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Research Article

Optimal Bounds for the Variance of Self-Intersection Local Times

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For a $\mathbb{Z}^d$-valued random walk $(S_n)_{n\in\mathbb{N}_0}$, let $l(n,x)$ be its local time at the site $x \in \mathbb{Z}^d$. For $\alpha \in \mathbb{N}$, define the α-fold self-intersection local time as $L_n(\alpha) := \sum_x l(n,x)^\alpha$. Also let $L_n^{\mathsf{SRW}}(\alpha)$ be the corresponding quantities for the simple random walk in $\mathbb{Z}^d$. Without imposing any moment conditions, we show that the variance of the self-intersection local time of any genuinely d-dimensional random walk is bounded above by the corresponding quantity for the simple symmetric random walk; that is, $\operatorname{var}(L_n(\alpha)) = O(\operatorname{var}(L_n^{\mathsf{SRW}}(\alpha)))$. In particular, for any genuinely d-dimensional random walk, with $d \geq 4$, we have $\operatorname{var}(L_n(\alpha)) = O(n)$. On the other hand, in dimensions $d \leq 3$ we show that if the behaviour resembles that of simple random walk, in the sense that $\liminf_{n\to\infty} \operatorname{var}(L_n(\alpha))/\operatorname{var}(L_n^{\mathsf{SRW}}(\alpha)) > 0$, then the increments of the random walk must have zero mean and finite second moment.

1. Introduction and Main Results

Let $X, X_1, X_2, \ldots$ be independent, identically distributed, $\mathbb{Z}^d$-valued random variables, and define the random walk $S_0 := 0$, $S_n = \sum_{j=1}^n X_j$, for $n \geq 1$. The special case with $\mathbb{P}(X_i = e) = 1/(2d)$, for all $e \in \mathbb{Z}^d$ with $|e| = 1$, is known as the *simple random walk* in $\mathbb{Z}^d$ and will be denoted by $(\mathsf{SRW}_n)_{n\in\mathbb{N}_0}$.

Let $l(n,x) = \sum_{j=1}^n \mathbb{1}(S_j = x)$ be the local time of $(S_n)_{n\in\mathbb{N}_0}$ at the site $x \in \mathbb{Z}^d$, and define for a positive integer α the α-fold *self-intersection local time*

$$L_n = L_n(\alpha) = \sum_{x\in\mathbb{Z}^d} l(n,x)^\alpha = \sum_{i_1,\ldots,i_\alpha=0}^{n} \mathbb{1}\left(S_{i_1} = \cdots = S_{i_\alpha}\right). \quad (1)$$

We will denote the corresponding quantities for simple random walk in $\mathbb{Z}^d$ by $L_n^{\mathsf{SRW}}(\alpha, d)$ or simply $L_n^{\mathsf{SRW}}(\alpha)$ when the dimension is clear from the context.

Let R^+ and R^- be, respectively, the semigroup and the group generated by the support of X,

$$R^+ := \left\{x \in \mathbb{Z}^d \mid \mathbb{P}(S_n = x) > 0 \text{ for some } n \geq 0\right\},$$
$$\overline{R} := \left\{x \in \mathbb{Z}^d \mid x = y - z \text{ for some } x, y \in R^+\right\}. \quad (2)$$

Following Spitzer [1], we call the random variable X and the random walk it generates *genuinely d-dimensional* if the group $\overline{R}$ is d-dimensional.

The quantity $L_n(\alpha)$ has received considerable attention in the literature due to its relation to *self-avoiding walks* and *random walks in random scenery*. In particular let the *random scenery* $\{\xi_x, x \in \mathbb{Z}^d\}$ be a collection of i.i.d. random variables, independent of $(S_n)_n$, and define the process $Z_0 = 0$, $Z_n = \sum_{i=1}^n \xi_{S_i}$. Then $(Z_n)_n$ is commonly referred to as *random walk in random scenery* and was introduced in Kesten and Spitzer [2], where functional limit theorems were obtained for $Z_{[nt]}$ under appropriate normalization for the case $d = 1$. The case $d = 2$, with X_i centered with nonsingular covariance matrix, was treated in [3] where it was shown that $Z_{[nt]}/\sqrt{n\log n}$ converges weakly to Brownian motion. As is obvious from the identities $Z_n = \sum_{x\in\mathbb{Z}^d} l(n,x)\xi_x$ and $\mathrm{var}(Z_n) = \mathrm{var}[L_n(2)]\,\mathrm{var}(\xi_x)$, limit theorems for $(Z_n)_n$ usually require asymptotic results for the local times of the random walk $(S_n)_n$.

Such asymptotic results are usually obtained from Fourier techniques applied to the characteristic function $f(t) = \mathbb{E}[\exp(it\cdot X)]$, under the additional assumption of a Taylor expansion of the form $f(t) = 1 - \langle \Sigma t, t\rangle + o(|t|^2)$, where Σ is a positive definite covariance matrix [3–7], which further requires that $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}X = 0$. Similar restrictions are also required for the application of local limit theorems such as in [8, 9].

In this paper, motivated by the results of Spitzer [1] for genuinely d-dimensional random walks and the approach of Becker and König [10], we will study the asymptotic behavior of $\mathrm{var}(L_n(\alpha))$ without imposing any moment assumptions on the random walk. The central idea behind our approach is to compare the self-intersection local times $L_n(\alpha)$ of a general d-dimensional walk with those of its symmetrised version. In addition we will compare the self-intersection local times of a general d-dimensional random walk with those of the d-dimensional simple symmetric random walk, $(\mathsf{SRW}_n)_{n\in\mathbb{N}_0}$. It is well known that, for some positive constants $K_{\alpha,d}$, $\mathrm{var}(L_n^{\mathsf{SRW}}(\alpha,d)) \sim K_{\alpha,d} v_{d,\alpha}(n)$ as $n\to\infty$, for

$$\begin{aligned} v_{1,\alpha}(n) &:= n^{1+\alpha}, \\ v_{2,\alpha}(n) &:= n^2 \log(n)^{2\alpha-4}, \\ v_{3,\alpha}(n) &:= n\log(n), \\ v_{d,\alpha}(n) &:= n, \quad d\ge 4. \end{aligned} \tag{3}$$

Several other cases have been treated in the literature, using a variety of methods.

A careful look at the literature reveals that the most difficult case in $d = 2$ is the *near transient recurrent* case, where $\mathbb{P}(S_n = 0) \sim C/n$, which corresponds to genuinely 2-dimensional symmetric recurrent random walks, which will be referred to as a critical case. Surprisingly enough, the variance of the self-intersection local times in the critical case is asymptotically the largest.

Theorem 1. *Let $X, X_1, X_2, \ldots$ be independent, identically distributed, and genuinely d-dimensional $\mathbb{Z}^d$-valued random variables, for any $d \ge 1$. Then, there exist positive constants $C_{\alpha,X} > c_{\alpha,X} > 0$, depending on α and the distribution of X, such that for all n large enough*

$$\mathrm{var}(L_n(\alpha)) \le c_{\alpha,X}\,\mathrm{var}\left(L_n^{SRW}(\alpha,d)\right) \le C_{\alpha,X} v_{d,\alpha}(n). \tag{4}$$

The result was motivated by [1, 10] and improves related results of Becker and König for $d = 3$ and $d = 4$. Several cases treated in [3, 4, 10–13] can then be obtained as particular cases.

Moreover, we also show the surprising converse. More precisely, we show that the right asymptotic behaviour of $\mathrm{var}(L_n)$ implies that the jumps must have zero mean and finite second moment.

