



King's Research Portal

DOI:

[10.1016/j.apal.2016.10.001](https://doi.org/10.1016/j.apal.2016.10.001)

Document Version

Peer reviewed version

[Link to publication record in King's Research Portal](#)

Citation for published version (APA):

Gabbay, M. J., & Gabbay, M. J. (2016). Representation and duality of the untyped λ -calculus in nominal lattice and topological semantics, with a proof of topological completeness. *ANNALS OF PURE AND APPLIED LOGIC*. <https://doi.org/10.1016/j.apal.2016.10.001>

Citing this paper

Please note that where the full-text provided on King's Research Portal is the Author Accepted Manuscript or Post-Print version this may differ from the final Published version. If citing, it is advised that you check and use the publisher's definitive version for pagination, volume/issue, and date of publication details. And where the final published version is provided on the Research Portal, if citing you are again advised to check the publisher's website for any subsequent corrections.

General rights

Copyright and moral rights for the publications made accessible in the Research Portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognize and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the Research Portal

Take down policy

If you believe that this document breaches copyright please contact librarypure@kcl.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

Accepted Manuscript

Representation and duality of the untyped λ -calculus in nominal lattice and topological semantics, with a proof of topological completeness

Murdoch J. Gabbay, Michael J. Gabbay

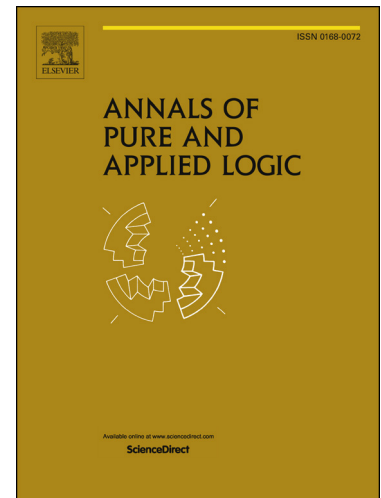
PII: S0168-0072(16)30124-5
DOI: <http://dx.doi.org/10.1016/j.apal.2016.10.001>
Reference: APAL 2532

To appear in: *Annals of Pure and Applied Logic*

Received date: 20 August 2013
Revised date: 7 September 2015
Accepted date: 1 October 2016

Please cite this article in press as: M.J. Gabbay, M.J. Gabbay, Representation and duality of the untyped λ -calculus in nominal lattice and topological semantics, with a proof of topological completeness, *Ann. Pure Appl. Logic* (2016), <http://dx.doi.org/10.1016/j.apal.2016.10.001>

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



Representation and duality of the untyped λ -calculus in nominal lattice and topological semantics, with a proof of topological completeness

Murdoch J. Gabbay^a, Michael J. Gabbay^b

^aHeriot-Watt University, Scotland, UK

^bKing's College London, UK

Abstract

We give a semantics for the λ -calculus based on a topological duality theorem in nominal sets. A novel interpretation of λ is given in terms of adjoints, and λ -terms are interpreted absolutely as sets (no valuation is necessary).

Keywords: Nominal algebras, fresh-finite limits, lambda-calculus, spectral spaces, lattices and order, variables, nominal techniques, mathematical foundations, Fraenkel-Mostowski set theory

Contents

1	Introduction	5
1.1	A very brief summary of the contributions of this paper	5
1.2	The point of duality results	6
1.3	Map of the paper	7
1.4	A list of interesting technical features	8
1.5	Why is this paper so long?	9
2	Background on nominal techniques	10
2.1	Basic definitions	10
2.2	Examples	12
2.2.1	Atoms and booleans	12
2.2.2	Cartesian product	12
2.2.3	Tensor product	13
2.2.4	Full function space	13
2.2.5	Small-supported function space	13
2.2.6	Full powerset	13
2.3	The principle of equivariance and the NEW quantifier	14
2.4	Two lemmas	17

URL: www.gabbay.org.uk (Murdoch J. Gabbay)

2.5	Further examples	17
2.5.1	Small-supported powerset	18
2.5.2	Strictly small-supported powerset	18
2.6	The NEW-quantifier for nominal sets	19
I	Nominal distributive lattices with quantification	20
3	Nominal algebras over nominal sets	20
3.1	Definition of a sigma-algebra (σ -algebra)	20
3.1.1	A termlike σ -algebra	20
3.1.2	A σ -algebra	21
3.2	Definition of an amgis-algebra	21
3.3	Duality I: sigma to amgis	22
3.4	Duality II: amgis to sigma	23
3.4.1	The pointwise sigma-action on subsets of an amgis-algebra	23
3.4.2	The σ -powerset $Pow_\sigma(\mathcal{P})$	24
3.4.3	Some further remarks	26
4	Nominal posets	27
4.1	Nominal posets and fresh-finite limits	27
4.2	Compatible σ -structure	28
4.3	Definition of a nominal distributive lattice with $\forall$	30
4.4	Impredicative nominal distributive lattices	32
5	The σ-powerset as a nominal distributive lattice with $\forall$	34
5.1	Basic sets operations	34
5.2	Sets quantification	35
II	Nominal spectral space representation	38
6	Completeness	38
6.1	Filters and prime filters	38
6.1.1	Filters	39
6.1.2	Growing filters	41
6.1.3	Growing ideals	42
6.1.4	Maximal and prime filters	42
6.1.5	The Zorn argument	44
6.2	The amgis-action on (prime) filters	47
6.3	Injecting $\mathcal{D}$ into the set of sets of prime filters	48
6.4	The map from x , to prime filters containing x , as a morphism	50

7	Nominal σ-topological spaces	51
7.1	The basic definition	51
7.2	The map F from distributive lattices to nominal σ -topological spaces	52
7.3	Technical interlude: two important propositions	52
7.4	Compactness	55
7.5	Coherent spaces: closure under σ , $\cap$ and $\bigcap^{\#a}$	56
7.6	Completely prime filters in a coherent space	57
7.7	Impredicativity	58
7.8	The map G from coherent spaces to distributive lattices	59
7.9	Sober spaces	60
7.10	Nominal spectral spaces	62
8	Morphisms of nominal spectral spaces	62
8.1	The definition of $\text{inSpect}\forall$, and F viewed as a functor to it	62
8.2	The action of G on morphisms in $\text{inSpect}\forall$	66
8.3	The equivalence	67
III	Adding application and its topological dual the combination operator	69
9	$\text{inDi}\forall_{\bullet}$ and $\text{inSpect}\forall_{\bullet}$	69
9.1	Adding $\bullet$ and $\multimap$ to $\text{inDi}\forall$ to get $\text{inDi}\forall_{\bullet}$	69
9.2	The combination operator $\circ$: a topological dual to $\bullet$ and $\multimap$	71
9.2.1	The basic definition	72
9.2.2	$\bullet$ and $\multimap$, and spectral spaces	72
9.2.3	Useful technical lemmas	74
9.3	Filters in the presence of $\bullet$ and $\multimap$	74
9.3.1	Combining filters I ($q \circ y$)	74
9.3.2	Combining filters II ($q \bullet x$)	75
9.4	The second representation theorem	77
9.4.1	The basic actions	77
9.4.2	Combining filters III ($p \bullet q$)	77
9.4.3	The representation theorem	78
9.4.4	Properties of the representation	79
9.5	Construction of the topological space $F(\mathcal{D})$, with $\bullet$ and $\multimap$	80
9.6	The duality, in the presence of $\bullet$ and $\multimap$	81
IV	Application to the λ-calculus	84
10	The λ-calculus	84
10.1	Syntax of the λ -calculus	84
10.2	λ , β , and η using adjoints	85
10.2.1	λ using $\forall$ and $\multimap$	85
10.2.2	λ as a sets operation in $F(\mathcal{D})$	86

10.3	Idioms	86
10.4	A sound denotation for the λ -calculus	88
10.5	Interlude: axiomatising the λ -calculus in nominal algebra	90
11	Representation of the λ-calculus in $\text{inSpect}\forall_\bullet$	91
11.1	Π -points and σ -freshness	92
11.2	Constructing Π -points, and their amgis-algebra structure	94
11.3	The left adjoint $p[a \mapsto u]$ to the amgis-action $p[u \mapsto a]$	95
11.3.1	Basic definition	96
11.3.2	Two characterisations of $p[a \mapsto u]$	96
11.3.3	Additional lemmas about the σ -action as an adjoint	98
11.4	The left adjoint $p[a \mapsto u]$ as a σ -action on points	98
11.4.1	It is indeed a σ -action	98
11.4.2	The σ -action distributes over union and subset	99
11.5	Some further operations on points	100
11.5.1	The operations: $\wedge$, $\vee$, $\bullet$, and $\multimap$ on points	100
11.5.2	$\multimap$ and $\vee$ make λ	101
11.6	How the σ -action on points commutes	102
11.7	Operations on sets of points	104
11.7.1	Basic definitions	104
11.7.2	Commutation properties	106
11.8	A topology	108
11.8.1	Giving Pnt_Π a topology	108
11.8.2	The compact open sets of Pnt_Π	109
11.8.3	Interaction of $\bullet$ and $\multimap$ with $\cup$ and $\subseteq$	110
11.8.4	Interaction of $\bigcap^{\#a}$ with unions	111
11.8.5	Proof that Pnt_Π is coherent	111
11.8.6	Proof that Pnt_Π is sober, thus spectral	113
11.9	Logical properties of the topology, and completeness	114
11.10	Interlude: an interesting disconnect	115
12	Conclusions	116
12.1	Related work	117
12.1.1	Algebraic semantics	117
12.1.2	Absolute semantics	118
12.1.3	η -expansion	119
12.1.4	Surjective pairing	119
12.1.5	Previous treatment of λ -calculus by the authors	119
12.1.6	No conflict with topological incompleteness results	120
12.1.7	Game models	120
12.1.8	Sheaves	121
12.1.9	Proof theory	121
12.1.10	In what universe does this paper take place?	122
12.1.11	Discussion of some decisions	122
12.2	Future work	123
12.2.1	Fine structure of the canonical model	123

12.2.2	Weaken the axioms	123
12.2.3	More structure in the existing axioms	124
12.2.4	Duality and cardinality	124
12.2.5	Size of permutations	125
A	More on fresh-finite limits	129
A.1	Nominal algebra axiomatisation of fresh-finite limits	129
A.2	More on fresh-finite limits	131
B	Additional properties of the canonical model	134
B.1	λ commutes with unions	134
B.2	An existential quantifier	135

1. Introduction

In this paper we build a topological duality result for the untyped λ -calculus in nominal sets and prove soundness, completeness, and topological completeness.

This means the following:

- We give a lattice-style axiomatisation of the untyped λ -calculus and prove it sound and complete.
- We define a notion of topological space whose compact open sets have notions of application and λ -abstraction.
- We prove that the categories of lattices-with- λ and topological-spaces-with- λ are dual.
- We give a complete topological semantics for the λ -calculus.

So this paper does what universal algebra and Stone duality do for Boolean algebras [Joh86], but for the λ -calculus. We do this by combining three threads of previous work:

- the *nominal algebra* of [GM09b];
- the study of the logical structure of *nominal powersets* from [Gab16] (applied in [Gab16] to first-order logic with equality, and in this paper to the λ -calculus); and
- the *Kripke-style models of the untyped λ -calculus* from [GG14, Gab14a].

1.1. A very brief summary of the contributions of this paper

We summarise some contributions of this paper; this list will be fleshed out in the rest of the Introduction:

1. No previous duality result for λ -calculus theories exists.
Duality results are interesting in themselves (see next subsection), and it is interesting to see how nominal techniques help to manage the technical demands of such a result.

We are extremely grateful to the editor Phil Scott and to the anonymous referees for their time and help in improving the mathematics and readability of what follows.

2. The lattice semantics of this paper is abstract, yet complete. The well-known domains-based denotational semantics are *incomplete* (whereas semantics such as term models quotiented by equivalence are complete, but concrete).
3. We prove a topological completeness result—but this should be impossible: topological *incompleteness* results exist in the literature.
This depends on the topology, so the fact that a notion of topology that ‘works’ for the λ -calculus exists, is surprising given the current state of the art. One would not expect this to work.
4. The topological representations are concrete, being based on nominal sets.
5. The representation of open terms does not use valuations; possibly open λ -terms are interpreted as open sets in a nominal topological space (this is sometimes called an *absolute* semantics).
Function application and also λ -abstraction get interpreted as concrete sets operations on nominal-style atoms.
6. β -reduction and η -expansion are exhibited as adjoint properties.
7. This paper is a nontrivial application of nominal ideas, and the techniques on nominal sets which we use are original and have independent technical interest.
8. We make nominal-style atoms—urelemente in Fraenkel-Mostowski set theory—behave like variables of the λ -calculus. Urelemente in set theory come equipped with very few properties; indeed, by design urelemente have virtually no properties at all. It is remarkable that they can nevertheless acquire such rich structure.¹
9. The fine structure of the canonical models (those of the form Pnt_{Π}) is very rich, as we shall see. The use of canonical models in this paper probably does not exhaust their interest.

