\mathbf{G}_{jd}(\mathbf{x}(t))$ such that the PFMB control system (6) is asymptotically stable, i.e. $\mathbf{x}(t) \rightarrow \mathbf{0}$ as time $t \rightarrow \infty$.

3 Stability conditions without membership function information

The stability of the PFMB control system (6) is investigated in this section. In the following, to lighten the notation, the time t associated with the variables is dropped for the situation without ambiguity, e.g., $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are denoted as \mathbf{x} and \mathbf{u} , respectively. Furthermore, we denote $\hat{\mathbf{x}}(\mathbf{x}(t))$, $w_i(\mathbf{x}(t))$ and $m_j(\mathbf{x}(t))$ as $\hat{\mathbf{x}}$, w_i and m_j , respectively.

From (6), denoting $\hat{\mathbf{x}} = [\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N]^T$,

$$\begin{aligned} \dot{\hat{\mathbf{x}}} = & \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \mathbf{T}(\mathbf{x}) \dot{\mathbf{x}} \\ = & \sum_{i=1}^p \sum_{j=1}^c w_i m_j (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{G}_{jd}(\mathbf{x})) \hat{\mathbf{x}} \quad \forall d \end{aligned} \quad (7)$$

where $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x}) \mathbf{A}_i(\mathbf{x})$, $\tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x}) \mathbf{B}_i(\mathbf{x})$ and $\mathbf{T}(\mathbf{x}) \in \mathfrak{R}^{N \times n}$ is polynomial matrix defined as

$$\mathbf{T}(\mathbf{x}) = \begin{bmatrix} \frac{\partial \hat{x}_1(\mathbf{x})}{\partial x_1} & \dots & \frac{\partial \hat{x}_1(\mathbf{x})}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \hat{x}_N(\mathbf{x})}{\partial x_1} & \dots & \frac{\partial \hat{x}_N(\mathbf{x})}{\partial x_n} \end{bmatrix} \quad (8)$$

Defining $\mathbf{K} = \{k_1, k_2, \dots, k_q\}$ as a set of row number that the entries of the entire row of $\tilde{\mathbf{B}}_i(\mathbf{x})$ are all zero for all i , and $\tilde{\mathbf{x}} = (x_{k_1}, x_{k_2}, \dots, x_{k_q})$. As $\hat{\mathbf{x}} = \mathbf{0}$ iff $\mathbf{x} = \mathbf{0}$, we consider the following polynomial Lyapunov function candidate to investigate the system stability of (6):

$$V(\mathbf{x}) = \hat{\mathbf{x}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}} \quad (9)$$

where $0 < \mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \mathfrak{R}^{N \times N}$ is a polynomial matrix. Note that all the sub-domains share the common Lyapunov function candidate (9).

In the following, we shall show that $\dot{V} < 0$ (excluding for $\hat{\mathbf{x}} = \mathbf{0}$) subject to $V > 0$ (excluding for $\hat{\mathbf{x}} = \mathbf{0}$), it is guaranteed by the Lyapunov stability theory that $\hat{\mathbf{x}} \rightarrow \mathbf{0}$ when time $t \rightarrow \infty$. As it is required that $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{0}$ iff $\mathbf{x}(t) = \mathbf{0}$, the stability of the PFMB control system (6) can thus be achieved.

From (6) and (9), we have

$$\begin{aligned} \dot{V} = & \hat{\mathbf{x}}^T \dot{\mathbf{X}}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \dot{\hat{\mathbf{x}}} + \hat{\mathbf{x}}^T \frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt} \hat{\mathbf{x}} \\ = & \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d w_i m_j \hat{\mathbf{x}}^T (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{G}_{jd}(\mathbf{x}))^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \\ & + \mathbf{X}(\tilde{\mathbf{x}})^{-1} (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{G}_{jd}(\mathbf{x})) \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt} \hat{\mathbf{x}} \end{aligned} \quad (10)$$

Define $\mathbf{z} = \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}$ and $\mathbf{G}_{jd}(\mathbf{x}) = \mathbf{N}_{jd}(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}})^{-1}$, where $\mathbf{N}_{jd}(\mathbf{x}) \in \mathfrak{R}^{m \times N}$, $j = 1, 2, \dots, c$, $d = 1, 2, \dots, D$, are polynomial matrices to be determined.

$$\xi_d = \begin{cases} 1 & \text{for } \mathbf{x} \in \text{sub-domain } d, \\ 0 & \text{otherwise} \end{cases}, \quad d = 1, 2, \dots, D.$$

From (10) we have

$$\dot{V} = \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d w_i m_j \mathbf{z}^T \mathbf{Q}_{ijd}(\mathbf{x}) \mathbf{z} \quad (11)$$

where

$$\begin{aligned} \mathbf{Q}_{ijd}(\mathbf{x}) = & \tilde{\mathbf{A}}_i(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}}) \tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{N}_{jd}(\mathbf{x}) \\ & + \mathbf{N}_{jd}(\mathbf{x})^T \tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}} \end{aligned}$$

for $i = 1, 2, \dots, p$, $j = 1, 2, \dots, c$ and $d = 1, 2, \dots, D$. $\mathbf{A}_i^k(\mathbf{x}) \in \mathfrak{R}^N$ denotes the k th row of $\mathbf{A}_i(\mathbf{x})$. It is required that $\sum_{d=1}^D \xi_d = 1$, which implies that there is only one active sub-domain at any instance.

The basic stability conditions based on the Lyapunov stability theory to guarantee the stability of the PFMB control system (6) are summarised in the following theorem.

Theorem 1: Consider the operating domain partitioned into D sub-domains, each sub-domain is characterised by $x_{\zeta d} \leq x_\zeta \leq \bar{x}_{\zeta d}$, $\zeta = 1, 2, \dots, n$, $d = 1, 2, \dots, D$. The PFMB control system (6) is asymptotically stable if there exist polynomial matrices $\mathbf{N}_{jd}(\mathbf{x}) \in \mathfrak{R}^{m \times N}$, $j = 1, 2, \dots, c$, $d = 1, 2, \dots, D$ and

$\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \mathfrak{R}^{N \times N}$, such that the following SOS-based conditions are satisfied:

$$v^T(\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}})\mathbf{I})v \text{ is SOS} \quad (12)$$

$$-v^T(\mathbf{Q}_{ijd}(\mathbf{x}) + \varepsilon_2(\mathbf{x})\mathbf{I})v \text{ is SOS } \forall i, j, d \quad (13)$$

where $v \in \mathfrak{R}^N$ is an arbitrary vector independent of \mathbf{x} , $\varepsilon_1(\tilde{\mathbf{x}}) > 0$ and $\varepsilon_2(\mathbf{x}) > 0$ are predefined scalar polynomials. The polynomial feedback gains are defined as $\mathbf{G}_{jd}(\mathbf{x}) = \mathbf{N}_{jd}(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$, $j = 1, 2, \dots, c$, $d = 1, 2, \dots, D$.

Remark 3: The information of the membership functions is not taken into consideration in Theorem 1 during stability analysis. Theorem 1 works on the PFMB control system with the same $\mathbf{Q}_{ijd}(\mathbf{x})$ but with any membership functions. It is the reason that the above theorem is conservative.

Remark 4: The PFMB control system (6) is asymptotic stable in the sense of the Lyapunov stability theory that it requires $V > 0$ and $\dot{V} < 0$ for $\dot{\mathbf{x}} \neq \mathbf{0}$. In Theorem 1, the condition (12) is to make sure that the Lyapunov function candidate (9) satisfying $V > 0$ and the condition (13) is to make sure that $\dot{V} < 0$ according to (11).

4 Main results

Section 3 reveals that the membership functions play an important role in the stability analysis. The membership functions of the fuzzy model are obtained through certain fuzzy modelling methods such as the sector-nonlinearity approach [11], the Taylor series approach [19] or the other system identification methods. For the imperfect premise matching, the membership functions of the polynomial fuzzy controller, which can be different from those of the polynomial fuzzy model, are designed by control engineers. Therefore, the membership functions of both polynomial fuzzy model and polynomial fuzzy controller are well defined before conducting stability analysis for the PFMB control system (6). As the membership functions are known, it can be found numerically that these membership functions can satisfy some inequalities such as in [37, 38]. Through considering these numerical relations, it effectively makes the family of systems represented by the fuzzy-model-based system become smaller in the stability analysis, therefore the stability conditions obtained are more relax [39]. The membership function approximation techniques introduced above are also intent to extract the membership function information. A group of functions are constructed and are used during the stability analysis to relax the stability conditions such as in [45]. However, taking the information of membership functions into account in the stability analysis is an open research topic so far as there is no systematic way to extract and bring the information into the stability analysis. In this section, we shall introduce some general forms to represent the information of membership functions, which will be used in the stability analysis resulting in more relaxed stability conditions.

4.1 Information of membership functions

The information of the membership functions is relatively limited when the whole operating domain is considered. To extract more information from the membership functions, the system operating domain can be partitioned into several sub-domains. In each sub-domain s_d , the following regional membership function information is considered.