Theorem 2. *Let $X, X_1, X_2, \ldots,$ be independent, identically distributed, and genuinely d-dimensional with $d \le 3$. If*

$$\liminf_{n\to\infty} \frac{\mathrm{var}(L_n(\alpha))}{\mathrm{var}(L_n^{SRW}(\alpha))} > 0, \tag{5}$$

then $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}X = 0$.

As it follows from Theorem 3 given below for $d = 2, 3$ and from Theorem 5.2.3 in Chen [12] for $d = 1$, if $\mathbb{E}X = 0$ and $0 < \mathbb{E}|X|^2 < \infty$, then $\liminf_n \mathrm{var}(L_n(\alpha))/v_{d,\alpha}(n) > 0$.

For any genuinely d-dimensional random walk with finite second moments and zero mean, the asymptotic behaviour of $\mathrm{var}(L_n(\alpha))$ is similar to that of the d-dimensional simple symmetric random walk. Also, as it follows from our general bounds (see Proposition 4 and Corollary 7) that the asymptotic results for the genuinely d-dimensional random walk can be reproduced by those of the symmetric *one-dimensional* random walk with appropriately chosen heavy tails, as was indicated by Kesten and Spitzer [2]. The proofs are based on adapting the Tauberian approach developed in [13].

Theorem 3. *Let $d = 1, 2, 3$, and suppose that for $t \in \Gamma := [-\pi,\pi]^d$ one has*

$$\begin{aligned} f(t) &= 1 - \gamma|t| + R(t), \quad \text{for } d = 1, \\ \text{or } f(t) &= 1 - \langle \Sigma t, t\rangle + R(t), \quad \text{for } d = 2, 3, \end{aligned} \tag{6}$$

where Σ is a nonsingular covariance matrix and $R(t) = o(|t|)$ for $d = 1$ and $o(|t|^2)$ for $d = 2, 3$ as $t \to 0$. Then

$$\mathrm{var}(L_n(\alpha)) \sim \begin{cases} \dfrac{(\pi^2+6)}{12}\dfrac{(\alpha!)^2(\alpha-1)^2}{(\gamma\pi)^{2\alpha-2}} n^2\log(n)^{2\alpha-4}, & \text{for } d = 1, \\[2ex] \dfrac{(\alpha!)^2(\alpha-1)^2}{2\left(2\pi\sqrt{|\Sigma|}\right)^{2\alpha-2}} n^2 \log(n)^{2\alpha-4}(\kappa+1), & \text{for } d = 2, \\[2ex] (\kappa_1+\kappa_2)\, n\log n, & \text{for } d = 3,\ \alpha = 2, \end{cases} \tag{7}$$

where

$$\kappa := \iint_0^\infty \mathrm{d}r\,\mathrm{d}s \left[(1+r)(1+s)\sqrt{(1+r+s)^2 - 4rs}\right]^{-1} - \frac{\pi^2}{6}, \tag{8}$$

and κ_1 and κ_2 are defined in (58) and (63), respectively.

Moreover, if $L'(n,\alpha)$ is the self-intersection local time of another random walk, independent of $(S_n)_n$, whose characteristic function also satisfies (6), then $\mathrm{var}(L_n'(\alpha)) = \mathrm{var}(L_n(\alpha))(1 + o(1))$.

2. Proofs

2.1. General Bounds. We first develop a technique to treat random walks with independent but not necessarily identically distributed increments.

Proposition 4 (general upper bound). *Assume that $X_1, X_2, \ldots$ are independent $\mathbb{Z}^d$-valued random variables and let $S_{u,v} := X_u + \cdots + X_{u+v}$. Suppose further that for all $n \in \mathbb{N}$ and integers $a, u, b, v \geq 0$, with $a + u \leq b$ and any $x \in \mathbb{Z}^d$, one has*

$$\mathbb{P}\left(S_{a,u} \pm S_{b,v} = x\right) \leq \phi(u+v), \quad \text{(A)}$$

$$\mathbb{P}\left(S_{a,u} = 0\right) - \mathbb{P}\left(S_{a,u} + S_{b,v} = 0\right) \leq \psi(u,v), \quad \text{(B)}$$

where $\phi(u)$ is nonincreasing and $\psi(u,v)$ is nonincreasing in u and is nondecreasing and subadditive in v in the sense that $\psi(u, v+w) \leq A_\psi[\psi(u,v) + \psi(u,w)]$, for some constant A_ψ independent of u, v, and w. Then, for some constant $K = cA_\psi(1+A_\psi)^{\alpha-2}$ depending only on α

$$\operatorname{var}\left(L_n(\alpha)\right) \leq Kn\left(\sum_{i=0}^{n-1}\phi(i)\right)^{2\alpha-4} \cdot \sum_{i,j,k=0}^{n-1}\left[\phi(j \vee i)\,\phi(k \vee i) + \phi(j)\,\psi(i+k, j)\right]. \quad (9)$$

Proof of Proposition 4. We first write out the variance as a sum

$$\operatorname{var}\left(L_n(\alpha)\right) = (\alpha!)^2 \cdot \sum_{k_1 \leq \cdots \leq k_\alpha} \sum_{l_1 \leq \cdots \leq l_\alpha} \left(\mathbb{P}\left[S_{k_1} = \cdots = S_{k_\alpha}, S_{l_1} = \cdots = S_{l_\alpha}\right] - \mathbb{P}\left[S_{k_1} = \cdots = S_{k_\alpha}\right]\mathbb{P}\left[S_{l_1} = \cdots = S_{l_\alpha}\right]\right). \quad (10)$$

An important role is played by the manner in which the two sequences are interlaced, since, for example, if $k_\alpha \leq l_1$ or $l_\alpha \leq k_1$, the term vanishes by the Markov property.

We will treat the sum over indices with $k_1 \leq l_1$. The sum over the remaining index set with $k_1 > l_1$ can be treated in a similar fashion and will contribute a constant factor. Therefore, we assume that $k_1 \leq l_1$ and we arrange the two sequences in an ordered sequence of combined length 2α which we denote as $(p_1, \ldots, p_{2\alpha})$; we also define $(\epsilon_1, \ldots, \epsilon_{2\alpha})$ where $\epsilon_i = 0$ if p_i came from $\mathbf{k} := \{k_1, \ldots, k_\alpha\}$ and $\epsilon_i = 1$ if p_i came from $\mathbf{l} := \{l_1, \ldots, l_\alpha\}$. Finally we define two new sequences $m_0, m_1, \ldots, m_{2\alpha-1}$, and $\delta_1, \ldots, \delta_{2\alpha-1}$, where $m_0 := p_1$, $m_i = p_{i+1} - p_i$, and $\delta_i = \epsilon_{i+1} - \epsilon_i$, for $i = 1, \ldots, 2\alpha - 1$. Notice that since we assume that $k_1 \leq l_1$, we have $p_1 = k_1$ and $\epsilon_1 = 0$. Let $\nu(\delta) := \sum_{i=1}^{2\alpha-1}|\delta_i|$ denote the *interlacement index*. The terms with $\nu = 1$ vanish, while the terms with $\nu = 2$ will be considered separately.