1.2. The point of duality results

What is the point of a duality result, and why bother doing it for the λ -calculus?

1. Duality results are a strong form of completeness: for a given class of abstract models, every model has a concrete topological representation (i.e. in terms of sets with a few consistency conditions) and every map between models has a concrete representation as a continuous map (i.e. a map that has to respect those conditions).

Intuitively, Boolean algebras, Heyting algebras, and distributive lattices look like they all have to do with powersets (negation is some kind of sets complement, conjunction is sets intersection, disjunction is sets union, and so on). But is this true? Is it possible to construct some sufficiently bizarre model such that for instance conjunction *must* mean something other than sets intersection?

The answer is no: duality theorems tell us that no matter how bizarre the model, it *can* be represented topologically. In topologies conjunction is sets intersection.

The same analysis is applicable to our duality result for the λ -calculus.

¹We carry out a similar programme for variables of first-order logic in [Gab14b, Gab16].

2. Of course, it is not obvious how conjunction enters into the untyped λ -calculus. Indeed, no duality result has been achieved for the λ -calculus before and it is not obvious even how to begin to go about this.

One contribution of this paper is that we embed the λ -calculus in an impredicative logic which we characterise in two ways: in nominal algebra, and using a nominal generalisation of finite limits which we call *fresh-finite limits* (Definition 4.1.2).

An example observation that comes out of this is that we exhibit β -reduction and η -expansion as adjoint maps (counit and unit respectively; see Proposition 10.2.4).

Another example is that λ is exhibited as a compound object made out of the fresh-finite limit $\forall$, and a right adjoint to application $\multimap$. See Notation 10.2.1 and the subsequent discussion.

The quantifier $\forall$ is itself an interesting entity, a kind of arbitrary conjunction, which relies heavily on nominal techniques. More on this later.

Thus, in the process of defining and proving our results—representation, duality, and completeness—we uncover a wealth of structure in the untyped λ -calculus (to add to the wealth of structure already known). The technical definitions and lemmas which our ‘main results’ depend on, are as interesting as the results themselves.

3. Finally, we note that our topological semantics for the λ -calculus is *complete* (Theorem 11.9.5).

This is remarkable because Salibra has shown that all known semantics for the λ -calculus based on partial orders, are incomplete [Sal03]. The fact that our semantics is topological (thus ordered) and complete, is unexpected.² We discuss this apparent paradox in Subsection 12.1.6.

In any case, new semantics for the untyped λ -calculus do not come along very often, and as mentioned above, no duality result for the λ -calculus has been proved before.

More interesting structure will be uncovered by this way of approaching the λ -calculus; the list above justifies why it is *a priori* interesting to try.

1.3. Map of the paper

Section 2 sets up some basic nominal theory. The reader might like to skim this at first, since the definitions might only make sense in terms of their application later on in the paper. Highlights are the notions of nominal set (Definition 2.1.6), small-supported and strictly-small-supported powersets (Subsections 2.5.1 and 2.5.2), equivariance properties of atoms and the $\mathcal{V}$ -quantifier (Theorem 2.3.2 and Definition 2.3.6), and the $\mathcal{V}$ -quantifier for sets (Subsection 2.6).

²Our paper handles only the case where we have η -expansion. We believe this could be generalised to the fully non-extensional case, at some cost in complexity. See Subsection 12.1.3 or [GG14].

Section 3 introduces σ - and τ -algebras. These are the basic building blocks from which our models will be constructed. The definitions are already non-trivial; highlights are the axioms of Figure 1 (which go back to [GM06a], where nominal algebra was introduced to axiomatise substitution) and [Gab14b] (which introduced τ -algebras), and the precise definition of the σ -action on nominal powersets in Definition 3.4.1, which uses the $\mathcal{N}$ -quantifier.

Section 4 considers lattices over nominal sets. The technical highlight here is the characterisation of universal quantification in terms of *fresh-finite limits* (Definition 4.1.1 and subsequent results). Combined with impredicativity and the σ -action we arrive at Definitions 4.4.1 and 4.4.9, which are the lattice-theoretic structure within which we eventually build models of the λ -calculus which we write $\text{inDi}\forall$ (pronounced ‘India’).

Section 5 shows that (simplifying) every nominal powerset is a model of Definition 4.4.1. Technical highlights here are Definition 5.2.1, Lemma 5.2.2, and Proposition 5.2.3 which show how to interpret universal quantification and check that it commutes with the σ -action.

Section 6 uses filters and prime filters to give a nominal sets representation of any $\text{inDi}\forall$, and Section 7 extends this to a full duality.

This duality is for a propositional logic with quantifiers. To handle the λ -calculus we need more: this happens in Section 9. The most important points are probably the introduction of $\bullet$ and its right adjoint $\multimap$ in Definition 9.1.1, and the observation that their topological dual is a *combination operator* $\circ$ in Definition 9.2.1.

It now becomes fairly easy to show that every $\text{inDi}\forall_\bullet$ is also a semantics for the untyped λ -calculus. This is Section 10, culminating in Definition 10.4.1 and Theorem 10.4.7; we include an interlude in Subsection 10.5 where we pause to take stock of what we have been doing so far.

Slightly harder is proving completeness, for which we must construct an object in $\text{inDi}\forall_\bullet/\text{inSpect}\forall_\bullet$ for any given λ -theory. This is Section 11: we have the tools (nominal and otherwise) required in principle to carry out the constructions (just build filters, etcetera)—but in practice the amount of detailed structure required to make this work is quite striking, involving amongst many other things the construction of $\bullet$ and $\multimap$ on points (Definition 11.5.1) and a left adjoint to the τ -algebra on filters (Definition 11.3.1). If two results should illustrate how tightly knit this part of the mathematics can be, then the technical results of Proposition 11.1.6 and Lemma 11.3.9 are good examples. The final Completeness result is Theorem 11.9.5.

We conclude in an Appendix with a nominal axiomatisation of $\forall$, to go with the lattice-theoretic one of Definition 4.1.2, and some nice additional observations on the structure of points.

1.4. A list of interesting technical features

This list is not of the major results, nor is it an exhaustive list of technical definitions. But one technical definition or proof looks very much like another, so here are suggestions of which technical highlights might be worth looking at first:

- We characterise universal quantification in various ways, most notably in Proposition 6.1.10, and see the connection made between universal quantification and the

$\mathcal{U}$ -quantifier in condition 4 of Definition 6.1.2. See also the discussion opening Appendix A.2.

- As mentioned above, β -reduction and η -expansion are derived from a counit and unit respectively in Proposition 10.2.4.
- We decompose λ into $\forall$ and $\multimap$ in Notation 10.2.1, and we decompose $\forall$ further using the $\mathcal{U}$ -quantifier and freshness in Proposition 6.1.10.
- Lemmas 11.4.2 and 11.8.13 are inherently surprising results.
- Three little proof gems are in Propositions 11.1.7 and 11.2.4, and Lemma 11.4.1.
- We need two notions of filter: one is Definition 6.1.2 (particularly condition 4, mentioned above), the other is Definition 11.1.3 (we call it a *point*). Both notions use the $\mathcal{U}$ -quantifier in interesting ways.
- Because of condition 4 of Definition 6.1.2, the proof of existence of maximal filters requires a delicate argument; see Remark 6.1.22 and Theorem 6.1.23.
- Propositions 7.3.4 and 7.3.5 are key to proving compactness in the duality proof.
- The pointwise definition of substitution is in Definition 3.4.1; a different approach to substitution than the reader has likely seen, which makes use of the $\mathcal{U}$ -quantifier and $\mathfrak{v}$ -algebra (Definition 3.2.1). The two concrete characterisations of it in Subsection 11.3.2 are also interesting.
- The reader will see much use made of the nominal $\mathcal{U}$ -quantifier, and of nominal equivariance properties and notions of small support and *strict* small support. We mention two (connected) examples: the use of $\mathcal{U}$ in defining the σ -action in Definition 3.4.1, and the treatment of the universal quantifier in condition 4 of Definition 6.1.2 and the dual condition 2 of Definition 7.9.1.

We use $\mathcal{U}$ deep in the proofs of many technical results: the interested reader could peruse for instance Lemma 5.2.2, Proposition 6.1.13, Lemma 6.2.1, Proposition 7.3.5, Lemma 11.8.13, and Proposition 11.8.23.

A search of the paper for uses of Corollary 2.3.5, Theorems 2.3.2 and 2.3.9, and Lemma 2.5.3 will find many uses made of equivariance and (strict) small support.

- A nominal algebra axiomatisation in Appendix A.1 underlies the design of the maths.

1.5. Why is this paper so long?

1. This paper contains a duality result *and* gives a semantics and completeness proof for the untyped λ -calculus.

It is a fact that duality results are hard, completeness proofs are also hard, and semantics for the untyped λ -calculus are not trivial to construct.³

2. We obtain our duality result by a far from obvious combination of nominal algebra [GM09b], logic in nominal powersets [Gab16], and the modal model theory of the lambda calculus [GG14]. Even individually these techniques are not well known, so to be self-contained and rigorous in combining them, this paper must be long and detailed. Where we can rely on the material being familiar, we will be more

³There are actually two duality results: one for impredicative distributive lattices (corresponding to a propositional logic with $\wedge$, $\vee$, and propositional quantification) from Part I, and the other when we add the combination structure in Part III. The second duality piggy-backs on the first, and is shorter.

brief.

So our starting point is urelements (atoms) of Fraenkel-Mostowski set theory; our target is the untyped λ -calculus, and between them there is a lot of ground to cover. Everything in this paper is there because it has to be, it is worth the effort, and the material has natural momentum which propels us from the first definitions to the final results.

2. Background on nominal techniques

A nominal set is a ‘set with names’. The notion of a name being ‘in’ an element is given by support $\text{supp}(x)$ (Definition 2.1.9). For more details of nominal sets, see [GP01, Gab11b, Pit13].

Here we just give necessary background information. The reader not interested in nominal techniques *per se* might like to read this section only briefly in the first instance, and use it as a reference for the later sections, where the ideas get applied.

For the reader’s convenience we take a moment to note the overall message of this section:

- *To the category-theorist* we say that we work mostly in the category of nominal sets, or equivalently in the Schanuel topos (more on this in [MM92, Section III.9], [Joh03, A.21, page 79], or [Gab11b, Theorem 9.14]), and occasionally also in the category of sets with a permutation action.
- *To the set-theorist* we say that our constructions can be carried out in Fraenkel-Mostowski set theory (**FM sets**) and Zermelo-Fraenkel set theory with atoms (**ZFA**). A discussion of such sets foundations, tailored to nominal techniques, can be found in [Gab11b, Section 10].
- *To the reader not interested in foundations* we say that the apparently inconsequential step of assuming names as primitive entities in Definition 2.1.1 yields a remarkable clutch of definitions and results, notably Theorem 2.1.12 and Corollary 2.3.5, and Theorems 2.3.2 and 2.3.9. These properties are phrased abstractly but will quickly make themselves very useful in the body of this paper. See previous work for more background [GP01, Gab11b, Gab13, Pit13].

2.1. Basic definitions

DEFINITION 2.1.1. For this paper we fix the following:

- Fix an infinite set of **atoms** $\mathbb{A}$ and write $\text{size}(\mathbb{A})$ for the cardinality of $\mathbb{A}$.
- Call any set whose cardinality is strictly lesser than $\text{size}(\mathbb{A})$ **small**.
- Call $A \subseteq \mathbb{A}$ **cosmall** when $\mathbb{A} \setminus A$ is small.
- We use a **permutative convention** that $a, b, c, \dots$ range over *distinct* atoms.

REMARK 2.1.2. Traditionally $\mathbb{A}$ is taken to be countable so that ‘small’ corresponds to being finite. It will be useful (and no harder) to work with the generalisation of the theory for potentially larger sets of atoms and larger notions of ‘small’.

DEFINITION 2.1.3. A **permutation** π is a bijection on atoms such that $\text{nontriv}(\pi) = \{a \mid \pi(a) \neq a\}$ is finite.

Write id for the **identity** permutation such that $\text{id}(a) = a$ for all a . Write $\pi' \circ \pi$ for composition, so that $(\pi' \circ \pi)(a) = \pi'(\pi(a))$. Write π^{-1} for inverse, so that $\pi^{-1} \circ \pi = \text{id} = \pi \circ \pi^{-1}$. Write $(a \ b)$ for the **swapping** (terminology from [GP01]) mapping a to b , b to a , and all other c to themselves, and take $(a \ a) = \text{id}$.