4.1.1 Regional boundary information: The following boundary information of membership functions [47] can be found and utilised in stability analysis. In each sub-domain s_d , the constant lower and upper bound of each membership function overlap term $w_i m_j$, denoted as $\underline{\gamma}_{ijd}$ and $\bar{\gamma}_{ijd}$, respectively, can be found to satisfy the following inequalities as

$$\underline{\gamma}_{ijd} \leq w_i m_j \leq \bar{\gamma}_{ijd} \quad \forall i, j, d \quad (14)$$

where $i = 1, 2, \dots, p$, $j = 1, 2, \dots, c$ and $d = 1, 2, \dots, D$.

4.1.2 Regional relation among overlap terms: The boundary information represented above is only the information of each single $w_i m_j$, which does not show the relationship with other membership functions for different values of i and j . As a result, the lower and upper bound $\underline{\gamma}_{ijd}$ and $\bar{\gamma}_{ijd}$ contain limited information of membership functions. The relation among all the membership functions will contain more information, which can help further relax the stability analysis results.

In each sub-domain s_d , we represent the relation among all the membership functions using R inequalities as

$$\sum_{i=1}^p \sum_{j=1}^c k_{ijdr} w_i m_j + \underline{c}_{dr} \geq 0 \quad \forall d, r \quad (15)$$

where k_{ijdr} and \underline{c}_{dr} , $d = 1, 2, \dots, D$, $r = 1, 2, \dots, R$, are the real constant scalars which can be found numerically.

For example, for a PFMB control system (6) in which $i = 1, 2$ and $j = 1, 2$, in a chosen operating sub-domain say s_3 , i.e. $d = 3$, we find the membership functions satisfy the following two inequalities: $w_1 m_1 - 0.3 w_2 m_1 \geq 0$ and $0.9 w_1 m_1 - 0.7 w_2 m_2 + 0.1 \geq 0$. The number of inequalities is two therefore $R = 2$. For the first inequality, i.e. $r = 1$, referring to (15), we denote $k_{1131} = 1$, $k_{2131} = -0.3$ and all the other overlap terms $k_{1231} = k_{2231} = 0$ and $\underline{c}_{31} = 0$ in this case. For the second inequality, i.e. $r = 2$, we denote $k_{1132} = 0.9$, $k_{2232} = -0.7$ and all other overlap terms $k_{1232} = k_{2132} = 0$ and $\underline{c}_{32} = 0.1$.

The information of membership functions in (14) and (15) will be introduced to the stability analysis through some slack matrices [50] using the S-procedure.

4.2 Stability analysis

We shall investigate the stability of the PFMB control system (6) using the Lyapunov stability theory subject to the control objective that is to determine the polynomial feedback gains $\mathbf{G}_{jd}(\mathbf{x})$ such that $\mathbf{x}(t) \rightarrow \mathbf{0}$ as time $t \rightarrow \infty$.

The information of the membership functions in Section 4.1 is then brought into the stability analysis using some slack matrices. Introducing the slack matrices $0 \leq \mathbf{R}_{ijd}(\mathbf{x}) = \mathbf{R}_{ijd}(\mathbf{x})^T \in \mathfrak{R}^{N \times N}$, $0 \leq \mathbf{S}_{ijd}(\mathbf{x}) = \mathbf{S}_{ijd}(\mathbf{x})^T \in \mathfrak{R}^{N \times N}$ and $0 \leq \mathbf{T}_{dr}(\mathbf{x}) = \mathbf{T}_{dr}(\mathbf{x})^T \in \mathfrak{R}^{N \times N}$, from (14) and (15), we have the following inequalities:

$$\sum_{i=1}^p \sum_{j=1}^c (w_i m_j - \underline{\gamma}_{ijd}) \mathbf{R}_{ijd}(\mathbf{x}) \geq 0 \quad \forall d \quad (16)$$

$$\sum_{i=1}^p \sum_{j=1}^c (\bar{\gamma}_{ijd} - w_i m_j) \mathbf{S}_{ijd}(\mathbf{x}) \geq 0 \quad \forall d \quad (17)$$

$$\sum_{r=1}^R \left(\sum_{i=1}^p \sum_{j=1}^c k_{ijdr} w_i m_j + \underline{c}_{dr} \right) \mathbf{T}_{dr}(\mathbf{x}) \geq 0 \quad \forall d \quad (18)$$

Adding (16)–(18) to (11), we have

$$\begin{aligned} \dot{V} \leq & \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d \mathbf{z}^T (\bar{\gamma}_{ijd} \mathbf{S}_{ijd}(\mathbf{x}) - \underline{\gamma}_{ijd} \mathbf{R}_{ijd}(\mathbf{x})) \\ & + \sum_{r=1}^R \underline{c}_{dr} \mathbf{T}_{dr}(\mathbf{x}) \mathbf{z} + \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d w_i m_j \mathbf{z}^T (\mathbf{Q}_{ijd}(\mathbf{x}) \\ & + \mathbf{R}_{ijd}(\mathbf{x}) - \mathbf{S}_{ijd}(\mathbf{x}) + \sum_{r=1}^R k_{ijdr} \mathbf{T}_{dr}(\mathbf{x})) \mathbf{z} \end{aligned} \quad (19)$$

Introducing the slack matrices $0 \leq \mathbf{Y}_{ijd}(\mathbf{x}) = \mathbf{Y}_{ijd}(\mathbf{x})^T \in \mathfrak{R}^{N \times N}$ which satisfies

$$\mathbf{Y}_{ijd}(\mathbf{x}) \geq \mathbf{Q}_{ijd}(\mathbf{x}) + \mathbf{R}_{ijd}(\mathbf{x}) - \mathbf{S}_{ijd}(\mathbf{x}) + \sum_{r=1}^R k_{ijdr} \mathbf{T}_{dr} \quad \forall i, j, d \quad (20)$$

Then, from (19) and (20), we have

$$\begin{aligned} \dot{V} &\leq \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d \mathbf{z}^T (\bar{\gamma}_{ijd} \mathbf{S}_{ijd}(\mathbf{x}) - \underline{\gamma}_{ijd} \mathbf{R}_{ijd}(\mathbf{x})) \\ &\quad + \sum_{r=1}^R \underline{c}_{dr} \mathbf{T}_{dr}(\mathbf{x}) \mathbf{z} + \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d w_r m_j \mathbf{z}^T \mathbf{Y}_{ijd}(\mathbf{x}) \mathbf{z} \\ &\leq \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d \mathbf{z}^T (\bar{\gamma}_{ijd} \mathbf{S}_{ijd}(\mathbf{x}) - \underline{\gamma}_{ijd} \mathbf{R}_{ijd}(\mathbf{x})) \\ &\quad + \sum_{r=1}^R \underline{c}_{dr} \mathbf{T}_{dr}(\mathbf{x}) + \bar{\gamma}_{ijd} \mathbf{Y}_{ijd}(\mathbf{x}) \mathbf{z}. \end{aligned} \quad (21)$$

Based on the Lyapunov stability theory, the PFMB control system (6) is asymptotically stable when both $V(t) > 0$ and $\dot{V}(t) < 0$ (excluding $\mathbf{z} = \mathbf{x} = \mathbf{0}$) hold, which can be achieved if the SOS-based stability conditions summarised in the following theorem are satisfied.

Theorem 2: Consider the operating domain partitioned into D sub-domains of which each is characterised by $\underline{x}_{\zeta d} \leq x_{\zeta} \leq \bar{x}_{\zeta d}$, $\zeta = 1, 2, \dots, n$, $d = 1, 2, \dots, D$. The PFMB control system (6) is asymptotically stable if there exist polynomial matrices $\mathbf{R}_{ijd}(\mathbf{x}) = \mathbf{R}_{ijd}(\mathbf{x})^T \in \mathfrak{R}^{N \times N}$, $\mathbf{S}_{ijd}(\mathbf{x}) = \mathbf{S}_{ijd}(\mathbf{x})^T \in \mathfrak{R}^{N \times N}$, $\mathbf{T}_{dr}(\mathbf{x}) = \mathbf{T}_{dr}(\mathbf{x})^T \in \mathfrak{R}^{N \times N}$, $\mathbf{Y}_{ijd}(\mathbf{x}) = \mathbf{Y}_{ijd}(\mathbf{x})^T \in \mathfrak{R}^{N \times N}$, $\mathbf{N}_{jd}(\mathbf{x}) \in \mathfrak{R}^{m \times N}$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, c$, $r = 1, 2, \dots, R$ and $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \mathfrak{R}^{N \times N}$, such that the following SOS-based conditions are satisfied.