Terms with $\nu \geq 3$. We first consider the sum I_n over the terms with $\nu \geq 3$ for which we drop the negative part and obtain the bound

$$\begin{aligned}
I_n &:= \sum_{\substack{k_1 \leq \cdots \leq k_\alpha \\ l_1 \leq \cdots \leq l_\alpha \\ k_1 \leq l_1, \nu(\delta) \geq 3}} \mathbb{P}\left[S_{k_1} = \cdots = S_{k_\alpha}, S_{l_1} = \cdots = S_{l_\alpha}\right] \\
&= \sum_{x,y \in \mathbb{Z}^d} \sum_{p_1 \leq \cdots \leq p_{2\alpha} \leq n} \sum_{\epsilon : \nu(\delta) \geq 3} \mathbb{P}\left[S_{p_1} = x, S_{p_2} = x + \epsilon_2 y, \ldots, S_{p_{2\alpha}} = x + \epsilon_{2\alpha} y\right] \\
&\leq \sum_{x,y \in \mathbb{Z}^d} \sum_{m_0, \ldots, m_{2\alpha-1} \leq n} \sum_{\delta : \nu(\delta) \geq 3} \mathbb{P}\left(S_{m_0} = x\right)\mathbb{P}\left(S_{m_0,m_1} = \delta_1 y\right) \cdots \mathbb{P}\left(S_{m_{2\alpha-2},m_{2\alpha-1}} = \delta_{2\alpha-1} y\right) \\
&= \sum_{y \in \mathbb{Z}^d} \sum_{m_0, \ldots, m_{2\alpha-1} \leq n} \sum_{\delta : \nu(\delta) \geq 3} \mathbb{P}\left(S_{m_0,m_1} = \delta_1 y\right) \cdots \mathbb{P}\left(S_{m_{2\alpha-2},m_{2\alpha-1}} = \delta_{2\alpha-1} y\right).
\end{aligned} \quad (11)$$

Summing over the free index m_0, it is clear that

$$I_n \leq (n+1) \cdot \sum_{m_1, \ldots, m_{2\alpha-1}} \sum_{y \in \mathbb{Z}^d} \sum_{\delta : \nu(\delta) \geq 3} \prod_{t=1}^{2\alpha-1} \sup_w \mathbb{P}\left(S_{w,m_t} = \delta_t y\right). \quad (12)$$

For any $\delta = (\delta_1, \ldots, \delta_{2\alpha-1})$ with $\nu(\delta) = \nu$, exactly $u := 2\alpha - 1 - \nu$ elements are equal to 0, and therefore by Assumption (A) with $x = 0$ we have

$$I_n \leq C(n+1)\sum_{\nu=3}^{\alpha}\left[\sum_{i=0}^{n}\phi(i)\right]^{2\alpha-1-\nu} \cdot \sum_{j_1, \ldots, j_\nu = 0}^{n} \sum_{y \in \mathbb{Z}^d} \sum_{\delta' \in \{-1,+1\}^\nu} \prod_{t=1}^{\nu} \sup_{w_t} \mathbb{P}\left(S_{w_t,j_t} = \delta_t y\right). \quad (13)$$

Letting $(\widetilde{S}_n)_{n \in \mathbb{N}_0}$ denote an independent copy of the random walk $(S_n)_{n \in \mathbb{N}_0}$ and assuming without loss of generality that $j_1 \leq \cdots \leq j_\nu$, we have that for any $\delta \in \{-1, +1\}^\nu$

$$\begin{aligned}
&\sum_{y \in \mathbb{Z}^d} \prod_{t=1}^{\nu} \sup_{w_t} \mathbb{P}\left(S_{w_t,j_t} = \delta_t y\right) \\
&\quad \leq \left(\prod_{t=2}^{\nu-1} \sup_y \sup_{w_t} \mathbb{P}\left(S_{w_t,j_t} = y\right)\right) \cdot \sup_{w_1,w_\nu} \mathbb{P}\left(S_{w_1,j_1} - \delta_\nu \widetilde{S}_{w_\nu,j_\nu} = 0\right) \leq \left[\prod_{t=2}^{\nu-1}\phi(j_t)\right] \\
&\quad \cdot \phi(j_1 + j_\nu) \leq \prod_{t=2}^{\nu}\phi(j_t \vee j_1).
\end{aligned} \quad (14)$$

Let $G_n := \sum_{i=0}^{n} \phi(i)$. Since ϕ is nonincreasing we have that

$$\begin{aligned}\Delta_{n,v} &:= \sum_{0\le j_1\le\cdots\le j_v\le n} \prod_{t=2}^{v} \phi(j_t \vee j_1) \\ &\le \sum_{j_v=0}^{n} \phi(j_v) \sum_{0\le j_1\le\cdots\le j_{v-1}\le n} \prod_{t=2}^{v-1} \phi(j_t \vee j_1) \\ &= G_n \Delta_{n,v-1},\end{aligned} \tag{15}$$

and iterating this procedure, for $v \ge 3$, we have that $\Delta_{n,v} \le \Delta_{n,3} G_n^{v-3}$. Combining the two bounds and summing over $v = 3, \ldots, 2\alpha - 1$, we have that

$$\begin{aligned}I_n &\le \sum_{v=3}^{2\alpha-1} c(\alpha)\, n G_n^{2\alpha-1-v} \Delta_{n,v} \le c(\alpha)\, n G_n^{2\alpha-1-v+v-3} \Delta_{n,3} \\ &= c(\alpha)\, n G_n^{2\alpha-4} \Delta_{n,3},\end{aligned} \tag{16}$$

where $c(\alpha)$ is a constant depending only on α.

Terms with $v = 2$. Next we consider the sum J_n over the terms with $v = 2$, which occurs when, for some j, the indices $l_1, \ldots, l_\alpha$ all lie in $[k_j, k_{j+1}]$. Then it is easy to see that this sum J_n is bounded above by

$$\begin{aligned}J_n &\le Cn \sup_{w_0,\ldots,w_{2\alpha-1}} \sum_{m_{\alpha+1},\ldots,m_{2\alpha-2}=0}^{n} \prod_{r=\alpha+1}^{2\alpha-2} \mathbb{P}\left(S_{w_r,m_r} = 0\right) \\ &\cdot \sum_{m_0,\ldots,m_\alpha=0}^{n} \left[\prod_{t=1}^{\alpha-1} \mathbb{P}\left(S_{w_t,m_t} = 0\right)\right] \left[\mathbb{P}\left(S_{w_0,m_0} + S_{w_\alpha,m_\alpha}\right.\right. \\ &\left.\left.= 0\right) - \mathbb{P}\left(S_{w_0,m_0} + \cdots + S_{w_\alpha,m_\alpha} = 0\right)\right] \le CnG_n^{\alpha-2} \\ &\cdot \sup_{w_0,\ldots,w_\alpha} \sum_{m_0,\ldots,m_\alpha=0}^{n} \left[\prod_{t=1}^{\alpha-1} \mathbb{P}\left(S_{w_t,m_t} = 0\right)\right] \\ &\cdot \left[\mathbb{P}\left(S_{w_0,m_0} + S_{w_\alpha,m_\alpha} = 0\right)\right. \\ &\left.- \mathbb{P}\left(S_{w_0,m_0} + \cdots + S_{w_\alpha,m_\alpha} = 0\right)\right] \\ &\le CnG_n^{\alpha-2} \sum_{m_0,\ldots,m_\alpha=0}^{n} \left[\prod_{t=1}^{\alpha-1} \phi(m_t)\right] \psi(m_0 + m_\alpha, m_1 \\ &+ \cdots + m_{\alpha-1}) \le C\alpha n G_n^{\alpha-2} A_\psi \left(1 + A_\psi\right)^{\alpha-2} \\ &\cdot \left(\sum_{m_2,\ldots,m_{\alpha-1}} \prod_{t=2}^{\alpha-1} \phi(m_t)\right) \sum_{m_0,m_1,m_\alpha} \phi(m_1)\, \psi(m_0 + m_\alpha, \\ &m_1) \le C\alpha A_\psi \left(1 + A_\psi\right)^{\alpha-2} n G_n^{2\alpha-4} \sum_{i,j,k=0}^{n} \phi(j)\, \psi(i \\ &+ k, j).\end{aligned} \tag{17}$$

□

The following corollary provides explicit bounds in the cases that are usually considered in the literature.