NOTATION 2.1.4. If $A \subseteq \mathbb{A}$ write

$$\text{fix}(A) = \{\pi \mid \forall a \in A. \pi(a) = a\}.$$

DEFINITION 2.1.5. 1. A **set with a permutation action** $\mathcal{X}$ is a pair $(|\mathcal{X}|, \cdot)$ of an **underlying set** $|\mathcal{X}|$ and a **permutation action** written $\pi \cdot_{\mathcal{X}} x$ or just $\pi \cdot x$ which is a group action on $|\mathcal{X}|$, so that $\text{id} \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ for all $x \in |\mathcal{X}|$ and permutations π and π' .

2. Say that $A \subseteq \mathbb{A}$ **supports** $x \in |\mathcal{X}|$ when $\forall \pi. \pi \in \text{fix}(A) \Rightarrow \pi \cdot x = x$.

3. If a small A supporting x exists, call x **small-supported**.

DEFINITION 2.1.6. Call a set with a permutation action $\mathcal{X}$ a **nominal set** when every $x \in |\mathcal{X}|$ has small support. $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ will range over nominal sets.

REMARK 2.1.7. Permutations are *finite* in Definition 2.1.3, yet support is *small* in Definition 2.1.6. Why are permutations not taken to be small (or even arbitrary) instead?

In this paper we are interested in modelling binders that abstract a single atom at a time. λ -syntax is finite so only finitely many binders are ever applied to an element, thus, we never need to α -rename more than finitely many atoms in any given element and we only need finite permutations.⁴

Generalisations to infinite permutations are possible, but they introduce complexity for no technical benefit to the specific concerns of this paper. More on this in Subsection 12.2.5.

For discussion of why we need small support rather than finite support, see Remark 4.4.2 and Subsection 12.2.4.

DEFINITION 2.1.8. Call a function f from $|\mathcal{X}|$ to $|\mathcal{Y}|$ **equivariant** when $\pi \cdot (f(x)) = f(\pi \cdot x)$ for all permutations π and $x \in |\mathcal{X}|$. In this case write $f : \mathcal{X} \rightarrow \mathcal{Y}$.

The category of nominal sets and equivariant functions between them is usually called the category of *nominal sets*.

DEFINITION 2.1.9. Suppose $\mathcal{X}$ is a nominal set and $x \in |\mathcal{X}|$. Define the **support** of x by

$$\text{supp}(x) = \bigcap \{A \mid A \text{ small and supports } x\}.$$

If $\text{supp}(x) = \emptyset$ we call x **equivariant**.

⁴This paper has many binders. For instance: $\lambda a, \nu a, \mathcal{U}a, \forall a, \lambda_{\mathcal{U}},$ and σ -actions ('substitutions') $[a \mapsto \cdot]$. Nevertheless, each of them only abstracts one atom at a time.

NOTATION 2.1.10. Write $a\#x$ as shorthand for $a \notin \text{supp}(x)$ and read this as a is **fresh** for x .

Given atoms $a_1, \dots, a_n$ and elements $x_1, \dots, x_m$ write $a_1, \dots, a_n\#x_1, \dots, x_m$ as shorthand for $\{a_1, \dots, a_n\} \cap \bigcup_{1 \leq j \leq m} \text{supp}(x_j) = \emptyset$, or to put it more plainly: $a_i\#x_j$ for every i and j .

PROPOSITION 2.1.11. *If $A \subseteq \mathbb{A}$ is small and supports x , and $a \in \mathbb{A}$ and $a\#x$, then $A \setminus \{a\}$ supports x .*

Proof. Suppose $\pi \in \text{fix}(A \setminus \{a\})$. We assumed $a\#x$ so choose an $A' \subseteq \mathbb{A}$ such that A' is small, A' supports x , and $a \notin A'$. Also, choose some fresh a' (so $a' \notin A \cup \{a\} \cup A'$).

Write $\tau = (a' a)$. Note that $\tau \cdot x = x$ by Definition 2.1.5(2), because A' supports x and $\tau \in \text{fix}(A')$.

It is a fact that $(\tau \circ \pi \circ \tau)(a) = a$ for every $a \in A$, so $\tau \circ \pi \circ \tau \in \text{fix}(A)$. Also by the group action $(\tau \circ \pi \circ \tau) \cdot x = \tau \cdot (\pi \cdot (\tau \cdot x))$. Since A supports x we have $\tau \cdot (\pi \cdot (\tau \cdot x)) = x$ by Definition 2.1.5(2).

We apply τ to both sides, recall that $\tau \cdot x = x$, and conclude that $\pi \cdot x = x$ as required. $\square$

THEOREM 2.1.12. *Suppose $\mathcal{X}$ is a nominal set and $x \in |\mathcal{X}|$. Then $\text{supp}(x)$ is the unique least small set of atoms that supports x .*

Proof. Consider a permutation $\pi \in \text{fix}(\text{supp}(x))$ and write $\{a_1, \dots, a_n\} = \text{nontriv}(\pi)$. Choose any small $A \subseteq \mathbb{A}$ that supports x , so by construction $\text{supp}(x) \subseteq A$.

By Proposition 2.1.11 $A \setminus \text{nontriv}(\pi)$ supports x . By construction $\pi \in \text{fix}(A \setminus \text{nontriv}(\pi))$, so by Definition 2.1.5(2) $\pi \cdot x = x$ as required. $\square$

2.2. Examples

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are nominal sets, and suppose $\mathcal{Z}$ is a set with a permutation action. We consider some examples of sets with a permutation action and of nominal sets. These will be useful later on in the paper.

2.2.1. Atoms and booleans

$\mathbb{A}$ is a nominal set with the *natural permutation action* $\pi \cdot a = \pi(a)$.

For the case of $\mathbb{A}$ only we will be lax about the difference between $\mathbb{A}$ (the set of atoms) and $(|\mathbb{A}|, \cdot)$ (the nominal set of atoms with its natural permutation action). In practice this means we write $a \in \mathbb{A}$ for $a \in |\mathbb{A}|$.

The only equivariant function from $\mathbb{A}$ to itself (Definition 2.1.8) is the identity map $a \mapsto a$. There are more small-supported maps from $\mathbb{A}$ to itself; see the *small-supported function space* below.

Write $\mathbb{B}$ for the nominal set of **Booleans**, which has elements $\{\perp, \top\}$ and the **trivial** permutation action that $\pi \cdot x = x$ for all π and $x \in |\mathbb{B}|$.

2.2.2. Cartesian product

$\mathcal{X} \times \mathcal{Y}$ is a nominal set with underlying set $\{(x, y) \mid x \in |\mathcal{X}|, y \in |\mathcal{Y}|\}$ and the *pointwise action* $\pi \cdot (x, y) = (\pi \cdot x, \pi \cdot y)$.

An equivariant $f : (\mathcal{X} \times \mathcal{Y}) \rightarrow \mathbb{B}$ corresponds to a relation $\mathcal{R}$ such that $x \mathcal{R} y$ if and only if $\pi \cdot x \mathcal{R} \pi \cdot y$.

2.2.3. Tensor product

$\mathcal{X} \otimes \mathcal{Y}$ is a nominal set with underlying set $\{(x, y) \mid x \in |\mathcal{X}|, y \in |\mathcal{Y}|, \text{supp}(x) \cap \text{supp}(y) = \emptyset\}$ and the pointwise action. For the pointwise action here to be well-defined depends on π being a permutation and the fact (Proposition 2.3.4 below) that $\text{supp}(\pi \cdot x) = \pi \cdot \text{supp}(x)$.

2.2.4. Full function space

Functions from $|\mathcal{X}|$ to $|\mathcal{Y}|$ form a set with a permutation action with underlying set all functions from $|\mathcal{X}|$ to $|\mathcal{Y}|$, and the **conjugation** permutation action

$$(\pi \cdot f)(x) = \pi \cdot (f(\pi^{-1} \cdot x)).$$

The conjugation action can be rephrased as ‘permutations distribute’ (cf. Theorem 2.3.2 below):

LEMMA 2.2.1. *If $f \in |\mathcal{X} \rightarrow \mathcal{Y}|$ then $\pi \cdot f(x) = (\pi \cdot f)(\pi \cdot x)$.*

Proof. By easy calculations. □

2.2.5. Small-supported function space

$\mathcal{X} \Rightarrow \mathcal{Y}$ is a nominal set with underlying set the functions from $|\mathcal{X}|$ to $|\mathcal{Y}|$ with small support under the conjugation action, and the conjugation permutation action.

A complete description of the small-supported functions from $\mathbb{A}$ to $\mathbb{A}$ is as follows:

1. Any function f such that there exists some small $U \subseteq \mathbb{A}$ such that if $b \notin U$ then $f(b) = b$ (so f is ‘eventually the identity’).
2. Any function f such that there exists some $a \in \mathbb{A}$ and some small $U \subseteq \mathbb{A}$ such that if $b \notin U$ then $f(b) = a$ (so f is ‘eventually constant’).

Looking ahead to Definition 2.3.6, we can write the possibilities as $\forall b. f(b) = b$ and $\exists a. \forall b. f(b) = a$ respectively.

LEMMA 2.2.2. *$f \in |\mathcal{X} \Rightarrow \mathcal{Y}|$ is equivariant in the sense of Definition 2.1.8 if and only if it is equivariant in the sense of Definition 2.1.9.*

Proof. We sketch the proof: If $\pi \cdot f = f$ then for any $x \in \mathcal{X}$ we have by Lemma 2.2.1 that $\pi \cdot (f(x)) = (\pi \cdot f)(\pi \cdot x) = f(\pi \cdot x)$. Conversely if for any $x \in \mathcal{X}$ we have $\pi \cdot (f(x)) = f(\pi \cdot x)$ then by the conjugation action $(\pi \cdot f)(x) = \pi \cdot (f(\pi^{-1} \cdot x)) = f(\pi \cdot (\pi^{-1} \cdot x)) = f(x)$. □

2.2.6. Full powerset

DEFINITION 2.2.3. Suppose $\mathcal{X}$ is a set with a permutation action. Give subsets $X \subseteq |\mathcal{X}|$ the **pointwise** permutation action

$$\pi \cdot X = \{\pi \cdot x \mid x \in X\}.$$

Then $\text{Pow}(\mathcal{X})$ (the full powerset of $\mathcal{X}$) is a set with a permutation action with

- underlying set $\{X \mid X \subseteq |\mathcal{X}|\}$ (the set of all subsets of $|\mathcal{X}|$), and

- the pointwise action $\pi \cdot X = \{\pi \cdot x \mid x \in X\}$.

A useful instance of the pointwise action is for sets of atoms. As discussed in Subsection 2.2.1 above, if $a \in \mathbb{A}$ then $\pi \cdot a = \pi(a)$. Thus if $A \subseteq \mathbb{A}$ then

$$\pi \cdot A \quad \text{means} \quad \{\pi(a) \mid a \in A\}.$$

In Definition 2.2.3 we gave $X \subseteq |\mathcal{X}|$ a permutation action. Thus by Definition 2.1.9 we also gave them a notion of equivariance (which X may or may not satisfy, of course). It is useful to unpack what this means:

LEMMA 2.2.4. *Suppose $\mathcal{X}$ is a set with a permutation action and $X \subseteq |\mathcal{X}|$. Then the following are equivalent:*

1. X is equivariant in the sense of Definition 2.1.9.
2. $\pi \cdot X = X$ for any permutation π .
3. $x \in X \Leftrightarrow \pi \cdot x \in X$ for any permutation π and $x \in X$.

Proof. If X is equivariant then X is supported by $\emptyset$ and by Theorem 2.1.12 and Definition 2.1.5(2) $\forall \pi. \pi \cdot X = X$. It follows by Definition 2.2.3 that $x \in X \Leftrightarrow x \in \pi^{-1} \cdot X \Leftrightarrow \pi \cdot x \in X$.

If $\forall \pi. \forall x \in |\mathcal{X}|. (x \in X \Leftrightarrow \pi \cdot x \in X)$ then also $x \in X \Leftrightarrow \pi^{-1} \cdot x \in X \Leftrightarrow x \in \pi \cdot X$, so that $\pi \cdot X = X$. This is for any π , so by Definition 2.1.5(2) $\emptyset$ supports X and by Definition 2.1.9 we have $\text{supp}(X) = \emptyset$. $\square$

LEMMA 2.2.5. *It is not the case that if $\mathcal{X}$ is a nominal set then $\text{Pow}(\mathcal{X})$ is a nominal set.*

Proof. It suffices to provide a counterexample. Take $\mathcal{X} = \mathbb{A}$ and enumerate atoms as $(a_\omega)_\omega$, and consider the set

$$\text{comb} = \{a_{2*\omega} \mid \omega\}$$

of ‘every other atom’. This does not have small support, though permutations still act on it pointwise. For more discussion of this point, see [Gab11b, Remark 2.18]. $\square$

We consider further examples in Subsection 2.5, including the small-supported and strictly small-supported powersets.