$$v^T (\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}}) \mathbf{I}) v \text{ is SOS} \quad (22)$$

$$v^T (\mathbf{R}_{ijd}(\mathbf{x}) - \varepsilon_2(\mathbf{x}) \mathbf{I}) v \text{ is SOS} \quad \forall i, j, d \quad (23)$$

$$v^T (\mathbf{S}_{ijd}(\mathbf{x}) - \varepsilon_3(\mathbf{x}) \mathbf{I}) v \text{ is SOS} \quad \forall i, j, d \quad (24)$$

$$v^T (\mathbf{Y}_{ijd}(\mathbf{x}) - \varepsilon_4(\mathbf{x}) \mathbf{I}) v \text{ is SOS} \quad \forall i, j, d \quad (25)$$

$$v^T (\mathbf{T}_{dr}(\mathbf{x}) - \varepsilon_5(\mathbf{x}) \mathbf{I}) v \text{ is SOS} \quad \forall d, r \quad (26)$$

$$\begin{aligned} v^T (\mathbf{Y}_{ijd}(\mathbf{x}) - \mathbf{Q}_{ijd}(\mathbf{x}) - \mathbf{R}_{ijd}(\mathbf{x}) + \mathbf{S}_{ijd}(\mathbf{x})) \\ - \sum_{r=1}^R k_{ijdr} \mathbf{T}_{dr} - \varepsilon_6(\mathbf{x}) \mathbf{I} v \text{ is SOS} \quad \forall i, j, d \end{aligned} \quad (27)$$

$$\begin{aligned} -v^T \left(\sum_{i=1}^p \sum_{j=1}^c (\bar{\gamma}_{ijd} \mathbf{S}_{ijd}(\mathbf{x}) - \underline{\gamma}_{ijd} \mathbf{R}_{ijd}(\mathbf{x})) + \sum_{r=1}^R \underline{c}_{dr} \mathbf{T}_{dr}(\mathbf{x}) \right. \\ \left. + \bar{\gamma}_{ijd} \mathbf{Y}_{ijd}(\mathbf{x}) + \varepsilon_7(\mathbf{x}) \mathbf{I} \right) v \text{ is SOS} \quad \forall d \end{aligned} \quad (28)$$

where $v \in \mathfrak{R}^N$ is an arbitrary vector independent of \mathbf{x} , $\varepsilon_l(\tilde{\mathbf{x}}) > 0$ and $\varepsilon_l(\mathbf{x}) > 0$, $l = 2, 3, \dots, 7$ are predefined scalar polynomials. $\bar{\gamma}_{ijd}$, $\underline{\gamma}_{ijd}$, k_{ijdr} and \underline{c}_{dr} , $i = 1, 2, \dots, p$, $j = 1, 2, \dots, c$, $d = 1, 2, \dots, D$ and $r = 1, 2, \dots, R$ are predefined constant scalars satisfying (14) and (15), respectively. The polynomial feedback gains are defined as $\mathbf{G}_{jd}(\mathbf{x}) = \mathbf{N}_{jd}(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}})^{-1}$, $j = 1, 2, \dots, c$ and $d = 1, 2, \dots, D$.

Remark 5: The predefined scalars $\bar{\gamma}_{ijd}$ and $\underline{\gamma}_{ijd}$, i.e. the bounds of the membership function overlap term, can be computed

numerically as the extrema of the $w_i(\mathbf{x})m_j(\mathbf{x})$ function in the corresponding sub-domain. The predefined scalars k_{ijdr} and \underline{c}_{dr} can be obtained numerically by some searching algorithms, e.g. genetic algorithm (GA).

Remark 6: It can be seen from Theorem 2 that the information of membership functions in the form of (14) and (15) are included in the stability conditions. The stability conditions are thus more dedicated to the nonlinear plant to be controlled. Consequently, more relaxed stability conditions are achieved compared with the basic stability conditions in Theorem 1.

Remark 7: The PFMB control system (6) is asymptotic stable in the sense of the Lyapunov stability theory that it requires $V > 0$ and $\dot{V} < 0$ for $\hat{\mathbf{x}} \neq \mathbf{0}$. In Theorem 2, the condition (22) is to make sure that the Lyapunov function candidate (9) satisfying $V > 0$ and the condition (28) is to make sure that $\dot{V} < 0$ according to (11), while the conditions (23) to (27) are to make sure that the slack matrices satisfying the prescribed requirement.

Remark 8: Comparing Theorems 1 and 2 in terms of computational complexity, as the number of stability conditions and decision variables to address the relationship among membership functions, the computational demand required to solve a feasible solution to the stability conditions in Theorem 2 will be increased. When more and more information/sub-domains is considered, although more relaxed stability analysis results can be achieved, the computational complexity will increase accordingly. To save computational power, it is thus suggested that the stability conditions in Theorem 1 can be tried first. If no feasible solution is obtained from Theorem 1, the stability conditions in Theorem 2 are considered that the number of relations/sub-domains is increased gradually to look for a feasible solution.

5 Simulation example

In this example, a polynomial fuzzy model with three rules in the form of (3) is considered to represent the nonlinear plant with the following parameters:

$$\mathbf{A}_1(x_2) = \begin{bmatrix} 1.59 - 1.66x_2 & -7.29 + 0.22x_2 - 0.25x_2^2 \\ 0 & -0.36 \end{bmatrix}$$

$$\mathbf{A}_2(x_2) = \begin{bmatrix} 0.02 + 2.72x_2 & 5.46 - 1.25x_2 - 0.25x_2^2 \\ 0 & -0.21 \end{bmatrix}$$

$$\mathbf{A}_3(x_2) = \begin{bmatrix} -a + 1.17x_2 & -4.33 - 3.36x_2 - 0.25x_2^2 \\ 0 & -0.05 \end{bmatrix}$$

$$\mathbf{B}_1(x_2) = \begin{bmatrix} 1 + x_2^2 \\ 0 \end{bmatrix}$$

$$\mathbf{B}_2(x_2) = \begin{bmatrix} 8 + x_2 \\ 0 \end{bmatrix}$$

$$\mathbf{B}_3(x_2) = \begin{bmatrix} -b + 6 + 2.36x_2^2 \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{x}} = \mathbf{x} = [x_1, x_2]^T$$

where a and b are scalar parameters to be determined. It is assumed that $x_2 \in [-10, 10]$.

The membership functions are chosen as

$$w_1(x_2) = \mu_{M_1}(x_2) = 1 - \frac{1}{1 + e^{-(x_2+4)}}$$

$$w_2(x_2) = \mu_{M_2}(x_2) = 1 - w_1(x_2) - w_3(x_2)$$

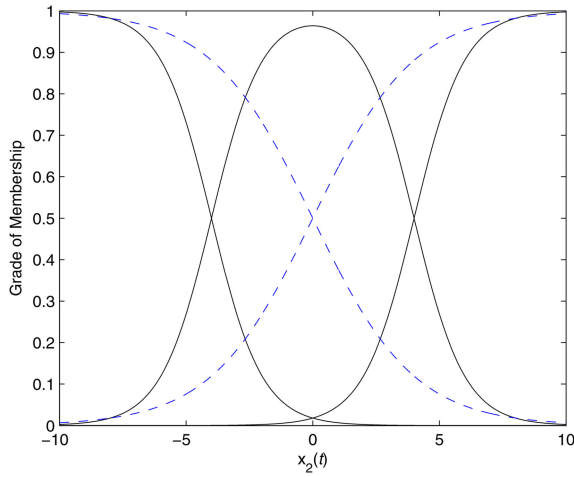


Fig. 1 Membership functions of polynomial fuzzy model (solid lines) and polynomial fuzzy controller (dotted lines)

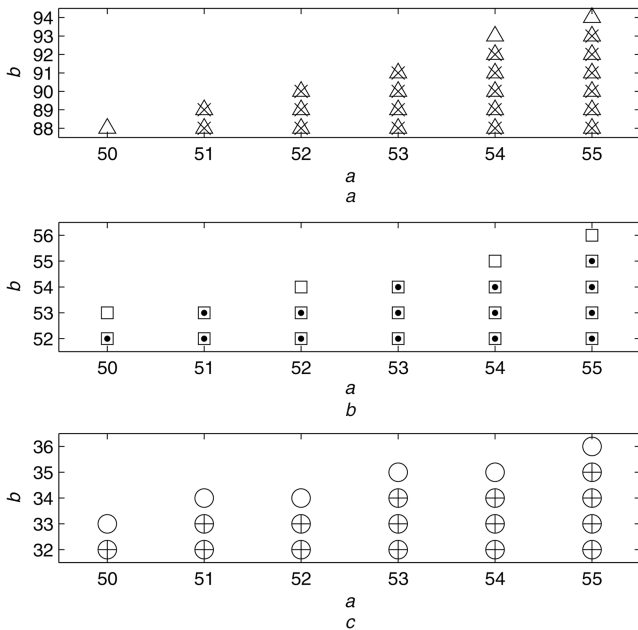


Fig. 2 Stability regions by Theorem 2

(a) Sub-domain controllers, four sub-domains ($D = 4$). Case 1: using the regional boundary information plus the regional relation among overlap terms (' Δ '), Case 2: using the regional boundary information only (' \times '), (b) Common controller, four sub-domains ($D = 4$). Case 3: using regional boundary information plus the regional relation among overlap terms (' \square '), Case 4: using the regional boundary information only (' \bullet '), (c) Common controller, one sub-domain ($D = 1$). Case 5: using the global boundary information plus the global relation among overlap terms (' \circ '), Case 6: no membership function information used (' $+$ ')

$$w_3(x_2) = \mu_{M_1^3}(x_2) = \frac{1}{1 + e^{-(x_2-4)}}$$

for $x_2 \in [-10, 10]$, which are shown in Fig. 1.