Corollary 5. *Assume that the conditions of Proposition 4 are satisfied with* $\phi(m) = Tm^{-r}$ *and* $\psi(m,k) = Tm^{-r-1}(k \wedge m)$. *Then,*

$$\operatorname{var}(L_n(\alpha)) \le c_\alpha T^{2\alpha-2} \begin{cases} n^2 \log(n)^{2\alpha-4}, & \text{if } r = 1, \\ n^{4-2r}, & \text{if } 1 < r < \dfrac{3}{2}, \\ n \log(n), & \text{if } r = \dfrac{3}{2}, \\ n, & \text{if } r > \dfrac{3}{2}. \end{cases} \tag{18}$$

It is straightforward to see that Corollary 5 includes random walks with mean zero and finite second moment; for example, $d = 2$ corresponds to $r = 1$ and $d = 3$ to $r = 3/2$. Therefore several relevant results in [3, 7–13] are obtained as a special case of Corollary 5 and extended to the case of independent but not necessarily identically distributed variables, for example, by applying the local limit theorem, as conducted in [8].

Also when the random walk increment X is in the domain of attraction of the one-dimensional symmetric Cauchy law [13, 14] or in the case of planar random walk with second moments [3, 7–9, 11], it is well known that the conditions of Proposition 4 are satisfied with $\phi(m) = T/m$ and $\psi(m,k) = Tm^{-2}(k \wedge m)$.

However, we can do better for symmetric variables and show that condition (A) implies (B), which together with the comparison technique motivates the following results. For a real number x, we write $[x]$ for the integer part of x.

Proposition 6 (bounds via comparison with characteristic function of symmetric random variables). *Let* $X_1, X_2, \ldots$, *be independent* $\mathbb{Z}^d$*-valued random variables and let* $f_i(t) := \mathbb{E}\exp(itX_i)$. *Assume that there exist a measurable function* $f : \Gamma \to [0,1]$ *and a positive nonincreasing sequence* $(\phi(m))_{m\in\mathbb{N}_0}$, *such that*

$$\begin{gathered}\left|1 - f_i(t)\right| \le Tf(t), \\ \left|f_i(\pm t)\right| \le f(t), \\ \int_\Gamma f(t)^m \,\mathrm{d}t \le \phi(m),\end{gathered} \tag{19}$$

for all integers $i, m \ge 0$, *all* $t \in \Gamma$, *and some positive constant* T. *Then there exists another positive constant* $K = c(\alpha, d, T)$ *such that*

$$\operatorname{var}(L_n(\alpha)) \le Kn \left(\sum_{i=0}^{n-1} \phi\left(\left[\frac{i}{2}\right]\right)\right)^{2\alpha-4} \sum_{j=0}^{n} j\phi\left(\left[\frac{j}{2}\right]\right) \sum_{k=j}^{2n} \phi\left(\left[\frac{k}{2}\right]\right). \tag{20}$$

Proof of Proposition 6. Using the notation of Proposition 4, for positive integers $a, u, b,$ and v, with $a + u \le b$, $\epsilon_j = \pm 1$, and any $x \in \mathbb{Z}^d$

$$\begin{aligned}\mathbb{P}\left(S_{a,u} + \epsilon \cdot S_{b,v} = x\right) &\le \frac{1}{(2\pi)^d} \int_\Gamma \prod_{j \in [a,a+u] \cup [b,b+v]} \left| f_j\left(\epsilon_j t\right)\right| \mathrm{d}t \\ &\le \frac{1}{(2\pi)^d} \int_\Gamma f(t)^{u+v} \mathrm{d}t \le \frac{1}{(2\pi)^d} \phi(u+v).\end{aligned} \tag{21}$$

To find $\psi(u, v)$, notice that since $f(t) \ge 0$,

$$\begin{aligned}\phi(m) &\ge \int_\Gamma f(t)^m \left[1 - f(t)^m\right] \mathrm{d}t \\ &= \sum_{j=0}^{m-1} \int_\Gamma f(t)^{m+j} (1 - f(t))\, \mathrm{d}t \\ &\ge m \int_\Gamma f(t)^{2m} (1 - f(t))\, \mathrm{d}t =: mQ(2m)\end{aligned} \tag{22}$$

whence $Q(m) \le 2\phi([m/2])/m$. Therefore,

$$\begin{aligned}&\left|\mathbb{P}\left(S_{a,u} = 0\right) - \mathbb{P}\left(S_{a,u} + S_{b,1} = 0\right)\right| \\ &\quad = \left| \frac{1}{(2\pi)^d} \int_\Gamma \left[\prod_{j=a}^{a+u} f_j(t)\right] \left(1 - f_{b+1}(t)\right) \mathrm{d}t \right| \\ &\quad \le CT \int_\Gamma |f(t)|^u \left|1 - f(t)\right| \mathrm{d}t \le \frac{CT\phi([u/2])}{u}.\end{aligned} \tag{23}$$

A telescoping argument implies that

$$\left|\mathbb{P}\left(S_{a,u} = 0\right) - \mathbb{P}\left(S_{a,u} + S_{b,v} = 0\right)\right| \le CT\phi\left(\left[\frac{u}{2}\right]\right)\frac{v}{u}. \tag{24}$$

On the other hand for $u \le v$ we can obtain a tighter bound through

$$\begin{aligned}&\mathbb{P}\left(S_{a,u} = 0\right) - \mathbb{P}\left(S_{a,u} + S_{b,v} = 0\right) \le \mathbb{P}\left(S_{a,u} = 0\right) \\ &\quad \le \phi(u).\end{aligned} \tag{25}$$

Combining the two bounds above it follows that (B) is satisfied with $\psi(u, v) := \phi([u/2]) \min(u, v)/u$. Thus all conditions of Proposition 4 are satisfied and the result follows. □

The following corollary allows for the case where $\phi(m)$ is regularly varying.

Corollary 7. *Assume that the conditions of Proposition 6 are satisfied with $\phi(m) = h(m)m^{-r}$, $r \ge 1$, where $h(\cdot)$ is slowly varying at ∞. Then,*

$$\operatorname{var}\left(L_n(\alpha)\right) \le K\Delta_n(\alpha, \phi)$$

$$\le c_\alpha T^{2\alpha-2} \begin{cases} n^2 \left[\sum_{k=1}^n \frac{h(k)}{k}\right]^{2\alpha-4}, & \textit{for } r = 1, \\ n^{4-2r} h^2(n), & \textit{for } 1 < r < \frac{3}{2}, \\ n \sum_{k=1}^n \frac{h(k)^2}{k}, & \textit{for } r = \frac{3}{2}, \\ n, & \textit{for } r > \frac{3}{2}. \end{cases} \tag{26}$$

Several results in [3, 7–13] are obtained as a special case of Corollary 7 and can be extended to dependent variables, for example, a random walk driven by a hidden Markov chain. In addition, following [2], we can construct a one-dimensional symmetric random walk with characteristic function $f(t) = 1 - c|t|^{1/r} + o(|t|^{1/r})$, where $r = 2/d$ for $d = 2, 3$ and $r = 1/2$ for $d \ge 4$, whose asymptotic behaviour is similar to that of genuinely d-dimensional random walk.

The following example of genuinely 2-dimensional recurrent walk with infinite variance was motivated by Spitzer [1, pp. 87].