2.3. The principle of equivariance and the NEW quantifier

We come to Theorem 2.3.2, a result which is central to the ‘look and feel’ of nominal techniques. It enables a particularly efficient management of renaming and α -conversion in syntax and semantics and captures why it is so useful to use *names* in the foundations of our semantics and not some other infinite set, such as numbers.

Names are by definition symmetric (i.e. can be permuted). Taking names and permutations as *primitive* implies that permutations propagate to the things we build using them. This is the *principle of equivariance* (Theorem 2.3.2 below; see also [Gab11b, Subsection 4.2] and [GP01, Lemma 4.7]).

The principle of equivariance implies that, provided we permute names uniformly in all the parameters of our definitions and theorems, we then get another valid set of definitions and theorems. This is not true of e.g. numbers because our mathematical

foundation equips numbers by construction with numerical properties such as *less than* or *equal to* $\leq$, which can be defined from first principles with no parameters.

So if we use (for instance) numbers for names then we do not care about $\leq$ because we just needed an infinite set of elements, but then we repeatedly have to *prove* that we did not use an asymmetric property like $\leq$. In contrast, with nominal foundations and atoms, we do not have to explicitly prove symmetry because we can just look at our mathematical foundation and note that it is naturally symmetric under permuting names; we reserve numbers for naturally *asymmetric* activities, such as counting.

This style of name management is characteristic of nominal techniques. The reader can find it applied often, e.g. in Lemmas 3.2.6 and 3.4.10, Propositions 3.3.5 and 4.2.5, and Lemma 5.1.3.

REMARK 2.3.1. The languages of ZFA set theory and FM set theory are identical: first-order logic with equality $=$ and sets membership $\in$.

THEOREM 2.3.2. *If $\bar{x}$ is a list $x_1, \dots, x_n$, write $\pi \cdot \bar{x}$ for $\pi \cdot x_1, \dots, \pi \cdot x_n$. Suppose $\Phi(\bar{x})$ is a predicate in the language of ZFA/FM set theory, with free variables $\bar{x}$. Suppose $\Upsilon(\bar{x})$ is a function specified in the language of ZFA/FM set theory, with free variables $\bar{x}$. Then we have the following principles:*

1. **Equivariance of predicates.** $\Phi(\bar{x}) \Leftrightarrow \Phi(\pi \cdot \bar{x})$.⁵
2. **Equivariance of functions.** $\pi \cdot \Upsilon(\bar{x}) = \Upsilon(\pi \cdot \bar{x})$ (cf. Definition 2.1.8).⁶
3. **Conservation of support.** *If $\bar{x}$ denotes elements with small support then $\text{supp}(\Upsilon(\bar{x})) \subseteq \text{supp}(x_1) \cup \dots \cup \text{supp}(x_n)$. If in addition Υ is injective, then $\text{supp}(\Upsilon(\bar{x})) = \text{supp}(x_1) \cup \dots \cup \text{supp}(x_n)$.*

Proof. See Theorem 4.4, Corollary 4.6, and Theorem 4.7 from [Gab11b]. □

REMARK 2.3.3. Theorem 2.3.2 is three fancy ways of observing that if a specification is symmetric in atoms, the the result must be at least as symmetric as the inputs. Using atoms (instead of e.g. numbers) to model names makes this a one-line argument.⁷

PROPOSITION 2.3.4. $\text{supp}(\pi \cdot x) = \pi \cdot \text{supp}(x)$ (which means $\{\pi(a) \mid a \in \text{supp}(x)\}$).

Using Notation 2.1.10, $a \# \pi \cdot x$ if and only if $\pi^{-1}(a) \# x$, and $a \# x$ if and only if $\pi(a) \# \pi \cdot x$.

Proof. Immediate consequence of part 2 of Theorem 2.3.2.⁸ □

⁵Here $\bar{x}$ is understood to contain *all* the variables mentioned in the predicate. It is not the case that $a = a$ if and only if $a = b$ —but it is the case that $a = b$ if and only if $b = a$.

⁶Parts 1 and 2 of Theorem 2.3.2 are morally the same result: by considering Φ to be a function from its arguments to the nominal set of Booleans $\mathbb{B}$ from Subsection 2.2; and by treating a function as a functional relation, i.e. as a binary predicate.

⁷The reasoning in this paper could in principle be fully formalised in a sets foundation with atoms, such as Zermelo-Fraenkel set theory with atoms **ZFA**. Nominal sets can be implemented in ZFA sets such that nominal sets map to equivariant elements (elements with empty support) and the permutation action maps to ‘real’ permutation of atoms in the model. See [Gab11b, Subsection 9.3] and [Gab11b, Section 4].

⁸There is also a nice proof of this fact by direct calculations; see [Gab11b, Theorem 2.19]. However, it just instantiates Theorem 2.3.2 to the particular Υ specifying support.

COROLLARY 2.3.5. *Suppose $\mathcal{X}$ is a set with a permutation action and suppose $A \subseteq \mathbb{A}$ supports x .*

1. *If $\pi(a) = a$ for all $a \in A$ then $\pi \cdot x = x$.
In particular, if x has small support and $\pi(a) = a$ for all $a \in \text{supp}(x)$, then $\pi \cdot x = x$.*
2. *If $\pi(a) = \pi'(a)$ for every $a \in A$ then $\pi \cdot x = \pi' \cdot x$.*
3. *If x has small support then $a \# x$ if and only if $\exists b. (b \# x \wedge (b \ a) \cdot x = x)$.*

Proof. 1. Direct from Definition 2.1.5(2) and Theorem 2.1.12.

2. By properties of the group action, $\pi \cdot x = \pi' \cdot x$ precisely when $x = (\pi^{-1} \circ \pi') \cdot x$. So it suffices by part 1 of this result that $a = \pi^{-1} \circ \pi'(a)$ for every $a \in A$, in other words, that $\pi(a) = \pi'(a)$.

3. Suppose x has small support. We prove two implications:

- Suppose $a \# x$, meaning by Notation 2.1.10 that $a \notin \text{supp}(x)$. Choose any other $b \notin \text{supp}(x)$; then $(a \ b) \in \text{fix}(\text{supp}(x))$ and by part 1 of this result, $(b \ a) \cdot x = x$.
- Suppose $b \# x$, meaning by Notation 2.1.10 that $b \notin \text{supp}(x)$, and suppose $(b \ a) \cdot x = x$. It follows by Proposition 2.3.4 that $(b \ a) \cdot \text{supp}(x) = \text{supp}(x)$ and by facts of sets, $a \notin \text{supp}(x)$. $\square$

DEFINITION 2.3.6. Write $\mathbb{I}a.\Phi(a)$ for ‘ $\{a \mid \neg\Phi(a)\}$ is small’. We call this the $\mathbb{I}$ **quantifier**.

REMARK 2.3.7. We can read $\mathbb{I}$ as ‘for all but a small number of a ’, ‘for fresh a ’, or ‘for new a ’. It captures a *generative* aspect of names, that for any x we can find plenty of atoms a such that $a \notin \text{supp}(x)$. $\mathbb{I}$ was designed in [GP01] to model the quantifier being used when we informally write “rename x in $\lambda x.t$ to be fresh”, or “emit a fresh channel name” or “generate a fresh memory cell”.

REMARK 2.3.8. $\mathbb{I}$ belongs to a family of ‘for most’ quantifiers [Wes89], and is a *generalised quantifier* [KW96, Section 1.2.1].

Specifically over nominal sets, however, $\mathbb{I}$ displays special properties. In particular, it satisfies the *some/any property* that to prove a $\mathbb{I}$ -quantified property we test it for *one* fresh atom; we may then use it for *any* fresh atom. This is Theorem 2.3.9, which appears in the literature for instance as [Gab11b, Theorem 6.5] and [GP01, Proposition 4.10]:

THEOREM 2.3.9. *Suppose $\Phi(\bar{z}, a)$ is a predicate in the language of ZFA/FM set theory, with free variables $\bar{z}, a$. Suppose $\bar{z}$ denotes elements with small support. Then the following are equivalent:*

$$\forall a \in \mathbb{A}. (a \# \bar{z} \Rightarrow \Phi(\bar{z}, a)) \quad \mathbb{I}a.\Phi(\bar{z}, a) \quad \exists a \in \mathbb{A}. (a \# \bar{z} \wedge \Phi(\bar{z}, a))$$

Proof. Where convenient we may write $\bar{z}$ as $z_1, \dots, z_n$.

- Suppose $\Phi(\bar{z}, a)$ holds for every atom $a \in \mathbb{A} \setminus \bigcup_{1 \leq i \leq n} \text{supp}(z_i)$.
By assumption $\bar{z}$ denotes elements with small support, and it is a fact that a finite union of small sets is small, so $\mathbb{A} \setminus \bigcup_{1 \leq i \leq n} \text{supp}(z_i)$ is cosmall.
It follows that $\forall a. \Phi(\bar{z}, a)$ holds.
- Suppose $A \subseteq \mathbb{A}$ is cosmall and $\Phi(\bar{z}, a)$ for every $a \in A$. As in the previous point, there exists some $a \in A$ such that $a \# z_i$ for every $1 \leq i \leq n$.
It follows that $\exists a \in \mathbb{A}. (a \# \bar{z} \wedge \Phi(\bar{z}, a))$.
- Now suppose $\Phi(\bar{z}, a)$ holds for some $a \in \mathbb{A} \setminus \bigcup_{1 \leq i \leq n} \text{supp}(z_i)$.
By Theorem 2.3.2(1) $\Phi((a' a) \cdot \bar{z}, a')$ holds for any $a' \in \mathbb{A}$. Choosing $a' \# \bar{z}$ we have by Corollary 2.3.5(1) that $(a' a) \cdot z_i = z_i$ for every $1 \leq i \leq n$.
Thus $\forall a \in \mathbb{A}. (a \# \bar{z} \Rightarrow \Phi(\bar{z}, a))$ holds. $\square$

We mention a characterisation of small support using $\mathbb{A}$:

LEMMA 2.3.10. *Suppose $\mathcal{X}$ is a set with a permutation action. Then*

$$x \in \mathcal{X} \text{ is small-supported} \quad \Leftrightarrow \quad \forall a. \forall b. (b a) \cdot x = x.$$

Proof. If x is small-supported then by Corollary 2.3.5(1) it suffices to choose $a, b \in \mathbb{A} \setminus \text{supp}(x)$.

Conversely suppose $A \subseteq \mathbb{A}$ is small and $(b a) \cdot x = x$ for all distinct $a, b \in \mathbb{A} \setminus A$. Consider any permutation $\pi \in \text{fix}(A)$ and note from Definition 2.1.3 that $\text{nontriv}(\pi)$ is finite, so we can write π as a finite composition of swappings of atoms not in A . It follows that $\pi \cdot x = x$ and so by Definition 2.1.5(2) that A supports x . $\square$

2.4. Two lemmas

We conclude with two technical but general lemmas (they are also beautiful; especially the first one) which will help us later in Lemma 9.4.10 and Proposition 9.3.3.

Lemma 2.4.1 goes back to [GM07, Lemma 5.2] and [GM09a, Corollary 4.30]; see [Gab13, Lemma 7.6.2] for a recent presentation:

LEMMA 2.4.1. *If Υ is an equivariant function (Definition 2.1.8) from $\mathcal{X}$ to $\mathcal{Y}$ and $a \# \Upsilon(x)$ then there exists some $x' \in |\mathcal{X}|$ such that $a \# x'$ and $\Upsilon(x) = \Upsilon(x')$.*

Proof. Choose fresh b (so $b \# x$). By Corollary 2.3.5(1) $(b a) \cdot \Upsilon(x) = \Upsilon(x)$ and by Definition 2.1.8 $(b a) \cdot \Upsilon(x) = \Upsilon((b a) \cdot x)$. We take $x' = (b a) \cdot x$. $\square$

LEMMA 2.4.2. *Suppose $F : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ is an equivariant function and $x \in |\mathcal{X}|$ and $y \in |\mathcal{Y}|$. Suppose further that $a, b \# y$. Then $F(x, (b a) \cdot y) = (b a) \cdot F(x, y)$.*

Proof. By equivariance $(b a) \cdot F(x, y) = F((b a) \cdot x, (b a) \cdot y)$. By Corollary 2.3.5(1) (since $a, b \# y$) also $(b a) \cdot y = y$. $\square$

2.5. Further examples

We now consider the small-supported powerset and the strictly small-supported powerset. These examples are more technically challenging and will be key to the later constructions.