A polynomial fuzzy controller with two rules in the form of (5) is proposed to control the nonlinear plant. The membership functions are chosen as

$$m_1(x_2) = \mu_{N_1^1}(x_2) = 1 - \frac{1}{1 + e^{\frac{-x_2}{2}}}$$

$$m_2(x_2) = \mu_{N_1^2}(x_2) = 1 - m_1(x_2)$$

which are also shown in Fig. 1.

Remark 9: It should be noted that the membership functions of both polynomial fuzzy model and polynomial fuzzy controller are not the same in this example. In addition, the entries of the second row of the input matrices $\mathbf{B}_i(\mathbf{x})$ are all zero for all i , therefore, $\tilde{\mathbf{x}} = (x_2)$.

The stability conditions of Theorem 2 are employed to check the stability region of the PFMB control system with $50 \leq a \leq 55$ and $32 \leq b \leq 94$ (both at the interval of 1). In this example, the solution is found numerically using SOSTOOLS [51] where the degrees of $\mathbf{R}_{ijd}(\mathbf{x})$, $\mathbf{S}_{ijd}(\mathbf{x})$, $\mathbf{Y}_{ijd}(\mathbf{x})$ and \mathbf{T}_{dr} are chosen as 0 in x_2 , the degrees of $\mathbf{X}(\tilde{\mathbf{x}})$ and $\mathbf{N}_{jd}(\mathbf{x})$ as 2 in x_2 , $\varepsilon_l = 0.001$, $l = 1, 2, \dots, 7$.

We partition the operating domain $x_2 \in [-10, 10]$ into four sub-domains, i.e. $D = 4$. The s_d sub-domain is characterised by $5(d-3) \leq x_2 \leq 5(d-2)$ and $d = 1, 2, 3, 4$. With the chosen membership functions, the regional boundary information $\underline{\gamma}_{ijd}$ and $\bar{\gamma}_{ijd}$ found numerically are shown in Table 1 in the Appendix. The regional relation among overlap terms \underline{c}_{dr} and k_{ijdr} found numerically by GA is shown in Tables 2 and 3 in the Appendix. All the parameters $\underline{\gamma}_{ijd}$, $\bar{\gamma}_{ijd}$, \underline{c}_{dr} and k_{ijdr} in this example are computed using MATLAB. Employing the sub-domain controllers in Theorem 2, consider two cases: Case 1 is with both regional boundary information and regional relation among overlap terms and Case 2 is with regional boundary information only, without regional relation among overlap terms. The stability regions obtained are shown in Fig. 2a indicated by ' Δ ' (Case 1) and ' \times ' (Case 2). It shows that larger stability region can be obtained with the regional relation among overlap terms as the membership function information. To highlight the result and show the boundary of the stability region, we choose the range of b in $88 \leq b \leq 94$ to show. It should be mentioned that the region for b in the range of 32 to 87 at the interval of 1 is stable for both Case 1 and Case 2.

To demonstrate the effect of the sub-domain controller, under the same operating domain partition as in Case 1 and Case 2, we employ the common controller instead of the sub-domain controller in above cases. The common controller is that $\mathbf{G}_f(\mathbf{x}) = \mathbf{G}_{jd}(\mathbf{x})$ for all $d = 1, 2, 3, 4$. We consider two cases: Case 3 is with both regional boundary information and regional relation among overlap terms and Case 4 is with regional boundary information only, without regional relation among overlap terms. As the operating domain partition are remain the same, the $\underline{\gamma}_{ijd}$, $\bar{\gamma}_{ijd}$, \underline{c}_{dr} and k_{ijdr} are the same as in Case 1 and Case 2. The stability regions obtained are shown in Fig. 2b indicated by ' \square ' (Case 3) and ' \bullet ' (Case 4). We can find the same effect of regional relation among overlap terms, i.e. relaxing the stability conditions indicated by larger stability region. Same as above, to highlight the boundary of the stability region, the range of b shown is chosen as $52 \leq b \leq 56$ although the stability region can be found for b in the range of 32 to 55 at the interval of 1 for both Case 3 and Case 4. Comparing the stability regions shown in Figs. 2a and 2b, it can be seen that Case 1 offers a larger stability region than Case 3 which demonstrates that the sub-domain polynomial fuzzy controller has a better feedback compensation capability than the common one. The same observation is applied to Case 2 and Case 4.

For the purpose of verifying the effect of the reducing of membership function information to the stability analysis results, we reduce the number of sub-domains from 4 to 1, i.e. $D = 1$. In this scenario, we only consider one single sub-domain which is actually the overall operating domain and the controller employed is common controller trivially. A group of new $\underline{\gamma}_{ijd}$ and $\bar{\gamma}_{ijd}$ which represent the global boundary information and a group of new \underline{c}_{dr} and k_{ijdr} taken as the global relation among overlap terms are obtained numerically and shown in Tables 4 and 5 in the Appendix, respectively. We consider two cases: Case 5 is with both global boundary information and global relation among overlap terms and Case 6 is without any membership function information. The stability regions obtained are shown in Fig. 2c indicated by ' \circ ' (Case 5) and ' $+$ ' (Case 6). It can be seen in Fig. 2c, comparing with Case 5 and Case 6, the membership function information can

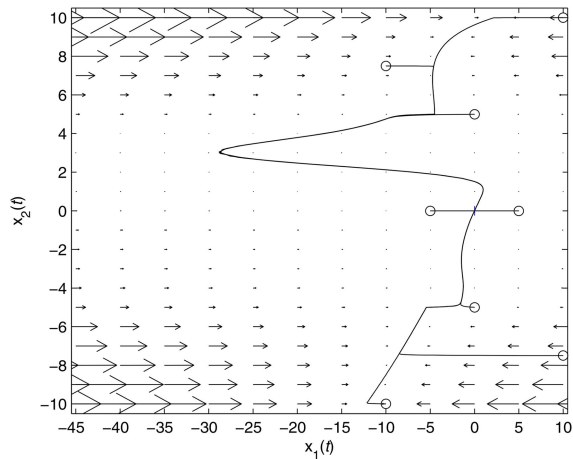


Fig. 3 Behaviours in x_1 - x_2 plane for Case 1 with $a = 55$ and $b = 94$. The initial conditions are indicated by 'o'

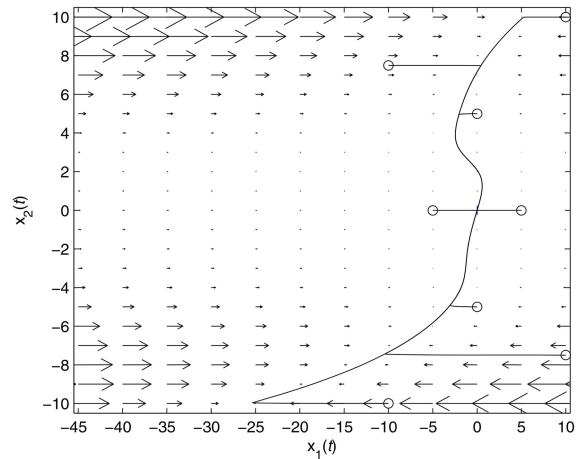


Fig. 6 Behaviours in x_1 - x_2 plane for Case 4 with $a = 51$, $b = 52$. The initial conditions are indicated by 'o'

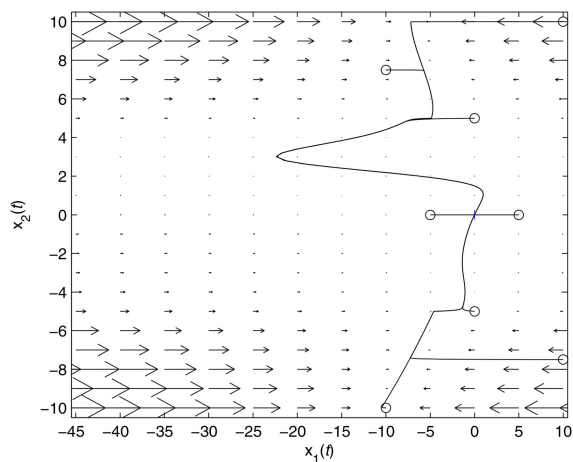


Fig. 4 Behaviours in x_1 - x_2 plane for Case 2 with $a = 51$ and $b = 88$. The initial conditions are indicated by 'o'