Example 8. Let $X_1, X_2, \ldots$ be independent, identically distributed, $\mathbb{Z}^2$-valued random variables, such that $\mathbb{P}(|X_1| = k) = c/(k^3 \log(k)^g)$, for $k \ge 4$ and $g \in [0, 1)$. Let $(S_n)_{n \in \mathbb{N}_0}$ be the corresponding random walk in $\mathbb{Z}^2$. Then we have

$$\begin{aligned}&\operatorname{var}\left(L_n(\alpha)\right) \\ &\quad \le cn^2 \max\left\{[\log n]^g, \log\log n\right\}^{2\alpha-4} \log n^{-2(1-g)},\end{aligned} \tag{27}$$

for $n \ge 10$. Under these assumptions we have that $\mathbb{P}(S_n = 0) \le c/n \log(n)^{1-g}$, which is in the *critical range*, where the random walk is recurrent, without second moment. To see why, we note that by a lengthy but straightforward calculation it can be shown that the characteristic function of X satisfies (19) with

$$\begin{aligned}\phi(n) &= \frac{c}{n \log(\mathrm{e} \vee n)^{1-g}}, \\ f(t) &= \exp\left[-A|t|^2 h\left(|t|^2\right)\right], \\ \text{where } h(r) &:= \left[1 + \log\left(\frac{1}{r}\right)_+\right]^{1-g}.\end{aligned} \tag{28}$$

The sequence $\phi(m)$ is identified via Fourier inversion, polar coordinates, and a Laplace argument,

$$\begin{aligned}\int_\Gamma f(t)^n \mathrm{d}t &\le c \int_0^1 \exp\left(-nr\left(1 + \log\left(\frac{1}{r}\right)\right)^{1-g}\right) \\ &\quad + O\left(\mathrm{e}^{-n}\right) \le \frac{c}{n \log(\mathrm{e} \vee n)^{1-g}} =: \phi(n).\end{aligned} \tag{29}$$

2.2. Bounds for Identically Distributed Variables

Proposition 9 (general upper bound for i.i.d.). *Let $X, X_1, X_2, \ldots,$ be independent, identically distributed,*

$\mathbb{Z}^d$-valued random variables. Suppose that for any $x \in \mathbb{Z}^d$ and all positive integers a, u, b, and v, with $a + u \le b$, it holds that

$$\mathbb{P}(S_{a,u} \pm S_{b,v} = x) \le \phi(u+v), \tag{30}$$

where $\{\phi(m)\}_{m\in\mathbb{N}_0}$ is a nonincreasing sequence. Then for some constant $K = c(\alpha)$ we have that

$$\begin{aligned} &\operatorname{var}(L_n(\alpha)) \\ &\le Kn\left(\sum_{i=0}^{n-1}\phi(i)\right)^{2\alpha-4}\sum_{j=0}^{n} j\phi(j)\sum_{k=j}^{[\alpha n]+1}\phi\left(\left[\frac{k}{\alpha}\right]\right). \end{aligned} \tag{31}$$

Proof of Proposition 9. By inspecting the proof of Proposition 6, we notice that we only need to bound the term J_n. Consider typical ordering

$$0 \le i_1 \le \cdots \le i_k \le j_1 \le \cdots \le j_\alpha \le i_{k+1} \le \cdots \le i_\alpha \le n, \tag{32}$$

and let us change variables to $(m_0, \ldots, m_{2\alpha})$ such that $m_0 + \cdots + m_{2\alpha} = n$. Then the contribution to J_n is given by

$$\begin{aligned} &\sum_{m_0,\ldots,m_{2\alpha}} \prod_{\substack{j\neq k,k+\alpha \\ 1\le j\le 2\alpha-1}} \mathbb{P}\left(S_{m_j} = 0\right) \\ &\quad\cdot\left[\mathbb{P}\left(S_{m_k+m_{k+\alpha}} = 0\right) - \mathbb{P}\left(S_{m_k+\cdots+m_{k+\alpha}} = 0\right)\right]. \end{aligned} \tag{33}$$

We keep m_j fixed for $j \neq \alpha, k+\alpha$ and we sum over $m = m_k + m_{k+\alpha}$ from 0 to some $M = M(n, \{m_j\}_{j\neq k,k+\alpha})$. Then for given $m_{k+1}, \ldots, m_{k+\alpha-1}$, the term in the sum is

$$\sum_{m=0}^{M}(m+1)\left[\mathbb{P}(S_m = 0) - \mathbb{P}\left(S_{m+q} = 0\right)\right], \tag{34}$$

where $q := m_{k+1} + \cdots + m_{k+\alpha-1}$. Then since $M \le n - q$, it is an easy exercise to show that this sum is bounded above by

$$\begin{aligned} &\sum_{m=0}^{M}(m+1)\left[\mathbb{P}(S_m = 0) - \mathbb{P}\left(S_{m+q} = 0\right)\right] \\ &\quad\le \sum_{m=0}^{q-1}(m+1)\,\mathbb{P}(S_m = 0) + q\mathbb{1}(n-q \ge q) \\ &\quad\cdot\sum_{m=q}^{n-q}\mathbb{P}(S_m = 0) \le \sum_{m=0}^{(\alpha m^*)\wedge n}(m+1)\,\mathbb{P}(S_m = 0) \\ &\quad+ \alpha m^* \sum_{m=m^*}^{n}\mathbb{P}(S_m = 0), \end{aligned} \tag{35}$$

where $m^* = \max\{m_{k+1}, \ldots, m_{k+\alpha-1}\}$. The result follows by summing over all indices apart from m^* and changing the order of summation. □

2.3. Proofs of Main Results

Proof of Theorem 1. We apply a comparison argument found to be useful in many areas (e.g., Montgomery-Smith and Pruss [15], and Lefèvre and Utev [16]). More specifically we bound the quantity $\operatorname{var}(L_n)$ by the corresponding quantity for the symmetrised random walk.

Following Spitzer's argument we notice that with $f(t) = \mathbb{E}[\exp(it\cdot X_1)]$

$$\begin{aligned} \mathbb{P}(S_{a,u} + \epsilon S_{b,v} = x) &\le c\int_\Gamma |f(t)|^u |f(-t)|^v\,\mathrm{d}t \\ &= c\int_\Gamma \left[|f(t)|^2\right]^{u/2}\left[|f(-t)|^2\right]^{v/2}\mathrm{d}t. \end{aligned} \tag{36}$$

Since $|f(t)|^2$ is the characteristic function of a symmetric random variable in $\mathbb{Z}^d$, for some positive λ, we have $1 - |f(t)|^2 \ge \lambda|t|^2$, and, hence,

$$\begin{aligned} \mathbb{P}(S_{a,u} + \epsilon S_{b,v} = x) &\le c\int_\Gamma \exp\left[-\frac{\lambda(u+v)}{2}|t|^2\right]\mathrm{d}t \\ &\le c(u+v)^{-d/2}. \end{aligned} \tag{37}$$

The result follows from Proposition 9 applied with $\phi(m) = m^{-d/2}$. □

The proof of Theorem 2 will be based on the following lemma.

Lemma 10. *Assume $X, X_1, X_2, \ldots$ are independent, identically distributed, genuinely d-dimensional random variables such that $\mathbb{E}|X|^2 = \infty$. Then there exists a monotone, slowly varying sequence $(h_n)_{n\in\mathbb{N}_0}$, such that $h_n \to 0$ as $n \to \infty$ and*

$$\sup_{x\in\mathbb{Z}^d}\mathbb{P}(S_n = x) \le c_d\int_\Gamma \left|\mathbb{E}e^{it\cdot X}\right|^n \mathrm{d}t \le h_n n^{-d/2}. \tag{38}$$

Proof of Lemma 10. Without loss of generality we assume that X is symmetric. Let $\sigma_{e,L} := \mathbb{E}[(e\cdot X)^2\mathbb{1}(|X| \le L)]$. Following Spitzer, since X is genuinely d-dimensional, we may assume that there exist positive constants c, W, such that for any unit vector $|e| = 1$ we have that $\sigma_{e,W} \ge c$ and $1 - f(t) \ge c|t|^2$ for all $t \in \Gamma$. Let λ_d be the d-dimensional Lebesgue measure on $\mathbb{R}^d$ and μ_d the Lebesgue-Haar measure on $S^{d-1} := \{e \in \Gamma : |e| = 1\}$. Notice that since $\mathbb{E}|X|^2 = \infty$, for any K, we have $\mu_d\{e : \sigma_{e,\infty} < K\} = 0$.