2.5.1. Small-supported powerset

Suppose $\mathcal{X}$ is a set with a permutation action (it does not have to be a nominal set). Then $Nom(\mathcal{X})$, the **nominal powerset**, is a nominal set, with

- underlying set those $X \in |Pow(\mathcal{X})|$ that are small-supported, and
- with the **pointwise** action $\pi \cdot X = \{\pi \cdot x \mid x \in X\}$ inherited from Definition 2.2.3.

Unpacking the definitions and using Corollary 2.3.5 and Lemma 2.3.10, $X \subseteq |\mathcal{X}|$ is small-supported when, equivalently:

- There exists small $A \subseteq \mathbb{A}$ such that if $\pi \in fix(A)$ then $\pi \cdot X = X$.
- There exists small $A \subseteq \mathbb{A}$ such that if $\pi \in fix(A)$ and $x \in X$ then $\pi \cdot x \in X$.
- $\forall a. \forall b. (b \# a) \cdot X = X$.
- $\forall a. \forall b. \forall x. (x \in X \Rightarrow (b \# a) \cdot x \in X)$.

For instance:

- $Nom(\mathbb{A})$ is the set of small and cosmall (Definition 2.1.1) sets of atoms.
- $X \in Nom(Pow(\mathbb{A}))$ is a set of sets of atoms with small support, though the elements $x \in X$ need not have small support.

For instance, if we set $x = comb$ from Lemma 2.2.5 then we can take $X = \{\pi \cdot x \mid \text{all permutations } \pi\}$. Here X has small (indeed, empty) support, even though none of its elements $\pi \cdot x$ have small support.

It is useful to formalise these observations as a lemma. A common source of confusion is to suppose that if A supports $X \in |Nom(\mathcal{X})|$ then A must support every $x \in X$. This is incorrect—and compare Lemma 2.5.1 with Lemma 2.5.3:

LEMMA 2.5.1. *It is not true in general that if $X \in |Nom(\mathcal{X})|$ and $x \in X$ then $supp(x) \subseteq supp(X)$.*

In other words, $a \# X$ and $x \in X$ does not imply $a \# x$.

Proof. It suffices to provide a counterexample. Take $\mathcal{X} = \mathbb{A}$ (the nominal set of atoms with the natural permutation action, from Subsection 2.2.1) and $X = \mathbb{A} \subseteq |\mathbb{A}|$ (the underlying set of the nominal set of all atoms, i.e. the set of all atoms!).

It is easy to check that $supp(X) = \emptyset$ and $a \in X$ and $supp(a) = \{a\} \not\subseteq \emptyset$. $\square$

2.5.2. Strictly small-supported powerset

Suppose $\mathcal{X}$ is a nominal set.

DEFINITION 2.5.2. Call $X \subseteq |\mathcal{X}|$ **strictly supported** by $A \subseteq \mathbb{A}$ when

$$\forall x \in X. supp(x) \subseteq A.$$

If there exists some small A that strictly supports X , then call X **strictly small-supported** (see [Gab11b, Theorem 2.29]).

Write $Strct(\mathcal{X})$ for the set of strictly small-supported $X \subseteq |\mathcal{X}|$. That is:

$$Strct(\mathcal{X}) = \{X \subseteq |\mathcal{X}| \mid \exists A \subseteq \mathbb{A}. A \text{ small} \wedge X \text{ strictly supported by } A\}$$

LEMMA 2.5.3. *If $X \in Strct(\mathcal{X})$ then:*

1. $\bigcup \{ \text{supp}(x) \mid x \in X \}$ is small.
2. $\bigcup \{ \text{supp}(x) \mid x \in X \} = \text{supp}(X)$.
3. If $X \subseteq |\mathcal{X}|$ is strictly small-supported then it is small-supported.
4. $x \in X$ implies $\text{supp}(x) \subseteq \text{supp}(X)$ (contrast this with Lemma 2.5.1).
Equivalently using Notation 2.1.10, if $a \in \mathbb{A}$ then $a\#X$ implies $a\#x$ for every $x \in X$.
5. $\text{Strct}(\mathcal{X})$ with the pointwise permutation action is a nominal set.

Proof. The first part is immediate since by assumption there is some small $A \subseteq \mathbb{A}$ that bounds $\text{supp}(x)$ for all $x \in X$. The second part follows by an easy calculation using Corollary 2.3.5(3); full details are in [Gab11b, Theorem 2.29], of which Lemma 2.5.3 is a special case. The other parts follow by definitions from the first and second parts. $\square$

EXAMPLE 2.5.4. 1. $\emptyset \subseteq |\mathbb{A}|$ is small-supported and strictly small-supported by $\emptyset$.
2. $\{a\}$ is small-supported by $\{a\}$ and also strictly small-supported by $\{a\}$.
3. $\mathbb{A} \subseteq |\mathbb{A}|$ is small-supported by $\emptyset$ but not strictly small-supported.
4. $\mathbb{A} \setminus \{a\}$ is small-supported by $\{a\}$ but not strictly small-supported.

LEMMA 2.5.5. If $X \subseteq |\mathcal{X}|$ is finite then $X \in \text{Strct}(\mathcal{X})$.

Proof. By Theorem 2.3.2(3) $\text{supp}(X) \subseteq \bigcup \{ \text{supp}(x) \mid x \in X \}$. $\square$

2.6. The NEW-quantifier for nominal sets

Suppose $\mathcal{X}$ is a set with a permutation action.

DEFINITION 2.6.1. Given a small-supported $X \subseteq |\mathcal{X}|$ define the **new-quantifier for (nominal) sets** by

$$\mathfrak{n}a.X = \{x \mid \forall b.(b \cdot a).x \in X\}.$$

LEMMA 2.6.2. Suppose $x \in |\mathcal{X}|$. Suppose $X \subseteq |\mathcal{X}|$ is small-supported. Then

$$x \in \mathfrak{n}a.X \Leftrightarrow \forall b.(b \cdot a).x \in X.$$

Proof. Immediate from Definition 2.6.1. $\square$

$\mathfrak{n}a.X$ was written $\mathfrak{n}a.X$ in [GLP11, Definition 5.2], and goes back to [Gab09b] where it was written $X - a$. We will use $\mathfrak{n}a$ in Lemma 11.3.8 to prove things about a σ -action.

LEMMA 2.6.3. $\text{supp}(\mathfrak{n}a.X) \subseteq \text{supp}(X) \setminus \{a\}$.

Proof. By a routine calculation using Corollary 2.3.5. $\square$

Recall from Subsection 2.5.1 the notion of nominal powerset $\text{Nom}(\mathcal{X})$.

LEMMA 2.6.4. If $X \in |\text{Nom}(\mathcal{X})|$ then $\mathfrak{n}a.X \in |\text{Nom}(\mathcal{X})|$.

Proof. This amounts to showing that if $X \subseteq |\mathcal{X}|$ has small support then so does $\mathfrak{n}a.X \subseteq |\mathcal{X}|$. This follows by Lemma 2.6.3 or direct from Theorem 2.3.2. $\square$

Part I

Nominal distributive lattices with quantification

3. Nominal algebras over nominal sets

3.1. Definition of a sigma-algebra (σ -algebra)

3.1.1. A termlike σ -algebra

Definitions 3.1.1, 3.1.4, and 3.2.1 assemble three key technical structures (see also Definitions 3.3.3 and 3.4.6).

DEFINITION 3.1.1. A **termlike σ -algebra** is a tuple $\mathcal{X} = (|\mathcal{X}|, \cdot, \text{sub}_{\mathcal{X}}, \text{atm}_{\mathcal{X}})$ of:

- a nominal set $(|\mathcal{X}|, \cdot)$ which we may write just as $\mathcal{X}$; and
- an equivariant σ -**action** $\text{sub}_{\mathcal{X}} : (\mathcal{X} \times \mathbb{A} \times \mathcal{X}) \rightarrow \mathcal{X}$, written $x[a \mapsto u]_{\mathcal{X}}$ or just $x[a \mapsto u]$; and
- an equivariant injection $\text{atm}_{\mathcal{X}} : \mathbb{A} \rightarrow \mathcal{X}$ written $a_{\mathcal{X}}$ or just a ,

such that the equalities $(\sigma\mathbf{a})$, $(\sigma\mathbf{id})$, $(\sigma\#)$, $(\sigma\alpha)$, and $(\sigma\sigma)$ of Figure 1 hold, where x , u , and v range over elements of $|\mathcal{X}|$.⁹

We may omit subscripts where $\mathcal{X}$ is understood.

REMARK 3.1.2. We unpack what equivariance from Definition 2.1.8 means for the σ -action from Definition 3.1.4: for every $x \in |\mathcal{X}|$, atom a , and $u \in |\mathcal{X}|$, and for every permutation π , we have that

$$\pi \cdot (x[a \mapsto u]) = (\pi \cdot x)[\pi(a) \mapsto \pi \cdot u].$$

Similarly for the equivariant $\mathfrak{v}$ -action in Definition 3.2.1 below.

REMARK 3.1.3. Definition 3.1.1 is abstract. It is an axiom system. We use *nominal algebra*, because the axioms require freshness side-conditions.

⁹Axiom $(\sigma\mathbf{id})$ might be more pedantically written as $x[a \mapsto a_{\mathcal{X}}] = x$.

$(\sigma\mathbf{a})$		$a[a \mapsto u] = u$
$(\sigma\mathbf{id})$		$x[a \mapsto a] = x$
$(\sigma\#)$	$a\#x \Rightarrow$	$x[a \mapsto u] = x$
$(\sigma\alpha)$	$b\#x \Rightarrow$	$x[a \mapsto u] = ((b \ a) \cdot x)[b \mapsto u]$
$(\sigma\sigma)$	$a\#v \Rightarrow$	$x[a \mapsto u][b \mapsto v] = x[b \mapsto v][a \mapsto u][b \mapsto v]$
$(\mathfrak{v}\sigma)$	$a\#v \Rightarrow$	$p[v \leftarrow b][u \leftarrow a] = p[u[b \mapsto v] \leftarrow a][v \leftarrow b]$

Figure 1: Nominal algebra axioms for σ and $\mathfrak{v}$

Examples of termlike σ -algebras include plenty of syntax: for instance the set of terms of first-order logic with substitution; or the syntax of the untyped λ -calculus quotiented by α -equivalence with capture-avoiding substitution; or the syntax of propositional logic with quantifiers (syntax generated by $\phi ::= a \mid \perp \mid \phi \Rightarrow \phi \mid \forall a.\phi$, with capture-avoiding substitution $[a:=\phi]$).

However, not all termlike σ -algebras are syntax. For a huge class of extremely non-syntactic termlike σ -algebras, consider models of FM sets [Gab09b].

3.1.2. A σ -algebra

DEFINITION 3.1.4. A **σ -algebra** is a tuple $\mathcal{X} = (|\mathcal{X}|, \cdot, \mathcal{X}^\partial, \text{sub})$ of:

- A nominal set $(|\mathcal{X}|, \cdot)$ which we may write just as $\mathcal{X}$.
- A termlike σ -algebra $\mathcal{X}^\partial$.
- An equivariant σ -**action** $\text{sub}_{\mathcal{X}} : (\mathcal{X} \times \mathbb{A} \times \mathcal{X}^\partial) \rightarrow \mathcal{X}$, written infix $x[a \mapsto u]_{\mathcal{X}}$ or $x[a \mapsto u]$.

such that the equalities (σid) , $(\sigma\#)$, $(\sigma\alpha)$, and $(\sigma\sigma)$ of Figure 1 hold,¹⁰ where x ranges over elements of $|\mathcal{X}|$ and u and v range over elements of $|\mathcal{X}^\partial|$.

As for termlike σ -algebras, we may omit the subscript $\mathcal{X}$. We may slightly informally say that $\mathcal{X}$ has a σ -algebra structure over $\mathcal{X}^\partial$.

REMARK 3.1.5. Every termlike σ -algebra is a σ -algebra over itself. The canonical ‘interesting’ example of a σ -algebra is the syntax of predicates of first-order logic, whose substitution action is not over predicates but over the termlike σ -algebra of terms.

Not all σ -algebras are syntactic. In this paper we will see many examples of non-syntactic σ -algebras, based on the σ -powersets of Definition 3.4.6.

3.2. Definition of an amgis-algebra

DEFINITION 3.2.1. An **τ -algebra** (spoken: **amgis-algebra**) is a tuple $\mathcal{P} = (|\mathcal{P}|, \cdot, \mathcal{P}^\partial, \text{amgis}_{\mathcal{P}})$ of:

- A set with a permutation action $(|\mathcal{P}|, \cdot)$ which we may write just as $\mathcal{P}$.
- A termlike σ -algebra $\mathcal{P}^\partial$.
- An equivariant **amgis**-action $\text{amgis}_{\mathcal{P}} : (\mathcal{P} \times \mathcal{P}^\partial \times \mathbb{A}) \rightarrow \mathcal{P}$, written infix $p[u \leftarrow a]_{\mathcal{P}}$ or $p[u \leftarrow a]$.

such that the equality $(\tau\sigma)$ of Figure 1 holds, where p ranges over elements of $|\mathcal{P}|$ and u and v range over elements of $|\mathcal{P}^\partial|$. We may omit the subscript $\mathcal{P}$.