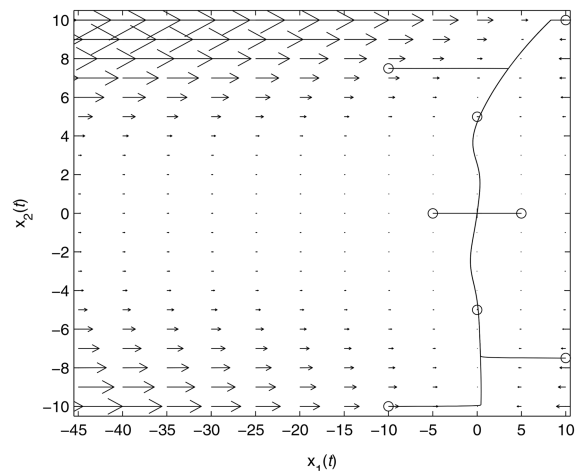


Fig. 7 Behaviours in x_1 - x_2 plane for Case 5 with $a = 55$ and $b = 35$. The initial conditions are indicated by 'o'

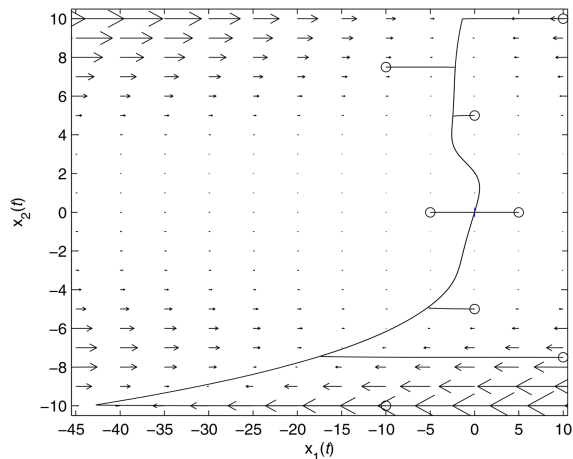


Fig. 5 Behaviours in x_1 - x_2 plane for Case 3 with $a = 54$ and $b = 55$. The initial conditions are indicated by 'o'

relax the stability conditions resulting in offering a larger stability region. The range of b is chosen as $32 \leq b \leq 36$ to highlight the boundary of the stability region. Since the region below $b = 52$ in Fig. 2b is stable for both Case 3 and Case 4, by comparing with Figs. 2b and 2c, it can be concluded that the more number of sub-domains is considered, the more the regional membership function information is extracted which will lead to more relaxed stability conditions.

It should be noted that the stability conditions used for Case 6 are the conditions in Theorem 1. It can be considered as the basic

stability conditions derived from Theorem 2 in [11] for the case of imperfect premise matching. For Case 4, with common fuzzy controller and without regional overlap term information, it is the case of the stability conditions of polynomial version for the case of imperfect premise matching derived from Theorem 2 in [40]. It can be seen from Fig. 2 that when regional relation among overlap terms is considered and apply the regional fuzzy control strategy, more relaxed results can be achieved.

For verification, we show the phase plots of system states of some stable points showing in Fig. 2 for all cases. For Case 1, considering $a = 55$ and $b = 94$, the phase plot is shown in Fig. 3. For Case 2, considering $a = 51$ and $b = 88$, the phase plot is shown in Fig. 4. For Case 3, considering $a = 54$ and $b = 55$, the phase plot is shown in Fig. 5. For Case 4, considering $a = 51$ and $b = 52$, the phase plot is shown in Fig. 6. For Case 5, considering $a = 55$ and $b = 35$, the phase plot is shown in Fig. 7. For Case 6, considering $a = 50$ and $b = 32$, the phase plot is shown in Fig. 8. The matrices $\mathbf{X}(\bar{\mathbf{x}})$ and $\mathbf{N}_{j_d}(\mathbf{x})$ of each case are obtained with SOSTOOLS [51] and are shown in Tables 6–8. It can be seen from all the phase plots that the polynomial fuzzy model is successfully stabilised by the polynomial fuzzy controller.

6 Conclusion

This paper has investigated the stability of PFMB control systems under imperfect premise matching based on the Lyapunov stability theory. It is not required that the rule number and membership functions of the polynomial fuzzy controller are the same as those of the polynomial fuzzy model. The membership function information has been taken account into the stability analysis through several inequality constraints which are formed by the

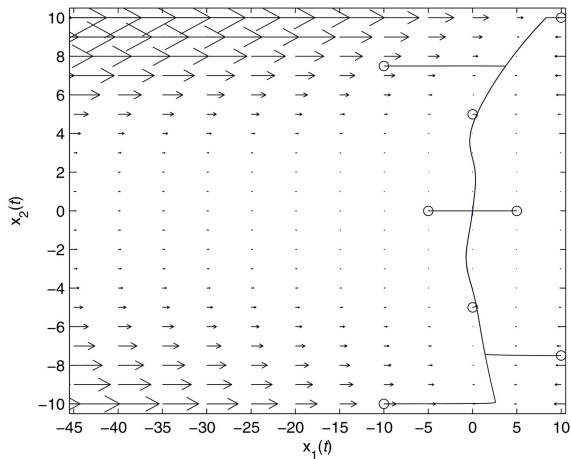


Fig. 8 Behaviours in x_1 - x_2 plane for Case 6 with $a = 50$ and $b = 32$. The initial conditions are indicated by 'o'

boundary information of each membership function and the numerical relationship among them. For the purpose of representing the membership functions more accurately, the operating domain is partitioned into several sub-domains to utilise the regional information of the membership functions. Based on the partition, a group of sub-domain polynomial fuzzy controllers are designed to enhance the feedback compensation capability and to obtain more relaxed stability conditions. A simulation example has been given to demonstrate the merit of the proposed approach.

7 Acknowledgments

The work described in this paper was partly supported by King's College London and Foreign Experts Scheme of China (grant no. GDW20151100010). Open access for this work was funded by King's College London.

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9 Appendix

Table 1 $\underline{\gamma}_{ijd}$ and $\bar{\gamma}_{ijd}$ for four sub-domains case in Section 5

s_d	$\underline{\gamma}_{ijd}$	$\bar{\gamma}_{ijd}$	
s_1	$\underline{\gamma}_{111} = 6.7560 \times 10^{-1}$	$\bar{\gamma}_{111} = 9.9085 \times 10^{-1}$	
	$\underline{\gamma}_{121} = 6.6763 \times 10^{-3}$	$\bar{\gamma}_{121} = 5.5457 \times 10^{-2}$	
	$\underline{\gamma}_{211} = 2.4552 \times 10^{-3}$	$\bar{\gamma}_{211} = 2.4843 \times 10^{-1}$	
	$\underline{\gamma}_{221} = 1.6543 \times 10^{-5}$	$\bar{\gamma}_{221} = 2.0392 \times 10^{-2}$	
	$\underline{\gamma}_{311} = 8.2596 \times 10^{-7}$	$\bar{\gamma}_{311} = 1.1403 \times 10^{-4}$	
	$\underline{\gamma}_{321} = 5.5653 \times 10^{-9}$	$\bar{\gamma}_{321} = 9.3605 \times 10^{-6}$	
	s_2	$\underline{\gamma}_{112} = 8.9931 \times 10^{-3}$	$\bar{\gamma}_{112} = 6.7560 \times 10^{-1}$
		$\underline{\gamma}_{122} = 8.9931 \times 10^{-3}$	$\bar{\gamma}_{122} = 5.9988 \times 10^{-2}$
		$\underline{\gamma}_{212} = 2.4843 \times 10^{-1}$	$\bar{\gamma}_{212} = 6.4274 \times 10^{-1}$
$\underline{\gamma}_{222} = 2.0392 \times 10^{-2}$		$\bar{\gamma}_{222} = 4.8201 \times 10^{-1}$	
$\underline{\gamma}_{312} = 1.1403 \times 10^{-4}$		$\bar{\gamma}_{312} = 8.9931 \times 10^{-3}$	
$\underline{\gamma}_{322} = 9.3605 \times 10^{-6}$		$\bar{\gamma}_{322} = 8.9931 \times 10^{-3}$	
s_3		$\underline{\gamma}_{113} = 9.3605 \times 10^{-6}$	$\bar{\gamma}_{113} = 8.9931 \times 10^{-3}$
		$\underline{\gamma}_{123} = 1.1403 \times 10^{-4}$	$\bar{\gamma}_{123} = 8.9931 \times 10^{-3}$
		$\underline{\gamma}_{213} = 2.0392 \times 10^{-2}$	$\bar{\gamma}_{213} = 4.8201 \times 10^{-1}$
	$\underline{\gamma}_{223} = 2.4843 \times 10^{-1}$	$\bar{\gamma}_{223} = 6.4274 \times 10^{-1}$	
	$\underline{\gamma}_{313} = 8.9931 \times 10^{-3}$	$\bar{\gamma}_{313} = 5.9988 \times 10^{-2}$	
	$\underline{\gamma}_{323} = 8.9931 \times 10^{-3}$	$\bar{\gamma}_{323} = 6.7560 \times 10^{-1}$	
	s_4	$\underline{\gamma}_{114} = 5.5653 \times 10^{-9}$	$\bar{\gamma}_{114} = 9.3605 \times 10^{-6}$
		$\underline{\gamma}_{124} = 8.2596 \times 10^{-7}$	$\bar{\gamma}_{124} = 1.1403 \times 10^{-4}$
		$\underline{\gamma}_{214} = 1.6543 \times 10^{-5}$	$\bar{\gamma}_{214} = 2.0392 \times 10^{-2}$
$\underline{\gamma}_{224} = 2.4552 \times 10^{-3}$		$\bar{\gamma}_{224} = 2.4843 \times 10^{-1}$	
$\underline{\gamma}_{314} = 6.6763 \times 10^{-3}$		$\bar{\gamma}_{314} = 5.5457 \times 10^{-2}$	
$\underline{\gamma}_{324} = 6.7560 \times 10^{-1}$		$\bar{\gamma}_{324} = 9.9085 \times 10^{-1}$	