Fix a small positive x such that $\sqrt{c/x} \ge 2W$, and for any $\epsilon > 0$ let $K = K(\epsilon) = \epsilon^{-d/2}$. Then there exists $L = L(\epsilon) > 0$ small enough so that $\mu_d\{e : \sigma_{e,L} < K\} \le \epsilon^{d/2}$. We partition S^{d-1} in two sets

$$\begin{aligned} A_{L,K} &= \{e \in S_d : \sigma_{e,L} \ge K\}, \\ \overline{A}_{L,K} &= \{e \in S_d : \sigma_{e,L} < K\}, \end{aligned} \tag{39}$$

so that, for any direction $e \in \overline{A}_{L,K}$,

$$\begin{aligned} \{z \in \mathbb{R} : 1 - f(ze) \le x\} &\subseteq \left\{z : cz^2 \le x\right\} \\ &\subseteq \left\{z : |z| \le \sqrt{\frac{x}{c}}\right\}. \end{aligned} \tag{40}$$

Hence, using d-dimensional spherical coordinates,

$$\lambda_d \left\{(z,e) \in \mathbb{R} \times \overline{A}_{L,K} : 1 - f(ez) \le x\right\} \le \mu_d \left\{\overline{A}_{L,K}\right\} \left(\frac{x}{c}\right)^{d/2} \left(\frac{1}{d}\right) \le \epsilon^{d/2} \left(\frac{x}{c}\right)^{d/2} \left(\frac{1}{d}\right). \quad (41)$$

On the other hand, for any t,

$$\begin{aligned} 1 - f(t) &= 2 \sum_{k\in Z^d} \sin\left(\frac{[t\cdot k]}{2}\right)^2 P(X = k) \\ &\ge \left(\frac{1}{4}\right) E\left[(t\cdot X)^2 I\left(|t\cdot X| \le \frac{1}{2}\right)\right] \\ &= \left(\frac{|t|^2}{4}\right) \sigma_{t/|t|,1/2|t|}. \end{aligned} \quad (42)$$

Now, assume that $\sqrt{c/x} \ge 2L$. Then for any direction $e \in A_{L,K}$, by choice of x and since $\sigma_{e,L}$ is increasing in L, for $cz^2 \le 1 - f(ez) \le x$ or $|z| \le \sqrt{x/c}$, it must be the case that

$$\begin{aligned} x &\ge 1 - f(ez) \ge \left(\frac{z^2}{4}\right) \sigma_{e,1/2z} \ge \left(\frac{z^2}{4}\right) \sigma_{e,L} \\ &\ge \left(\frac{z^2}{4}\right) K, \end{aligned} \quad (43)$$

implying that, on the set $A_{L,K}$, it must be that $|z| \le 2\sqrt{x/K}$. Changing to d-dimensional polar coordinates, we find that

$$\lambda_d \left\{(z,e) \in \mathbb{R} \times A_{L,K} : 1 - f(ez) \le x\right\} \le \int_{A_{L,K}} \int_0^{\sqrt{4x/K}} r^{d-1} \mathrm{d}r \, \mathrm{d}e \le C_d \epsilon^{d/2} x^{d/2}. \quad (44)$$

Overall, for $x \le c/4L^2$, $\lambda_d\{t : 1 - f(t) \le x\} \le c_d(x\epsilon)^{d/2}$, and hence $\{t \in \Gamma : 1 - f(t) \le x\}$ has Lebesgue measure $o(x^{d/2})$.

Let $F(x)$ be the cumulative distribution function of the random variable $\log(1/f(\cdot))$ defined on the probability space Γ with normalised Lebesgue measure. Then F is continuous at $x = 0$ and supported on $\mathbb{R}^+$. Moreover, we have that $F(x) = o(x^{d/2})$ as $x \downarrow 0$. Therefore, for some positive sequence $(\epsilon_n)_{n\in\mathbb{N}_0}$ with $\epsilon_n \to 0$, we have that

$$\begin{aligned} \frac{1}{(2\pi)^2} \int_\Gamma f(t)^n \, \mathrm{d}t &= \int_0^\infty \mathrm{e}^{-nx} \mathrm{d}F(x) \\ &= n \int_0^\infty \mathrm{e}^{-nx} F(x) \, \mathrm{d}x \le n^{-d/2} \epsilon_n. \end{aligned} \quad (45)$$

It remains to show that there exists a positive, monotone, slowly varying sequence $(h_n)_{n\in\mathbb{N}_0}$, such that $\epsilon_n \le h(n) \to 0$ as $n \to \infty$. Let $\delta_n = \sup_{j\ge n} \epsilon_j$ and $a_0 := 0$ and for $n \ge 1$ define a_n recursively by $a_n = \min(2a_{2^{r-1}}, 1/\delta_n)$, for $2^{r-1} < n \le 2^r$, so that $a_n \to \infty$ is monotone, $a_{2^r} \le 2a_{2^{r-1}}$ implying that $a_{2n} \le 4a_n$, and $1/a_n \ge \delta_n \ge \epsilon_n$. Finally, take $h_n := 1/\max(a_0, \log a_n)$. □

Proof of Theorem 2. Assume that $\mathbb{E}|X|^2 = \infty$ and $d = 2$ or $d = 3$. Then, by Lemma 10 there exists a slowly varying sequence $h_n \to 0$ as $n \to \infty$ such that $\int_\Gamma |\mathbb{E}\exp(it\cdot X)|^n \mathrm{d}t \le h_n n^{-d/2}$. Applying Corollary 7 with $r = 1$ and $r = 3/2$ we, respectively, find that

$$\operatorname{var}(L_n(\alpha)) \le \begin{cases} Kn^2 \left(\sum_{k=1}^n \frac{h(k)}{k}\right)^{2\alpha-4} = o\left(n^2 (\log n)^{2\alpha-4}\right), & \text{for } d = 2, \\ Kn \left(\sum_{k=1}^n \frac{h(k)^2}{k}\right) = o(n \ln n), & \text{for } d = 3. \end{cases} \quad (46)$$

Finally assume that $\mathbb{E}|X|^2 < \infty$ and $E[X] = \mu \ne 0$. Then $\mathbb{P}(S_n = 0) = \mathbb{P}(S'_n = -n\mu)$ whence it follows that $\mathbb{P}(S_n = 0) = o(n^{-d/2})$ (see, e.g., [17, Theorem 2.3.10]). Then inspecting the proof of Proposition 4, one can readily obtain the desired bound for the J_n term, while with slight modification the bound for the I_n term also follows.