REMARK 3.2.2. $[u \leftarrow a]$ looks like $[a \mapsto u]$ written backwards, and a casual glance at $(\tau\sigma)$ suggests that it is just $(\sigma\sigma)$ written backwards. This is not quite true: we have $u[b \mapsto v]$ on the right in $(\tau\sigma)$ and not ‘ $u[v \leftarrow b]$ ’ (and this would make no sense, since $\mathcal{P}^\partial$ in Definition 3.2.1 is a σ -algebra and is not equipped with an amgis-action).

Discussion of the origin of the axioms of τ -algebras is in Subsections 3.3 and 3.4; see also Proposition 3.3.5 and Subsection 3.4.3.

¹⁰That is, the σ axioms except $(\sigma\alpha)$, since we do not assume a function $\text{atm}_{\mathcal{X}}$. Axiom (σid) can be more pedantically written as $x[a \mapsto a_{\mathcal{X}^\partial}] = x$.

REMARK 3.2.3. In this paper we will see three main classes of $\mathfrak{v}$ -algebras in the sense of Definition 3.2.1:

- $\mathfrak{v}$ -algebras constructed directly from a σ -powerset. See Subsection 3.3. We do not require p to have small support (and this does not prevent us from getting all the results that we need).
- $\mathfrak{v}$ -algebras underlying spaces of the form $F(\mathcal{D})$ in Definition 8.1.5 (which is really a specialisation of the previous case). In Theorem 6.1.23 we may make many choices when we construct the points of these spaces, so we cannot require p to have small support, and in general this will not hold.
- $\mathfrak{v}$ -algebras underlying the concrete model Pnt_{Π} in Definition 11.1.3. These do have small support. See the discussion opening Subsection 11.1 and Definition 11.3.1 and the surrounding results.

We conclude with three technical lemmas which will be useful later:

LEMMA 3.2.4. *If $\mathcal{X}$ is a σ -algebra and $b \# x$ then $x[a \mapsto b] = (b \ a) \cdot x$.*

Proof. By $(\sigma\alpha)$ $x[a \mapsto b] = ((b \ a) \cdot x)[b \mapsto b]$. We use (σid) . $\square$

REMARK 3.2.5. In the two papers that introduced nominal algebra [GM06a, GM06b], Lemma 3.2.4 was taken as an axiom (it was called $(ren \mapsto)$) and (σid) was the lemma. In the presence of the other axioms of substitution, the two are equivalent.

LEMMA 3.2.6. *If $a \# u$ then $a \# x[a \mapsto u]$.*

Proof. Choose fresh b (so $b \# x, u$). By $(\sigma\alpha)$ $x[a \mapsto u] = ((b \ a) \cdot x)[b \mapsto u]$. Also by Corollary 2.3.5(1) $(b \ a) \cdot u = u$ and by Theorem 2.3.2 $(b \ a) \cdot (x[a \mapsto u]) = ((b \ a) \cdot x)[b \mapsto (b \ a) \cdot u]$. We put this all together and we deduce that $(b \ a) \cdot (x[a \mapsto u]) = x[a \mapsto u]$. It follows by Corollary 2.3.5(3) that $a \notin \text{supp}(x[a \mapsto u])$. $\square$

LEMMA 3.2.7. *If $c \# x$ then $x[a \mapsto c][b \mapsto a][c \mapsto b] = (b \ a) \cdot x$.*

Proof. We use Lemma 3.2.4, Proposition 2.3.4, and Corollary 2.3.5(2) to reason as follows:

$$x[a \mapsto c][b \mapsto a][c \mapsto b] = ((c \ a) \cdot x)[b \mapsto a][c \mapsto b] = (c \ b) \cdot (b \ a) \cdot ((c \ a) \cdot x) = (b \ a) \cdot x \quad \square$$

In Subsection 3.3 we move from a σ -algebra to an $\mathfrak{v}$ -algebra using nominal powersets. In Subsection 3.4 we move from an $\mathfrak{v}$ -algebra to a σ -algebra, again using nominal powersets.

3.3. Duality I: sigma to amgis

Given a σ -algebra we generate an $\mathfrak{v}$ -algebra out of its subsets. This is Proposition 3.3.5.

DEFINITION 3.3.1. Suppose $\mathcal{X} = (|\mathcal{X}|, \cdot, \mathcal{X}^\partial, \text{sub}_{\mathcal{X}})$ is a σ -algebra.

Give subsets $p \subseteq |\mathcal{X}|$ **pointwise** actions as follows:

$$\begin{aligned} \pi \cdot p &= \{\pi \cdot x \mid x \in p\} \\ p[u \leftarrow a] &= \{x \mid x[a \mapsto u] \in p\} \quad u \in |\mathcal{X}^\partial| \end{aligned}$$

PROPOSITION 3.3.2. Suppose $\mathcal{X}$ is a σ -algebra and $p \subseteq |\mathcal{X}|$ and $u \in |\mathcal{X}^\partial|$. Then:

- $x \in \pi \cdot p$ if and only if $\pi^{-1} \cdot x \in p$.
- $x \in p[u \leftarrow a]$ if and only if $x[a \mapsto u] \in p$.

Proof. By easy calculations on the pointwise actions in Definition 3.3.1. $\square$

DEFINITION 3.3.3. Suppose $\mathcal{X}$ is a σ -algebra. Define the $\mathfrak{v}$ -powerset algebra $\text{Pow}_{\mathfrak{v}}(\mathcal{X})$ by setting:

- $|\text{Pow}_{\mathfrak{v}}(\mathcal{X})|$ is the set of subsets $p \subseteq |\mathcal{X}|$ with permutation action $\pi \cdot p$ following Definition 3.3.1.
- $(\text{Pow}_{\mathfrak{v}}(\mathcal{X}))^\partial = \mathcal{X}^\partial$.
- The amgis-action $p[u \leftarrow a]$ follows Definition 3.3.1.

REMARK 3.3.4. In Definition 3.3.3 p is *not* necessarily small-supported. We will need this: the construction of Theorem 6.1.23 will generate p with potentially large support.

PROPOSITION 3.3.5. If $\mathcal{X}$ is a σ -algebra then $\text{Pow}_{\mathfrak{v}}(\mathcal{X})$ from Definition 3.3.3 is an $\mathfrak{v}$ -algebra.

Proof. By Theorem 2.3.2 the operations are equivariant. We verify rule $(\mathfrak{v}\sigma)$ from Figure 1:

- *Property $(\mathfrak{v}\sigma)$.* Suppose $a \# v$. Then:

$$\begin{aligned} x \in p[v \leftarrow b][u \leftarrow a] &\Leftrightarrow x[a \mapsto u][b \mapsto v] \in p && \text{Proposition 3.3.2} \\ &\Leftrightarrow x[b \mapsto v][a \mapsto u[b \mapsto v]] \in p && (\sigma\sigma), a \# v \\ &\Leftrightarrow x \in p[u[b \mapsto v] \leftarrow a][v \leftarrow b] && \text{Proposition 3.3.2} \quad \square \end{aligned}$$

3.4. Duality II: amgis to sigma

3.4.1. The pointwise sigma-action on subsets of an amgis-algebra

DEFINITION 3.4.1. Suppose $\mathcal{P} = (|\mathcal{P}|, \cdot, \mathcal{P}^\partial, \text{amgis}_{\mathcal{P}})$ is an $\mathfrak{v}$ -algebra. Give subsets $X \subseteq |\mathcal{P}|$ **pointwise** actions as follows:

$$\begin{aligned} \pi \cdot X &= \{\pi \cdot p \mid p \in X\} \\ X[a \mapsto u] &= \{p \mid \forall c. p[u \leftarrow c] \in (c \ a) \cdot X\} \quad u \in |\mathcal{P}^\partial| \end{aligned}$$

REMARK 3.4.2. We call the action in Definition 3.4.1 *pointwise*. We should note:

- In the case of $\pi \cdot X$ this is true in the conventional sense; see Proposition 3.4.3(3).
- However, in the case of $X[a \mapsto u]$ this is true only subject to a freshness side-condition that is encoded in the use of $\forall$ above; see Proposition 3.4.3(1&2). The extra technical machinery of the $\forall c$ and the freshening permutation $(c \ a)$ is needed to give us Lemma 3.4.4.

PROPOSITION 3.4.3. *Suppose $\mathcal{P}$ is an $\mathfrak{v}$ -algebra and $X \subseteq |\mathcal{P}|$. Suppose $p \in |\mathcal{P}|$ and $u \in |\mathcal{P}^\partial|$. Then:*

1. $p \in X[a \mapsto u]$ if and only if $\forall c. p[u \leftarrow c] \in (c a) \cdot X$.
2. If furthermore $p \in |\mathcal{P}|$ has small support and $a \# u, p$, then we can simplify part 1 of this result to $p \in X[a \mapsto u]$ if and only if $p[u \leftarrow a] \in X$.
3. $p \in \pi \cdot X$ if and only if $\pi^{-1} \cdot p \in X$.

Proof. 1. Direct from Definition 3.4.1.

2. Suppose $a \# u, p$. From part 1 of this result $p \in X[a \mapsto u]$ if and only if $\forall c. p[u \leftarrow c] \in (c a) \cdot X$. By Corollary 2.3.5(1) $(c a) \cdot u = u$ and $(c a) \cdot p = p$, so (applying $(c a)$ to both sides of the equality) this is if and only if $\forall c. p[u \leftarrow a] \in X$, which means that $p[u \leftarrow a] \in X$.

3. Direct from Theorem 2.3.2. $\square$

LEMMA 3.4.4 (α -equivalence). *Suppose $\mathcal{P}$ is an $\mathfrak{v}$ -algebra and $X \subseteq |\mathcal{P}|$. Then*

$$b \# X \text{ implies } X[a \mapsto u] = ((b a) \cdot X)[b \mapsto u].$$

Proof. By part 1 of Proposition 3.4.3 $p \in X[a \mapsto u]$ if and only if $\forall c. p[u \leftarrow c] \in (c a) \cdot X$, and $p \in ((b a) \cdot X)[b \mapsto u]$ if and only if $\forall c. p[u \leftarrow c] \in (c b) \cdot ((b a) \cdot X)$. By Corollary 2.3.5(1) $(c a) \cdot X = (c b) \cdot ((b a) \cdot X)$ since $b \# X$. The first part follows. $\square$

Proposition 3.4.5 is useful, amongst other things, in Lemma 3.4.10. On syntax it is known as the *substitution lemma*, but here it is about an action on sets X , and the proof is different:

PROPOSITION 3.4.5. *Suppose $\mathcal{P}$ is an $\mathfrak{v}$ -algebra and $X \subseteq |\mathcal{P}|$. Suppose $u, v \in |\mathcal{P}^\partial|$. Then*

$$a \# v \text{ implies } X[a \mapsto u][b \mapsto v] = X[b \mapsto v][a \mapsto u[b \mapsto v]].$$

Proof. We reason as follows, where we write $\pi = (a' a) \circ (b' b)$:

$$\begin{aligned} p \in X[a \mapsto u][b \mapsto v] &\Leftrightarrow \forall a', b'. p[v \leftarrow b'][(b' b) \cdot u \leftarrow a'] \in \pi \cdot X && \text{Proposition 3.4.3} \\ &\Leftrightarrow \forall a', b'. p[((b' b) \cdot u)[b' \mapsto v] \leftarrow a'] [v \leftarrow b'] \in \pi \cdot X && (\mathfrak{v}\sigma) a' \# v \\ &\Leftrightarrow \forall a', b'. p[u[b \mapsto v] \leftarrow a'] [v \leftarrow b'] \in \pi \cdot X && (\sigma\alpha) b' \# u \\ &\Leftrightarrow \forall a'. p[u[b \mapsto v] \leftarrow a'] \in ((a' a) \cdot X)[b \mapsto v] && \text{Proposition 3.4.3} \\ &\Leftrightarrow \forall a'. p[u[b \mapsto v] \leftarrow a'] \in ((a' a) \cdot X)[b \mapsto (a' a) \cdot v] && \text{Cor 2.3.5 } a', a \# v \\ &\Leftrightarrow \forall a'. p[u[b \mapsto v] \leftarrow a'] \in (a' a) \cdot (X[b \mapsto v]) && \text{Theorem 2.3.2} \\ &\Leftrightarrow p \in X[b \mapsto v][a \mapsto u[b \mapsto v]] && \text{Proposition 3.4.3} \end{aligned}$$

$\square$

3.4.2. The σ -powerset $Pow_\sigma(\mathcal{P})$

Recall from Subsection 2.5.1 the *small-supported powerset* $Nom(\mathcal{X})$ of a nominal set $\mathcal{X}$.