Table 2 \underline{c}_{dr} for four sub-domains case in Section 5

s_d	\underline{c}_{dr}
s_1	$\underline{c}_{11} = 0.0000, \underline{c}_{12} = 3.5580 \times 10^{-1}$
	$\underline{c}_{13} = 0.0000, \underline{c}_{14} = 3.6960 \times 10^{-1}$
	$\underline{c}_{15} = 1.1464, \underline{c}_{16} = 2.0600 \times 10^{-2}$
s_2	$\underline{c}_{21} = 0.0000, \underline{c}_{22} = 0.0000$
	$\underline{c}_{23} = 0.0000, \underline{c}_{24} = 0.0000$
	$\underline{c}_{25} = 8.0640 \times 10^{-1}, \underline{c}_{26} = 6.6380 \times 10^{-1}$
s_3	$\underline{c}_{31} = 7.1310 \times 10^{-1}, \underline{c}_{32} = 8.2850 \times 10^{-1}$
	$\underline{c}_{33} = 0.0000, \underline{c}_{34} = 0.0000$
	$\underline{c}_{35} = 5.6440 \times 10^{-1}, \underline{c}_{36} = 0.0000$
s_4	$\underline{c}_{41} = 1.7400 \times 10^{-2}, \underline{c}_{42} = 9.3600 \times 10^{-2}$
	$\underline{c}_{43} = 2.5770 \times 10^{-1}, \underline{c}_{44} = 0.0000$
	$\underline{c}_{45} = 0.0000, \underline{c}_{46} = 0.0000$

Table 3a k_{ijdr} for four sub-domains case in Section 5

s_d	k_{ijdr}
s_1	$k_{1111} = 1.0000, k_{1112} = -3.6080 \times 10^{-1}, k_{1113} = -2.2000 \times 10^{-3},$ $k_{1114} = -3.6790 \times 10^{-1}, k_{1115} = -1.1519, k_{1116} = -1.0000 \times 10^{-3}$ $k_{1211} = -1.7500, k_{1212} = 1.0000, k_{1213} = -2.0000 \times 10^{-3}$ $k_{1214} = -1.5700 \times 10^{-2}, k_{1215} = -1.0000 \times 10^{-3},$ $k_{1216} = -1.0000 \times 10^{-3}$ $k_{2111} = -2.0411, k_{2112} = -1.0000 \times 10^{-3}, k_{2113} = 1.0000$ $k_{2114} = -1.0000 \times 10^{-3}, k_{2115} = -1.0000 \times 10^{-3},$ $k_{2116} = -5.9700 \times 10^{-2}$ $k_{2211} = -3.2588, k_{2212} = -1.1437, k_{2213} = -1.2056 \times 10^1$ $k_{2214} = 1.0000, k_{2215} = -1.0000 \times 10^{-3}, k_{2216} = -1.0000 \times 10^{-3}$ $k_{3111} = -6.9110 \times 10^{-1}, k_{3112} = -1.0000 \times 10^{-3}, k_{3113} = -2.2842$ $k_{3114} = -1.0000 \times 10^{-3}, k_{3115} = 1.0000, k_{3116} = -1.0000 \times 10^{-3}$ $k_{3211} = -5.5390 \times 10^{-1}, k_{3212} = -8.5110 \times 10^{-1},$ $k_{3213} = -5.4030 \times 10^{-1}$ $k_{3214} = -1.0000 \times 10^{-3}, k_{3215} = -6.9830 \times 10^{-1}, k_{3216} = 1.0000$
s_2	$k_{1121} = 0.0000, k_{1122} = 0.0000, k_{1123} = 0.0000$ $k_{1124} = 0.0000, k_{1125} = -1.1638, k_{1126} = -1.9540 \times 10^{-1}$ $k_{1221} = 0.0000, k_{1222} = 0.0000, k_{1223} = 0.0000$ $k_{1224} = 0.0000, k_{1225} = -1.0000 \times 10^{-3}, k_{1226} = -1.0000 \times 10^{-3}$ $k_{2121} = 0.0000, k_{2122} = 0.0000, k_{2123} = 0.0000$ $k_{2124} = 0.0000, k_{2125} = -6.0900 \times 10^{-2}, k_{2126} = -9.8940 \times 10^{-1}$ $k_{2221} = 0.0000, k_{2222} = 0.0000, k_{2223} = 0.0000$ $k_{2224} = 0.0000, k_{2225} = -1.0000 \times 10^{-3}, k_{2226} = -1.0000 \times 10^{-3}$ $k_{3121} = 0.0000, k_{3122} = 0.0000, k_{3123} = 0.0000$ $k_{3124} = 0.0000, k_{3125} = 1.0000, k_{3126} = -1.4857$ $k_{3221} = 0.0000, k_{3222} = 0.0000, k_{3223} = 0.0000$ $k_{3224} = 0.0000, k_{3225} = -1.0007, k_{3226} = 1.0000$

Table 4 $\underline{\gamma}_{ijd}$ and $\bar{\gamma}_{ijd}$ for one sub-domain case in Section 5

s_d	$\underline{\gamma}_{ijd}$	$\bar{\gamma}_{ijd}$
s_1	$\underline{\gamma}_{111} = 5.5653 \times 10^{-9}$ $\underline{\gamma}_{121} = 8.2596 \times 10^{-7}$ $\underline{\gamma}_{211} = 1.6543 \times 10^{-5}$ $\underline{\gamma}_{221} = 1.6543 \times 10^{-5}$ $\underline{\gamma}_{311} = 8.2596 \times 10^{-7}$ $\underline{\gamma}_{321} = 5.5653 \times 10^{-9}$	$\bar{\gamma}_{111} = 9.9085 \times 10^{-1}$ $\bar{\gamma}_{211} = 6.4274 \times 10^{-1}$ $\bar{\gamma}_{121} = 5.9988 \times 10^{-2}$ $\bar{\gamma}_{221} = 6.4274 \times 10^{-1}$ $\bar{\gamma}_{311} = 5.9988 \times 10^{-2}$ $\bar{\gamma}_{321} = 9.9085 \times 10^{-1}$

Table 3b

s_3	$k_{1131} = 1.0000, k_{1132} = 0.0000, k_{1133} = -8.8135$ $k_{1134} = -3.9795, k_{1135} = -1.0000 \times 10^{-3}, k_{1136} = -9.4800 \times 10^{-2}$ $k_{1231} = -9.6050 \times 10^{-1}, k_{1232} = 1.0000, k_{1233} = -5.0030$ $k_{1234} = -6.0010, k_{1235} = -9.5110 \times 10^{-1}, k_{1236} = -1.0000 \times 10^{-3}$ $k_{2131} = -9.3030 \times 10^{-1}, c_{2132} = 0.0000, k_{2133} = 1.0000$ $k_{2134} = -7.5100 \times 10^{-1}, k_{2135} = -1.0000 \times 10^{-3},$ $k_{2136} = -1.0000 \times 10^{-3}$ $k_{2231} = -5.0050 \times 10^{-1}, k_{2232} = -2.0000 \times 10^{-1},$ $k_{2233} = -1.0000 \times 10^{-3}$ $k_{2234} = 1.0000, k_{2235} = -9.1750 \times 10^{-1}, k_{2236} = -6.2000 \times 10^{-3}$ $k_{3131} = -1.5007, k_{3132} = 0.0000, k_{3133} = -2.5100 \times 10^{-1}$ $k_{3134} = -2.9733, k_{3135} = 1.0000, k_{3136} = -5.1060 \times 10^{-1}$ $k_{3231} = -5.8660 \times 10^{-1}, k_{3232} = -1.1455, k_{3233} = -8.2000 \times 10^{-3}$ $k_{3234} = -9.9400 \times 10^{-2}, k_{3235} = -1.0000 \times 10^{-3}, k_{3236} = 1.0000$
s_4	$k_{1141} = 1.0000, k_{1142} = 0.0000, k_{1143} = -1.8171$ $k_{1144} = -1.0954, k_{1145} = 0.0000, k_{1146} = 0.0000$ $k_{1241} = -1.0000 \times 10^{-3}, k_{1242} = 1.0000, k_{1243} = -8.5840 \times 10^{-1}$ $k_{1244} = -3.9385, k_{1245} = 0.0000, k_{1246} = 0.0000$ $k_{2141} = -5.6170 \times 10^{-1}, k_{2142} = -3.7530 \times 10^{-1}, k_{2143} = 1.0000$ $k_{2144} = -1.1945 \times 10^1, k_{2145} = 0.0000, k_{2146} = 0.0000$ $k_{2241} = -1.0000 \times 10^{-3}, k_{2242} = -3.2620 \times 10^{-1}, k_{2243} = -1.0960$ $k_{2244} = 1.0000, k_{2245} = 0.0000, k_{2246} = 0.0000$ $k_{3141} = -1.0000 \times 10^{-3}, k_{3142} = 0.0000, k_{3143} = -1.0000 \times 10^{-3}$ $k_{3144} = -5.7700 \times 10^{-2}, k_{3145} = 0.0000, k_{3146} = 0.0000$ $k_{3241} = -1.0000 \times 10^{-3}, k_{3242} = 0.0000, k_{3243} = -1.0000 \times 10^{-3}$ $k_{3244} = -1.0000 \times 10^{-3}, k_{3245} = 0.0000, k_{3246} = 0.0000$