Note that for $d = 1$ the situation is much simpler since then $\operatorname{var}(L_n^{\mathsf{SRW}}(\alpha)) \sim C[\mathbb{E}L_n^{\mathsf{SRW}}(\alpha,d)]^2$ and if $\mathbb{E}|X|^2 = \infty$ or $\mathbb{E}[X] \ne 0$, $\mathbb{E}L_n^{\mathsf{SRW}}(\alpha,d) = o(n^{(1+\alpha)/2})$. □

Proof of Theorem 3. We first give the proof for the case $d = 1$. As in the proof of Proposition 4 we begin from expression (10) and define the sequences p_i and δ_i for $i = 1,\ldots,2\alpha-1$, and the quantity $\nu(\delta) = \sum_{i=1}^{2\alpha-1} |\delta_i|$. Recall that $\nu(\delta)$ measures the interlacement of the two sequences $k_1,\ldots,k_\alpha$ and $l_1,\ldots,l_\alpha$. For example, $\nu(\delta) = 1$ occurs when either $k_\alpha \le l_1$ or $l_\alpha \le k_1$, in which case the contribution vanishes by the Markov property. On the other hand $\nu(\delta) = 2$ when, for example, $l_1,\ldots,l_\alpha \in [k_i, k_{i+1}]$ for some i. Finally $\nu(\delta) = 3$ occurs when, for example,

$$\begin{aligned} k_1 &\le \cdots \le k_r \le l_1 \le \cdots \le l_s \le k_{r+1} \le \cdots \le k_\alpha \le l_{s+1} \\ &\le \cdots \le l_\alpha \le n. \end{aligned} \quad (47)$$

From the proof of Proposition 4, and using the bound $\mathbb{P}(S_n = 0) \le c/n$, the terms of the sum are bounded above by $n^2 \log(n)^{2\alpha-1-\nu(\delta)}$, and thus the leading term appears when either $\nu(\delta) = 2, 3$, with other terms giving strictly lower order. We will therefore analyze these two situations in detail in order to derive the exact asymptotic constants. When $\nu = 3$, the two terms in the difference individually give the correct order and will be treated by the classical Tauberian theory. However for $\nu = 2$, the two terms only give the correct order when considered together. This however forbids the use of Karamata's Tauberian theorem since the monotonicity restriction would require roughly that X_i is symmetric. Thus the complex Tauberian approach, as developed in [13], is required to justify the answer.

Case 1 ($\nu(\delta) = 3$). Assume that part of the sequence $\mathbf{l} = \{l_1,\ldots,l_\alpha\}$ lies between k_r and k_{r+1} and the rest between k_s and k_{s+1}. Then using the change of variables

$$\begin{aligned} i_1 &= m_0, \\ i_2 &= m_0 + m_1, \\ &\vdots \\ i_r &= m_0 + \cdots + m_{r-1} \\ j_1 &= m_0 + \cdots + m_r, \\ j_2 &= m_0 + \cdots + m_{r+1}, \\ &\vdots \\ j_s &= m_0 + \cdots + m_{r+s-1}, \\ i_{r+1} &= m_0 + \cdots + m_{r+s}, \\ i_{r+2} &= m_0 + \cdots + m_{r+s+1}, \\ &\vdots \\ i_\alpha &= m_0 + \cdots + m_{\alpha+s-1}, \\ j_{s+1} &= m_0 + \cdots + m_{\alpha+s}, \\ j_{s+2} &= m_0 + \cdots + m_{\alpha+s+1}, \\ &\vdots \\ j_\alpha &= m_{2\alpha-1}, \\ n &= m_0 + \cdots + m_{2\alpha}, \end{aligned} \tag{48}$$

we rewrite the positive term in (10) as

$$\begin{aligned} a(n) &= \sum \mathbb{P}\left[S(i_1) = \cdots = S(i_\alpha); S(j_1) = \cdots = S(j_\alpha)\right] \\ &= \sum_{m_0,\ldots,m_{2\alpha-1}} \left[\prod_{\substack{j=1 \\ j\neq r,r+s,\alpha+s}}^{2\alpha-1} \mathbb{P}\left(S_{m_j} = 0\right) \right] \\ &\cdot \mathbb{P}\left(S_{m_r} + S'_{m_{r+s}} = S'_{m_{r+s}} + S''_{m_{\alpha+s}} = 0\right). \end{aligned} \tag{49}$$

Notice that from [13] we have that $\sum_{n\geq 0} \lambda^n \mathbb{P}(S_n = 0) \sim \log(1/(1-\lambda))/\pi\gamma$. Let

$$\begin{aligned} a(\lambda) &= (1-\lambda)^{-3} \left[-\log(1-\lambda)\right]^{2\alpha-4}, \\ c_\gamma &= (\pi\gamma)^{-2\alpha+4}. \end{aligned} \tag{50}$$

Then, by direct calculations and Fourier inversion formula

$$\begin{aligned} \sum_{n\geq 0} \lambda^n a(n) &= c_\gamma (1-\lambda) a(\lambda) \\ &\cdot \sum_{x\in\mathbb{Z}} \sum_{k_1,k_2,k_3\geq 0} \lambda^{k_1+k_2+k_3} \mathbb{P}\left(S_{k_1} = x\right) \mathbb{P}\left(S_{k_2} = -x\right) \\ &\cdot \mathbb{P}\left(S_{k_3} = x\right) = c_\gamma (1-\lambda) a(\lambda) \frac{1}{(2\pi)^2} \\ &\cdot \iint_\Gamma \frac{\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t+s))} \\ &\sim c_\gamma (1-\lambda) a(\lambda) \frac{1}{(2\pi)^2 \gamma^2} \frac{1}{1-\lambda} \\ &\cdot \iint_{\mathbb{R}^2} \frac{\mathrm{d}x\,\mathrm{d}y}{(1+|x|)(1+|y|)(1+|x+y|)} \sim \left(\frac{1}{4\gamma^2}\right) \\ &\cdot c_\gamma a(\lambda). \end{aligned} \tag{51}$$

Next we consider the negative term in (10)

$$\begin{aligned} b(n) &\coloneqq \sum_{m_0,\ldots,m_{2\alpha-1}} \mathbb{P}\left[S_{m_1} = \cdots = S_{m_{r-1}} = S_{m_r} + \cdots\right. \\ &\left.+ S_{m_{r+s}} = S_{m_{r+s+1}} = \cdots = S_{m_{\alpha+s-1}} = 0\right] \mathbb{P}\left[S_{m_{r+1}} = \cdots\right. \\ &\left.= S_{m_{r+s}} + \cdots + S_{m_{\alpha+s}} = S_{m_{\alpha+s+1}} = \cdots = S_{m_{2\alpha-1}} = 0\right]. \end{aligned} \tag{52}$$

By direct calculations and (6),

$$\begin{aligned} \sum_n \lambda^n b(n) &= \left(\frac{1}{\pi\gamma} \log\left(\frac{1}{1-\lambda}\right)\right)^{\alpha-s+r-2} (1-\lambda)^{-2} \\ &\cdot \sum_{m_r,\ldots,m_{\alpha+s}=0}^{\infty} \lambda^{m_r+\cdots+m_{\alpha+s}} \\ &\cdot \prod_{\substack{t=r+1,\ldots,\alpha+s-1 \\ t\neq r+s}} \mathbb{P}\left(S_{m_t} = 0\right) \\ &\cdot \mathbb{P}\left(S_{m_r} + \cdots + S_{m_{r+s}} = 0\right) \\ &\cdot \mathbb{P}\left(S_{m_{r+s}} + \cdots + S_{m_{\alpha+s}} = 0\right), \end{aligned} \tag{53}$$

and using Fourier inversion and (6) the internal sum behaves as

$$\begin{aligned} &(2\pi)^{-\alpha-s+r} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (1-\lambda\phi(x))^{-1} (1-\lambda\phi(x)\phi(y))^{-1} (1-\lambda\phi(y))^{-1} \\ &\cdot \left[\prod_{j=r+1}^{r+s-1} \prod_{k=r+s+1}^{\alpha+s-1} \left(1-\lambda\phi(x)\phi(t_j)\right)^{-1} (1-\lambda\phi(y)\phi(t_k))^{-1} \mathrm{d}t_j\,\mathrm{d}t_k \right] \mathrm{d}x\,\mathrm{d}y \sim (\pi\gamma)^{-\alpha-s+r} (1-\lambda)^{-1} \\ &\cdot \log\left(\frac{1}{1-\lambda}\right)^{\alpha-r+s-2} \frac{\pi^2}{6}. \end{aligned} \tag{54}$$