DEFINITION 3.4.6. Suppose $\mathcal{P}$ is an $\mathfrak{v}$ -algebra. Define the σ -**powerset** algebra $Pow_\sigma(\mathcal{P})$ by setting:

- $|Pow_\sigma(\mathcal{P})|$ is those $X \in |Nom(\mathcal{P})|$ with the actions $\pi \cdot X$ and $X[a \mapsto u]$ from Definition 3.4.1, satisfying conditions 1 and 2 below.

- $Pow_\sigma(\mathcal{P})^\partial = \mathcal{P}^\partial$.

The $X \in |Nom(\mathcal{P})|$ above are restricted with conditions as follows, where $u \in |\mathcal{P}^\partial|$ and $p \in |\mathcal{P}|$:

1. $\forall u. \forall a. \forall p. (p[u \leftarrow a] \in X \Leftrightarrow p \in X)$.
In other words: if $a \# X$, u then $p[u \leftarrow a] \in X \Leftrightarrow p \in X$.
2. $\forall a. \forall b. \forall p. (p[b \leftarrow a] \in X \Leftrightarrow (b \ a) \cdot p \in X)$.
In other words: if $b \# X$ then $p[b \leftarrow a] \in X \Leftrightarrow (b \ a) \cdot p \in X$.

Lemma 3.4.7 reformulates the conditions of Definition 3.4.6 by packaging up some complexity using the σ -action on subsets of an $\mathfrak{o}$ -algebra from Definition 3.4.1:

LEMMA 3.4.7. *Continuing the notation of Definition 3.4.6, if $X \in |Pow_\sigma(\mathcal{P})|$ then*

1. *If $a \# X$ then $X[a \rightarrow u] = X$.*
2. *If $b \# X$ then $X[a \rightarrow b] = (b \ a) \cdot X$.*

Proof. 1. Suppose $a \# X$. By part 1 of Proposition 3.4.3 $p \in X[a \rightarrow u]$ if and only if $\forall c. p[u \leftarrow c] \in (c \ a) \cdot X$. By Corollary 2.3.5(1) $(c \ a) \cdot X = X$ and by condition 1 of Definition 3.4.6 $p[u \leftarrow c] \in X$ if and only if $p \in X$, so this is if and only if $\forall c. (p \in X)$, that is $p \in X$.
2. We combine Proposition 3.4.3 with condition 2 of Definition 3.4.6, since $a \# b$. $\square$

We continue the discussion from Remark 3.4.2:

REMARK 3.4.8. Condition 1 of Definition 3.4.6 corresponds to part 1 of Lemma 3.4.7 for $i \in \{1, 2\}$; Definition 3.4.6 uses the $\mathfrak{o}$ -action and Lemma 3.4.7 uses the corresponding σ -action for sets from Definition 3.4.1.

Lemma 3.4.7 is more than just a rephrasing of Definition 3.4.6, because of the freshness conditions discussed in Remark 3.4.2: condition 1 of Definition 3.4.6 insists that $a \# u$ (because we have $\forall u. \forall a$), whereas part 1 of Lemma 3.4.7 does not insist on $a \# u$.

Thus, Lemma 3.4.7 is slightly stronger than Definition 3.4.6 (harder to prove; easier to apply).

The conditions of Definition 3.4.6 are optimised for a situation where we want to prove that some X is in $Pow_\sigma(\mathcal{P})$ (because we may assume $a \# u$, and so have a relatively weaker proof-obligation), whereas Lemma 3.4.7 is optimised for a situation where we are told that $X \in |Pow_\sigma(\mathcal{P})|$ and we want to manipulate it (because we do not need to assume $a \# u$ to apply part 1 of the Lemma). We will freely use whichever form is most convenient in context.

COROLLARY 3.4.9. *Suppose $X \in |Pow_\sigma(\mathcal{P})|$. Then $X[a \rightarrow a] = X$.*

Proof. Suppose $b \# X$. By Lemma 3.4.4 $X[a \rightarrow a] = ((b \ a) \cdot X)[b \rightarrow a]$. By Proposition 2.3.4 $a \# (b \ a) \cdot X$. By part 2 of Lemma 3.4.7 $((b \ a) \cdot X)[b \rightarrow a] = (b \ a) \cdot ((b \ a) \cdot X) = X$. $\square$

LEMMA 3.4.10. *If $X \in |Pow_\sigma(\mathcal{P})|$ and $u \in |\mathcal{P}^\partial|$ then also $X[a \rightarrow u] \in |Pow_\sigma(\mathcal{P})|$.*

As a corollary, in Definition 3.4.6, $|Pow_\sigma(\mathcal{P})|$ is closed under the σ -action from Definition 3.4.1.

Proof. By construction $X[a \mapsto u] \subseteq |\mathcal{P}|$, so we now check the properties listed in Definition 3.4.6.

By assumption in Definition 3.4.6, X is small-supported. Small support of $X[a \mapsto u]$ is from Theorem 2.3.2.

We check the conditions of Definition 3.4.6 for $X[a \mapsto u]$:

1. For fresh b (so $b \# u, X$), $X[a \mapsto u][b \mapsto v] = X[a \mapsto u]$.

We use Lemma 3.4.4 to assume without loss of generality that $a \# u$. It suffices to reason as follows:

$$\begin{aligned} X[a \mapsto u][b \mapsto v] &= X[b \mapsto v][a \mapsto u[b \mapsto v]] && \text{Proposition 3.4.5, } a \# v \\ &= X[b \mapsto v][a \mapsto u] && (\sigma \#), b \# u \\ &= X[a \mapsto u] && \text{Lemma 3.4.7(1), } b \# X \end{aligned}$$

2. For fresh b' (so $b' \# u, v, X$) $X[a \mapsto u][b \mapsto b'] = (b' b) \cdot (X[a \mapsto u])$.

It suffices to reason as follows:

$$\begin{aligned} X[a \mapsto u][b \mapsto b'] &= X[b \mapsto b'][a \mapsto u[b \mapsto b']] && \text{Proposition 3.4.5, } a \# b' \\ &= ((b' b) \cdot X)[a \mapsto (b' b) \cdot u] && \text{Lemma 3.4.7, } b' \# u, X \\ &= (b' b) \cdot (X[a \mapsto u]) && \text{Theorem 2.3.2(2)} \quad \square \end{aligned}$$

PROPOSITION 3.4.11. *If $\mathcal{P}$ is an $\mathfrak{v}$ -algebra then $\text{Pow}_\sigma(\mathcal{P})$ (Definition 3.4.6) is indeed a σ -algebra.*

Proof. By Lemma 3.4.10 the σ -action does indeed map to $|\text{Pow}_\sigma(\mathcal{P})|$. By Theorem 2.3.2 so does the permutation action. It remains to check validity of the axioms from Definition 3.1.4.

- Axiom (σid) is Corollary 3.4.9.
- Axiom $(\sigma \#)$ is part 1 of Lemma 3.4.7.
- Axiom $(\sigma \alpha)$ is Lemma 3.4.4.
- Axiom $(\sigma \sigma)$ is Proposition 3.4.5. $\square$

3.4.3. Some further remarks

The $\mathfrak{v}$ -action $p[u \leftarrow a]$ in Definition 3.3.1 is the *functional preimage* of an underlying σ -action, and the σ -action in $X[a \mapsto u]$ in Definition 3.4.1 is obtained by first binding a using a $\mathfrak{U}$ -quantifier—and then taking the functional preimage of an underlying $\mathfrak{v}$ -action. This binding is designed to make α -equivalence (Lemma 3.4.4) a structural fact of the definition—that is, $(\sigma \alpha)$ from Figure 1.

We can suggest the following intuitions for the amgis- and sigma-actions from Definitions 3.3.1 and 3.4.1:

- $p[u \leftarrow a]$ is intuitively “ p , reprogrammed to believe that a is equal to u ”.
- $X[a \mapsto u]$ is intuitively “ X , reprogrammed to believe that a is equal to u , then bind a ”.

In the case that $\mathcal{P}$ is small-supported, so that every $p \in |\mathcal{P}|$ has small support, then Definition 3.3.1 can be simplified as described in part 2 of Proposition 3.4.3. This version appeared as a definition in [Gab14b, Gab16]. We arrived at Definition 3.4.1 as a modification and generalisation of the first definition to the case where we cannot

assume that points have small support (because of the infinitely many choices made in Theorem 6.1.23).

Interestingly, only $(\sigma\sigma)$ comes directly from the structure of the underlying $\mathfrak{o}$ -algebra (from $(\mathfrak{o}\sigma)$). Other axioms are forced— $(\sigma\alpha)$ from the definition (Lemma 3.4.4), and $(\sigma\#)$ and (σid) from conditions 1 and 2 in Definition 3.4.6.

4. Nominal posets

4.1. Nominal posets and fresh-finite limits

DEFINITION 4.1.1. A **nominal poset** is a tuple $\mathcal{L} = (|\mathcal{L}|, \cdot, \leq)$ where

1. $(|\mathcal{L}|, \cdot)$ is a nominal set, and
2. The relation $\leq \subseteq |\mathcal{L}| \times |\mathcal{L}|$ is an equivariant partial order.¹¹

DEFINITION 4.1.2. Say a nominal poset $\mathcal{L}$ is **finitely fresh-complete** or has **fresh-finite limits** when:

- $\mathcal{L}$ has a top element $\top$.
- $\mathcal{L}$ has conjunctions $x \wedge y$ (a greatest lower bound for x and y).
- $\mathcal{L}$ has a -fresh limits $\forall a.x$, where $\forall a.x$ is greatest amongst elements x' such that $x' \leq x$ and $a\#x'$.

Say $\mathcal{L}$ is **finitely cocomplete**¹² or say it has **finite colimits** when:

- $\mathcal{L}$ has a bottom element $\perp$.
- $\mathcal{L}$ has disjunctions $x \vee y$ (a least upper bound for x and y).

Lemmas 4.1.3, 4.1.4, and 4.1.5 will be useful later:

LEMMA 4.1.3. If $b\#x$ then $\forall a.x = \forall b.(b a).x$.

Proof. By assumption $a\#\forall a.x$ and by Theorem 2.3.2 also $b\#\forall a.x$, so by Corollary 2.3.5(1) $\forall a.x = (b a).\forall a.x$. By Theorem 2.3.2(2) $(b a).\forall a.x = \forall b.(b a).x$. $\square$

LEMMA 4.1.4. $\top$, $\perp$, $x \wedge y$, $x \vee y$, and $\forall a.x$ are unique if they exist.

Proof. Since for a partial order, $x \leq y$ and $y \leq x$ imply $x = y$. $\square$

LEMMA 4.1.5. $\forall a.(x_1 \wedge \dots \wedge x_n) = (\forall a.x_1) \wedge \dots \wedge (\forall a.x_n)$.

Proof. Both the left-hand and right-hand sides specify a greatest element z such that $a\#z$ and $z \leq x_i$ for $1 \leq i \leq n$. $\square$

LEMMA 4.1.6. $\forall a.\forall b.x = \forall b.\forall a.x$.

Proof. Both the left-hand and right-hand sides specify a greatest element z such that $a\#z$ and $b\#z$ and $z \leq x$. $\square$

¹¹So $\leq$ is transitive, reflexive, and antisymmetric, and $x \leq y$ if and only if $\pi \cdot x \leq \pi \cdot y$.

¹²There is also a notion of finitely fresh-cocomplete, but we will not need it.

LEMMA 4.1.7. Suppose $\mathcal{L}$ is a finitely fresh-complete nominal poset. Then $x \leq x'$ implies $\forall a.x \leq \forall a.x'$.

Proof. Suppose $x \leq x'$. By assumption $\forall a.x \leq x$ and $a\# \forall a.x$. But then also $\forall a.x \leq x'$ and $a\# \forall a.x$, so $\forall a.x \leq \forall a.x'$. $\square$

NOTATION 4.1.8. We may write $\forall a_1, \dots, a_n.x$ for $\forall a_1 \dots \forall a_n.x$.

4.2. Compatible σ -structure

DEFINITION 4.2.1. Say that a finitely fresh-complete and finitely cocomplete nominal poset $\mathcal{L} = (|\mathcal{L}|, \cdot, \leq)$ has a **compatible** σ -algebra structure when it is also a σ -algebra $(|\mathcal{L}|, \cdot, \mathcal{L}^\partial, \text{sub}_{\mathcal{L}})$ and in addition

$$\begin{aligned} (x \wedge y)[a \mapsto u] &= (x[a \mapsto u]) \wedge (y[a \mapsto u]) \\ (x \vee y)[a \mapsto u] &= (x[a \mapsto u]) \vee (y[a \mapsto u]) \\ b\#u \Rightarrow (\forall b.x)[a \mapsto u] &= \forall b.(x[a \mapsto u]) \end{aligned}$$

where $x, y \in |\mathcal{L}|$ and $u \in |\mathcal{L}^\partial|$, where $\wedge, \vee$, and $\forall b$ exist.