Table 5 \underline{c}_{dr} and k_{ijdr} for one sub-domain case in Section 5

s_d	\underline{c}_{dr}	k_{ijdr}
s_1	$\underline{c}_{11} = 0.0000$ $\underline{c}_{12} = 0.0000$ $\underline{c}_{13} = 0.0000$	$k_{1111} = 1.0000, k_{1112} = -1.3000 \times 10^{-3}$ $k_{1113} = -1.0000 \times 10^{-3}, k_{1211} = -2.7000 \times 10^{-3}$ $k_{1212} = -3.2500 \times 10^{-2}, k_{1213} = -1.1000 \times 10^{-3}$ $k_{2111} = -1.0000 \times 10^{-3}, k_{2112} = 1.0000$ $k_{2113} = -1.0000 \times 10^{-3}, k_{2211} = -1.0000 \times 10^{-3}$ $k_{2212} = -1.0000 \times 10^{-3}, k_{2213} = 1.0000$ $k_{3111} = -1.2000 \times 10^{-3}, k_{3112} = -1.0000 \times 10^{-3}$ $k_{3113} = -8.8600 \times 10^{-2}, k_{3211} = -1.0000 \times 10^{-3}$ $k_{3212} = -1.0000 \times 10^{-3}, k_{3213} = -1.0000 \times 10^{-3}$

Table 6 Feedback gains $\mathbf{X}(x_2)$ and $\mathbf{N}_{jd}(x_2)$ for Case 1 with $a = 55$ and $b = 94$ in Section 5

$$\mathbf{X}(x_2) = \begin{bmatrix} \mathbf{X}_{11}(x_2) & \mathbf{X}_{12}(x_2) \\ \mathbf{X}_{21}(x_2) & \mathbf{X}_{22}(x_2) \end{bmatrix}$$

$$\mathbf{X}_{11}(x_2) = 2.2790 \times 10^{-3}x_2^2 + 7.1140 \times 10^{-3}x_2 + 1.1080 \times 10^{-1}$$

$$\mathbf{X}_{12}(x_2) = \mathbf{X}_{21}^T(x_2) = 2.0050 \times 10^{-5}x_2^2 - 9.8560 \times 10^{-6}x_2 + 4.9070 \times 10^{-4}$$

$$\mathbf{X}_{22}(x_2) = 3.9630 \times 10^{-6}x_2^2 - 1.4000 \times 10^{-4}x_2 + 1.2340 \times 10^{-3}$$

$$s_d \mathbf{N}_{jd}(x_2) = [\mathbf{N}_{jd1}(x_2) \quad \mathbf{N}_{jd2}(x_2)]$$

$$s_1 \mathbf{N}_{111}(x_2) = -5.9890 \times 10^{-3}x_2^2 - 2.3934 \times 10^{-3}x_2 - 2.6829 \times 10^{-1}$$

$$\mathbf{N}_{112}(x_2) = -2.0124 \times 10^{-5}x_2^2 - 2.6215 \times 10^{-4}x_2 + 3.2901 \times 10^{-3}$$

$$\mathbf{N}_{211}(x_2) = -5.7650 \times 10^{-3}x_2^2 - 3.6044 \times 10^{-3}x_2 - 7.0685 \times 10^{-2}$$

$$\mathbf{N}_{212}(x_2) = -9.5058 \times 10^{-6}x_2^2 - 1.0412 \times 10^{-4}x_2 + 2.4881 \times 10^{-3}$$

$$s_2 \mathbf{N}_{121}(x_2) = -5.8849 \times 10^{-3}x_2^2 - 4.2751 \times 10^{-3}x_2 - 1.5270 \times 10^{-1}$$

$$\mathbf{N}_{122}(x_2) = -4.3962 \times 10^{-6}x_2^2 - 8.7609 \times 10^{-7}x_2 + 2.1779 \times 10^{-4}$$

$$\mathbf{N}_{221}(x_2) = -5.7924 \times 10^{-3}x_2^2 - 4.3266 \times 10^{-3}x_2 - 6.9215 \times 10^{-2}$$

$$\mathbf{N}_{222}(x_2) = -6.7048 \times 10^{-6}x_2^2 - 2.7756 \times 10^{-5}x_2 - 2.4929 \times 10^{-4}$$

$$s_3 \mathbf{N}_{131}(x_2) = -6.0313 \times 10^{-3}x_2^2 - 4.9202 \times 10^{-3}x_2 - 4.8730 \times 10^{-2}$$

$$\mathbf{N}_{132}(x_2) = -6.4740 \times 10^{-6}x_2^2 - 1.7660 \times 10^{-5}x_2 - 6.2832 \times 10^{-4}$$

$$\mathbf{N}_{231}(x_2) = -6.1337 \times 10^{-3}x_2^2 - 4.9391 \times 10^{-3}x_2 - 4.7349 \times 10^{-2}$$

$$\mathbf{N}_{232}(x_2) = -1.6135 \times 10^{-5}x_2^2 - 2.6247 \times 10^{-5}x_2 - 5.0413 \times 10^{-4}$$

$$s_4 \mathbf{N}_{141}(x_2) = -5.7780 \times 10^{-3}x_2^2 - 3.9018 \times 10^{-3}x_2 - 5.8616 \times 10^{-2}$$

$$\mathbf{N}_{142}(x_2) = -1.7569 \times 10^{-5}x_2^2 - 2.7354 \times 10^{-5}x_2 - 4.4848 \times 10^{-4}$$

$$\mathbf{N}_{241}(x_2) = -5.7676 \times 10^{-3}x_2^2 - 4.3167 \times 10^{-3}x_2 - 3.0887 \times 10^{-2}$$

$$\mathbf{N}_{242}(x_2) = -2.9687 \times 10^{-5}x_2^2 - 3.4835 \times 10^{-5}x_2 - 3.1681 \times 10^{-4}$$

Table 7 Feedback gains $\mathbf{X}(x_2)$ and $\mathbf{N}_{jd}(x_2)$ for Case 2 with $a = 51$ and $b = 88$ in Section 5

$$\mathbf{X}(x_2) = \begin{bmatrix} \mathbf{X}_{11}(x_2) & \mathbf{X}_{12}(x_2) \\ \mathbf{X}_{21}(x_2) & \mathbf{X}_{22}(x_2) \end{bmatrix}$$

$$\mathbf{X}_{11}(x_2) = 3.5910 \times 10^{-3}x_2^2 + 1.1820 \times 10^{-2}x_2 + 1.5640 \times 10^{-1}$$

$$\mathbf{X}_{12}(x_2) = \mathbf{X}_{21}^T(x_2) = 2.9150 \times 10^{-5}x_2^2 - 4.8560 \times 10^{-5}x_2 + 5.7200 \times 10^{-4}$$

$$\mathbf{X}_{22}(x_2) = 5.1490 \times 10^{-6}x_2^2 - 2.2750 \times 10^{-4}x_2 + 2.4680 \times 10^{-3}$$

$$s_d \mathbf{N}_{jd}(x_2) = [\mathbf{N}_{jd1}(x_2) \quad \mathbf{N}_{jd2}(x_2)]$$

$$s_1 \mathbf{N}_{111}(x_2) = -9.5099 \times 10^{-3}x_2^2 - 4.2653 \times 10^{-3}x_2 - 3.6991 \times 10^{-1}$$

$$\mathbf{N}_{112}(x_2) = -3.3067 \times 10^{-5}x_2^2 - 2.7368 \times 10^{-4}x_2 + 6.4091 \times 10^{-3}$$

$$\mathbf{N}_{211}(x_2) = -9.1014 \times 10^{-3}x_2^2 - 6.5546 \times 10^{-3}x_2 - 1.0229 \times 10^{-1}$$