Then, we have $\sum_n \lambda^n b(n) \sim (\pi^2/6(\pi\gamma)^{2\alpha-2})a(\lambda)$, whence the Tauberian theorem implies that $a(n) - b(n) \sim n^2\log(n)^{2\alpha-4}/24\pi^{2\alpha-4}\gamma^{2\alpha-2}$. Most importantly we see that the lengths and locations of the chains, r and s, do not affect the asymptotic behaviour. Noting that if $1 \leq r, s \leq \alpha - 1$, we can partition $2\alpha = r + s + (\alpha - r) + (\alpha - s)$ in $(\alpha-1)^2$ ways, and thus overall the total contribution from terms with $v = 3$ is

$$\left[\frac{(\alpha!\,(\alpha-1))^2}{12\pi^{2\alpha-4}\gamma^{2\alpha-2}}\right] n^2 \log(n)^{2\alpha-4}. \tag{55}$$

Case 2 ($v(\delta) = 2$). The typical term $c(n)$ was introduced in (33) in the proof of Proposition 9. Now we let $\lambda \in \mathbb{C}$, with $|\lambda| < 1$. By lengthy but direct calculations we can derive an expression of the form

$$\sum_n \lambda^n c(n) = \frac{\alpha-1}{(\gamma\pi)^{2\alpha-2}} a(\lambda) + o(a(\lambda)), \quad \lambda \longrightarrow 1. \tag{56}$$

The approach developed in [13] can then be used to bound the error terms and show that $c(n) \sim [(\alpha-1)/2(\gamma\pi)^{2\alpha-2}]n^2\log(n)^{2\alpha-4}$.

Finally taking into account the fact that $l_1, \ldots, l_\alpha$ can be in any of the $\alpha - 1$ intervals $[k_i, k_{i+1}]$, for $i = 1, \ldots, \alpha - 1$, the result follows the overall contribution of terms with $v(\delta) = 2$

$$\frac{(\alpha-1)^2}{2(\gamma\pi)^{2\alpha-2}} n^2 \log(n)^{2\alpha-4}. \tag{57}$$

The case for $d = 2$ is very similar, so we move on to the case $d = 3$.

Case 3 ($d = 3$ and $\alpha = 2$). Using the same notation as before, we have three terms to consider $a(n)$, $b(n)$, and $c(n)$. We first consider $c(n)$. Letting $K := \epsilon/\sqrt{1-\lambda}$ and using the usual power series construction and spherical coordinates

$$\begin{aligned}
\sum_n \lambda^n c(n) &= (1-\lambda)^{-2}(2\pi)^{-6} \\
&\cdot \iint_{J^3\times J^3} \frac{\lambda f(y)(1-f(x))\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x) f(y))} \\
&\sim 2(2\pi)^{-4}|\Sigma|^{-1}(1-\lambda)^{-2} \\
&\cdot \iint_0^K \frac{r^4 s^2\,\mathrm{d}r\,\mathrm{d}s}{(1+r^2)^2(1+s)^2(1+r^2+s^2)} \sim 2(2\pi)^{-4}|\Sigma|^{-1} \\
&\cdot \frac{\pi}{2}(1-\lambda)^{-2}\log\left(\frac{1}{1-\lambda}\right) =: \kappa_1 (1-\lambda)^{-2}\log\left(\frac{1}{1-\lambda}\right),
\end{aligned} \tag{58}$$

and thus $c(n) \sim \kappa_1 n \log n$, where $\kappa_1 > 0$, where the answer can be justified following [13].

The term $a(n) - b(n)$ is trickier to compute. As usual we consider the power series

$$\begin{aligned}
\sum_{n\geq 0} \lambda^n (a(n) - b(n)) &= (1-\lambda)^{-2}(2\pi)^{-6} \\
&\cdot \iint_{B(\epsilon)} \frac{\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))(1-\lambda f(y))(1-\lambda f(x+y))} \\
&- (1-\lambda)^{-2}(2\pi)^{-6} \\
&\cdot \iint_{B(\epsilon)} \frac{\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))(1-\lambda f(y))(1-\lambda f(x) f(y))} \\
&= (1-\lambda)^{-2}(2\pi)^{-6}(I_1(\lambda) - I_2(\lambda)).
\end{aligned} \tag{59}$$

Let $A \in [-1, 1]$ be the cosine of the angle between x and y, which in spherical coordinates is

$$\begin{aligned}
A &= A(\theta_1, \theta_2, \phi_1, \phi_2) \\
&= \cos(\phi_1 - \phi_2)\sin(\theta_1)\sin(\theta_2) \\
&\quad + \cos(\theta_1)\cos(\theta_2).
\end{aligned} \tag{60}$$

Then as $0 < \lambda \uparrow 1$, using the expansion (6)

$$\begin{aligned}
I_1(\lambda) &\sim |\Sigma|^{-1} \int_{r,s=0}^{\epsilon} \int_{\phi_{1,2}=0}^{2\pi} \int_{\theta_1,\theta_2=0}^{\pi} \frac{r^2 s^2 \sin(\theta_1)\sin(\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\mathrm{d}r\,\mathrm{d}s}{(1-\lambda+\lambda r^2)(1-\lambda+\lambda s^2)[1-\lambda+\lambda(r^2+s^2+2Ars)]} \\
&= |\Sigma|^{-1} \int_{\theta_1,\theta_2=0}^{\pi} \int_{\phi_1,\phi_2=0}^{2\pi} \int_{r,s=0}^{K} \frac{\sin(\theta_1)\sin(\theta_2) r^2 s^2\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{(1+r^2)(1+s^2)[1+r^2+s^2+2Ars]} \\
&\sim |\Sigma|^{-1}\log(K) \int_{\theta_1,\theta_2=0}^{\pi} \int_{\phi_1,\phi_2=0}^{2\pi} \sin(\theta_1)\sin(\theta_2) \frac{\arccos(A(\theta_1,\theta_2,\phi_1,\phi_2))}{\sqrt{1-A(\theta_1,\theta_2,\phi_1,\phi_2)^2}}\mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.
\end{aligned} \tag{61}$$

The other integral is slightly easier

$$\begin{aligned}
I_2(\lambda) &\sim |\Sigma|^{-1}\frac{\pi}{2}\log K \\
&\cdot \int_{\theta_1,\theta_2=0}^{\pi} \int_{\phi_1,\phi_2=0}^{2\pi} \sin(\theta_1)\sin(\theta_2)\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2,
\end{aligned} \tag{62}$$

and thus overall we must have that

$$\begin{aligned}
(I_1 - I_2)(\lambda) &\sim \frac{1}{2}(2\pi)^{-6}|\Sigma|^{-1}(1-\lambda)^{-2}\log\left(\frac{1}{1-\lambda}\right) \\
&\cdot \int_{\theta_1,\theta_2=0}^{\pi} \int_{\phi_1,\phi_2=0}^{2\pi} \left[\frac{\arccos(A)}{\sqrt{1-A^2}} - \frac{\pi}{2}\right]\sin(\theta_1)
\end{aligned}$$

$$\cdot \sin(\theta_2)\, d\phi_1\, d\phi_2\, d\theta_1\, d\theta_2 =: \kappa_2 (1 - \lambda)^{-2} \log\left(\frac{1}{1-\lambda}\right), \tag{63}$$

whence it follows that $\mathrm{var}(L_n(2)) \sim (\kappa_1 + \kappa_2) n \log n$.

To prove the last claim let $S_n' = X_1' + \cdots + X_n'$ be another random walk, independent of S_n, such that its characteristic function $f'(t) = \mathbb{E}[\exp(itX_i')]$ also satisfies the expansion (6). Then using [13, Lemma 3.1], one can adapt the proof of [13, Theorem 2.1] to show that $L_n'(\alpha) = L_n(\alpha) + o(L_n(\alpha))$. □

Competing Interests

The authors declare that they have no competing interests.

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