$$\mathbf{N}_{212}(x_2) = -1.4738 \times 10^{-5}x_2^2 - 1.2790 \times 10^{-4}x_2 + 5.2481 \times 10^{-3}$$

$$s_2 \mathbf{N}_{121}(x_2) = -9.1915 \times 10^{-3}x_2^2 - 6.7481 \times 10^{-3}x_2 - 2.0012 \times 10^{-1}$$

$$\mathbf{N}_{122}(x_2) = 8.4487 \times 10^{-6}x_2^2 + 3.6431 \times 10^{-5}x_2 + 6.0266 \times 10^{-4}$$

$$\mathbf{N}_{221}(x_2) = -9.0924 \times 10^{-3}x_2^2 - 7.1165 \times 10^{-3}x_2 - 9.2380 \times 10^{-2}$$

$$\mathbf{N}_{222}(x_2) = 2.4669 \times 10^{-6}x_2^2 - 3.5544 \times 10^{-5}x_2 - 2.4507 \times 10^{-4}$$

$$s_3 \mathbf{N}_{131}(x_2) = -9.3374 \times 10^{-3}x_2^2 - 7.9303 \times 10^{-3}x_2 - 6.4856 \times 10^{-2}$$

$$\mathbf{N}_{132}(x_2) = -1.5222 \times 10^{-6}x_2^2 - 3.5818 \times 10^{-5}x_2 - 9.6682 \times 10^{-4}$$

$$\mathbf{N}_{231}(x_2) = -9.6093 \times 10^{-3}x_2^2 - 8.1046 \times 10^{-3}x_2 - 6.4100 \times 10^{-2}$$

$$\mathbf{N}_{232}(x_2) = -1.0876 \times 10^{-5}x_2^2 - 4.2272 \times 10^{-5}x_2 - 8.4239 \times 10^{-4}$$

$$s_4 \mathbf{N}_{141}(x_2) = -9.1041 \times 10^{-3}x_2^2 - 6.8544 \times 10^{-3}x_2 - 8.2304 \times 10^{-2}$$

$$\mathbf{N}_{142}(x_2) = -1.4327 \times 10^{-5}x_2^2 - 4.9133 \times 10^{-5}x_2 - 6.3889 \times 10^{-4}$$

$$\mathbf{N}_{241}(x_2) = -9.1474 \times 10^{-3}x_2^2 - 7.2984 \times 10^{-3}x_2 - 5.0482 \times 10^{-2}$$

$$\mathbf{N}_{242}(x_2) = -4.5419 \times 10^{-5}x_2^2 - 6.1082 \times 10^{-5}x_2 - 4.4539 \times 10^{-4}$$

Table 8 Feedback gains $\mathbf{X}(x_2)$ and $\mathbf{N}_{jd}(x_2)$ for Case 3, Case 4, Case 5 and Case 6 in Section 5

$$\mathbf{X}(x_2) = \begin{bmatrix} \mathbf{X}_{11}(x_2) & \mathbf{X}_{12}(x_2) \\ \mathbf{X}_{21}(x_2) & \mathbf{X}_{22}(x_2) \end{bmatrix}, \mathbf{N}_j(x_2) = [\mathbf{N}_{j1}(x_2) \quad \mathbf{N}_{j2}(x_2)]$$

Case 3 $\mathbf{X}_{11}(x_2) = 8.1690 \times 10^{-3}x_2^2 + 6.4310 \times 10^{-2}x_2 + 3.0110 \times 10^{-1}$
with $a = 54$
and $b = 95$

$$\mathbf{X}_{12}(x_2) = \mathbf{X}_{21}^T(x_2) = 1.4740 \times 10^{-5}x_2^2 + 5.3990 \times 10^{-5}x_2 + 3.1750 \times 10^{-4}$$

$$\mathbf{X}_{22}(x_2) = 6.8440 \times 10^{-7}x_2^2 + 1.8730 \times 10^{-5}x_2 + 4.3380 \times 10^{-4}$$

$$\mathbf{N}_{11}(x_2) = -2.2295 \times 10^{-2}x_2^2 - 5.4350 \times 10^{-2}x_2 - 5.7727 \times 10^{-1}$$

$$\mathbf{N}_{12}(x_2) = 1.0664 \times 10^{-6}x_2^2 - 5.8404 \times 10^{-5}x_2 - 2.7992 \times 10^{-4}$$

$$\mathbf{N}_{21}(x_2) = -2.1715 \times 10^{-2}x_2^2 - 6.6221 \times 10^{-2}x_2 - 3.0713 \times 10^{-1}$$

$$\mathbf{N}_{22}(x_2) = -2.9942 \times 10^{-5}x_2^2 - 7.5285 \times 10^{-5}x_2 - 3.9447 \times 10^{-4}$$

Case 4 $\mathbf{X}_{11}(x_2) = 7.7430 \times 10^{-3}x_2^2 + 5.5730 \times 10^{-2}x_2 + 2.5890 \times 10^{-1}$
with $a = 51$
and $b = 52$

$$\mathbf{X}_{12}(x_2) = \mathbf{X}_{21}^T(x_2) = 1.1420 \times 10^{-5}x_2^2 + 1.3590 \times 10^{-5}x_2 + 7.3510 \times 10^{-5}$$

$$\mathbf{X}_{22}(x_2) = 3.4420 \times 10^{-7}x_2^2 + 1.4890 \times 10^{-5}x_2 + 5.6580 \times 10^{-4}$$

$$\mathbf{N}_{11}(x_2) = -2.1083 \times 10^{-2}x_2^2 - 4.6793 \times 10^{-2}x_2 - 5.0298 \times 10^{-1}$$

$$\mathbf{N}_{12}(x_2) = -1.4954 \times 10^{-6}x_2^2 - 1.6145 \times 10^{-5}x_2 + 2.8551 \times 10^{-4}$$

$$\mathbf{N}_{21}(x_2) = -2.0674 \times 10^{-2}x_2^2 - 5.7567 \times 10^{-2}x_2 - 2.6635 \times 10^{-1}$$

$$\mathbf{N}_{22}(x_2) = -8.8375 \times 10^{-6}x_2^2 - 3.8180 \times 10^{-5}x_2 - 1.7115 \times 10^{-4}$$

Case 5 $\mathbf{X}_{11}(x_2) = 4.7360 \times 10^{-2}x_2^2 + 1.2380 \times 10^{-1}x_2 + 4.0220 \times 10^{-1}$
with $a = 55$
and $b = 35$

$$\mathbf{X}_{12}(x_2) = \mathbf{X}_{21}^T(x_2) = 1.6930 \times 10^{-6}x_2^2 + 3.3500 \times 10^{-3}x_2 + 5.6760 \times 10^{-4}$$

$$\mathbf{X}_{22}(x_2) = 9.1070 \times 10^{-9}x_2^2 + 6.8950 \times 10^{-7}x_2 + 6.0320 \times 10^{-4}$$

$$\mathbf{N}_{11}(x_2) = -1.2838 \times 10^{-1}x_2^2 - 2.0800 \times 10^{-1}x_2 - 7.7226 \times 10^{-1}$$

$$\mathbf{N}_{12}(x_2) = 5.1177 \times 10^{-7}x_2^2 - 1.5836 \times 10^{-6}x_2 - 1.1238 \times 10^{-3}$$

$$\mathbf{N}_{21}(x_2) = -1.2836 \times 10^{-1}x_2^2 - 2.1003 \times 10^{-1}x_2 - 7.4651 \times 10^{-1}$$

$$\mathbf{N}_{22}(x_2) = 5.5884 \times 10^{-7}x_2^2 - 1.8026 \times 10^{-6}x_2 - 1.1322 \times 10^{-3}$$

Case 6 $\mathbf{X}_{11}(x_2) = 9.2420 \times 10^0x_2^2 + 2.3810 \times 10^1x_2 + 6.9010 \times 10^1$
with $a = 50$
and $b = 32$

$$\mathbf{X}_{12}(x_2) = \mathbf{X}_{21}^T(x_2) = 1.0400 \times 10^{-4}x_2^2 + 3.4520 \times 10^{-3}x_2 + 5.4280 \times 10^{-2}$$

$$\mathbf{X}_{22}(x_2) = 2.9410 \times 10^{-7}x_2^2 + 6.8980 \times 10^{-5}x_2 + 5.3610 \times 10^{-2}$$

$$\mathbf{N}_{11}(x_2) = -2.5095 \times 10^1x_2^2 - 4.0912 \times 10^1x_2 - 1.3200 \times 10^2$$

$$\mathbf{N}_{12}(x_2) = 7.2167 \times 10^{-6}x_2^2 + 1.5115 \times 10^{-5}x_2 - 1.1037 \times 10^{-1}$$

$$\mathbf{N}_{21}(x_2) = -2.5095 \times 10^1x_2^2 - 4.0912 \times 10^1x_2 - 1.3200 \times 10^2$$

$$\mathbf{N}_{22}(x_2) = 7.2167 \times 10^{-6}x_2^2 + 1.5115 \times 10^{-5}x_2 - 1.1037 \times 10^{-1}$$