Call the σ -action **monotone** when

$$x \leq y \text{ implies } x[a \mapsto u] \leq y[a \mapsto u].$$

LEMMA 4.2.2. Continuing Definition 4.2.1, if the σ -structure is compatible then it is monotone.

Proof. It is a fact that $x \leq y$ if and only if $x \wedge y = x$. The result follows. $\square$

Our main source of nominal posets with a compatible or monotone σ -action, in this paper, will be the nominal distributive lattices with $\forall$ of the later Definition 4.3.3.

LEMMA 4.2.3. Suppose $\mathcal{L}$ is a nominal poset with a monotone σ -action, and suppose $a\#z$ for $z \in |\mathcal{L}|$.

Then if $z \leq x$ then z is a lower bound for $\{x[a \mapsto u] \mid u \in |\mathcal{L}^\partial|\}$, and as a particular corollary,

$$\forall a.x \leq x[a \mapsto u]$$

for every $x \in |\mathcal{L}|$ and $u \in |\mathcal{L}^\partial|$.

Proof. By monotonicity $z[a \mapsto u] \leq x[a \mapsto u]$ for every $u \in |\mathcal{L}^\partial|$. By $(\sigma\#)$ also $z = z[a \mapsto u]$. The corollary follows just noting that by the definition of fresh-finite limit in Definition 4.1.2, $\forall a.x \leq x$ and $a\# \forall a.x$. $\square$

LEMMA 4.2.4. $a\#\{x[a \mapsto u] \mid u \in |\mathcal{L}^\partial|\}$ and $a\#\{x[a \mapsto n_{\mathcal{L}^\partial}] \mid n \in \mathbb{A}\}$.

Proof. We use Corollary 2.3.5(3). Choose fresh b (so $b\#x$). Then we reason as follows:

$$\begin{aligned} (b a) \cdot \{x[a \mapsto u] \mid u \in |\mathcal{L}^\partial|\} &= \{(b a) \cdot (x[a \mapsto u]) \mid u \in |\mathcal{L}^\partial|\} && \text{Pointwise action} \\ &= \{((b a) \cdot x)[b \mapsto (b a) \cdot u] \mid u \in |\mathcal{L}^\partial|\} && \text{Theorem 2.3.2} \\ &= \{((b a) \cdot x)[b \mapsto u] \mid u \in |\mathcal{L}^\partial|\} && \pi \cdot |\mathcal{L}^\partial| = |\mathcal{L}^\partial| \\ &= \{x[a \mapsto u] \mid u \in |\mathcal{L}^\partial|\} && (\sigma\alpha), b\#x \end{aligned}$$

The reasoning for $a\#\{x[a \mapsto n_{\mathcal{L}^\partial}] \mid n \in \mathbb{A}\}$ is similar. $\square$

In Definition 4.1.2 we characterised universal quantification as a fresh-finite limit. However, in the presence of a σ -action we also have an intuition that universal quantification is an infinite intersection (so $\forall a.x$ should mean ‘ $x[a \mapsto u]$ for every u ’). Proposition 4.2.5 makes this intuition formal:

PROPOSITION 4.2.5. *Suppose $\mathcal{L}$ is a nominal poset with a monotone σ -action (Definition 4.2.1), and suppose $x \in |\mathcal{L}|$. Then:*

1. *If $\forall a.x$ exists in $\mathcal{L}$ then so does $\bigwedge_{u \in |\mathcal{L}^\partial|} x[a \mapsto u]$ the limit for $\{x[a \mapsto u] \mid u \in |\mathcal{L}^\partial|\}$, and they are equal. In symbols:*

$$\forall a.x = \bigwedge_{u \in |\mathcal{L}^\partial|} x[a \mapsto u].$$

2. *If $\bigwedge_u x[a \mapsto u]$ exists in $\mathcal{L}$ then so does $\forall a.x$, and they are equal.*

Proof. By Lemma 4.2.3 $\forall a.x$ is a lower bound for $\{x[a \mapsto u] \mid u \in |\mathcal{L}^\partial|\}$.

Now suppose z is any other lower bound, that is: $z \leq x[a \mapsto u]$ for every $u \in |\mathcal{L}^\partial|$. Note that we do not know *a priori* that $a \# z$.

Choose b fresh (so $b \# z, x$) and take $u = b$. Then $z \leq x[a \mapsto b] \stackrel{\text{Lem 3.2.4}}{=} (b \ a) \cdot x$. Since $b \# z$ it follows that $z \leq \forall b. (b \ a) \cdot x \stackrel{\text{Lem 4.1.3}}{=} \forall a.x$. So $\forall a.x = \bigwedge_u x[a \mapsto u]$.

Now suppose that $\bigwedge_u x[a \mapsto u]$ exists. By Lemma 4.2.4 and part 2 of Theorem 2.3.2 we have that $a \# \bigwedge_u x[a \mapsto u]$. Also by assumption $\bigwedge_u x[a \mapsto u] \leq x[a \mapsto a] \stackrel{(\sigma \text{id})}{=} x$. Thus $\bigwedge_u x[a \mapsto u]$ is an *a*-lower bound for x .

Now suppose $z \leq x$ and $a \# z$; we need to show that $z \leq \bigwedge_u x[a \mapsto u]$. This is direct from Lemma 4.2.3. $\square$

We also mention a characterisation of $\forall a.x$ using a ‘smaller’ conjunction which does not depend on most of $\mathcal{L}^\partial$ (see also Remark 4.2.7):

PROPOSITION 4.2.6. *Suppose $\mathcal{L}$ is a nominal poset with a monotone σ -action (Definition 4.2.1) and $x \in |\mathcal{L}|$. Then:*

1. *If $\forall a.x$ exists in $\mathcal{L}$ then so does $\bigwedge_n x[a \mapsto n]$ where n ranges over all atoms, and they are equal.*
2. *If $\bigwedge_n x[a \mapsto n]$ exists in $\mathcal{L}$ then so does $\forall a.x$, and they are equal.¹³*

Proof. The proof is just like the proof of Proposition 4.2.5, but we are careful and still give full details. The important point is that by Lemma 4.2.4 and part 2 of Theorem 2.3.2 we have $a \# \bigwedge_n x[a \mapsto n]$.

Using Lemma 4.2.3 we see that $\forall a.x$ is a lower bound for $\{x[a \mapsto n] \mid n \in \mathbb{A}\}$.

Now suppose z is any other lower bound, that is: $z \leq x[a \mapsto n]$ for every $n \in \mathbb{A}$. Choose b fresh (so $b \# z, x$) and take $n = b$. Then $z \leq x[a \mapsto b] \stackrel{\text{Lem 3.2.4}}{=} (b \ a) \cdot x$. Since $b \# z$ it follows that $z \leq \forall b. (b \ a) \cdot x \stackrel{\text{Lem 4.1.3}}{=} \forall a.x$. So $\forall a.x = \bigwedge_n x[a \mapsto n]$.

¹³More fully this is $\bigwedge_n x[a \mapsto \text{atm}_{\mathcal{L}^\partial}(n)]$. See the notation in Definition 3.1.1.

Now suppose that $\bigwedge_n x[a \mapsto n]$ exists. By Lemma 4.2.4 and part 2 of Theorem 2.3.2 we have that $a \# \bigwedge_n x[a \mapsto n]$. Also by assumption $\bigwedge_n x[a \mapsto n] \leq x[a \mapsto a] \stackrel{(\sigma \text{id})}{=} x$. Thus $\bigwedge_n x[a \mapsto n]$ is an $a \#$ lower bound for x .

Now suppose $z \leq x$ and $a \# z$; we need to show that $z \leq \bigwedge_n x[a \mapsto n]$. This is direct from Lemma 4.2.3. $\square$

REMARK 4.2.7. Proposition 4.2.6 is important: later on when we consider morphisms, we will need to know that a map g^{-1} in Proposition 8.2.2 commutes with quantification. But g^{-1} can change the underlying termlike σ -algebra—that is, the domain of substitution and of quantification $\mathcal{L}^\partial$ can change.

Could this interfere with the universal quantifier, since extra elements might make what was a true universal quantification, into a false one? If we look at Proposition 4.2.5, it seems that this might be the case; Proposition 4.2.6 says that *this cannot happen*.

It asserts that the notion of morphism used in Proposition 8.2.2 must take any extra elements into account. In the language of [Sel02], Proposition 4.2.6 implies that our models are *well-pointed*.

A dedicated discussion of such issues in the context of nominal models of the λ -calculus is in [GM11, Subsection 3.4]. More on this in Remark 8.2.3.

4.3. Definition of a nominal distributive lattice with $\forall$

DEFINITION 4.3.1. Suppose $\mathcal{L}$ is a fresh-finitely complete and finitely cocomplete nominal poset. Call $\mathcal{L}$ **distributive** when

$(\text{distrib}\wedge) \quad x\forall(y\wedge z) = (x\forall y) \wedge (x\forall z)$	
$(\text{distrib}\forall) \quad a \# x \Rightarrow x\forall\forall a.y = \forall a.(x\forall y)$	for every $x, y, z \in \mathcal{L} $.

REMARK 4.3.2. Definition 4.3.1 generalises the usual notion of distributivity; $\forall$ distributes over $\wedge$ and also over $\forall a$ (subject to a typical nominal algebra freshness side-condition), which we have seen exhibited as an infinite intersection in Proposition 4.2.5.¹⁴

An elegant, though arguably less readable, version of Definition 4.3.1 unifies $(\text{distrib}\wedge)$ and $(\text{distrib}\forall)$ to a single axiom which we could write as $S \# x \Rightarrow x\forall\forall S.Y = \forall S.\{x\forall y \mid y \in Y\}$, where $S \subseteq \mathbb{A}$ and $Y \subseteq |\mathcal{L}|$ are finite. Or in English: $\forall$ distributes over fresh-finite limits.

Note in passing that in Subsection A.1, $(\text{distrib}\wedge)$ and $(\text{distrib}\forall)$ will feature as part of a purely nominal algebraic axiomatisation of fresh-finite limits.

¹⁴A dual version of part 1 of Definition 4.3.1 is $x \wedge (y\forall z) = (x\wedge y)\forall(x\wedge z)$ and by a standard argument [DP02, Lemma 4.3] the two are equivalent.

DEFINITION 4.3.3. A **nominal distributive lattice with $\forall$** is a tuple $\mathcal{D} = (|\mathcal{D}|, \cdot, \leq, \mathcal{D}^\partial, \text{sub}_{\mathcal{D}})$ such that:

- $(|\mathcal{D}|, \cdot, \leq)$ is a nominal poset (Definition 4.1.1).
- $\mathcal{D}$ has fresh-finite limits and finite colimits (Definition 4.1.2), and is distributive (Definition 4.3.1).
- $(|\mathcal{D}|, \cdot, \mathcal{D}^\partial, \text{sub}_{\mathcal{D}})$ is a σ -algebra (Definition 3.1.4), and the σ -algebra structure is compatible (Definition 4.2.1).

Lemma 4.3.4 is a technical lemma which we use later in Lemma 6.1.9 and Proposition 7.3.5. We mention it now as an example, since it illustrates the extra structure that nominal distributive lattices with $\forall$ have, compared with ‘ordinary’ distributive lattices:

LEMMA 4.3.4. *Suppose $\mathcal{D}$ is a nominal distributive lattice with $\forall$. Suppose $x, y, z \in |\mathcal{D}|$. Then:*

1. *If $a \# z$ and $z \leq x$ then $z \leq \forall a.x$.*
Think: “if $\Gamma \vdash \phi$ and a is not free in Γ then $\Gamma \vdash \forall a.\phi$ ”.
2. *If $b \# z, x$ and $z \leq (b a).x$ then $z \leq \forall a.x$.*
Think: “if $\Gamma \vdash \phi[b/a]$ for some b not free in Γ or ϕ then $\Gamma \vdash \forall a.\phi$ ”.
3. *If $b \# z, y, x$ and $z \leq y \forall ((b a).x)$ then $z \leq y \forall \forall a.x$.*
Think: “if $\Gamma \vdash \phi[b/a], \Psi$ and b is not free in Γ, ϕ , or Ψ then $\Gamma \vdash \forall a.\phi, \Psi$ ”.

Proof. We consider each part in turn:

1. Suppose $a \# z$ and $z \leq x$. By Lemma 4.1.7 $\forall a.z \leq \forall a.x$ and since $a \# z$ we have $\forall a.z = z$.
2. Suppose $b \# z, x$ and $z \leq (b a).x$. By part 1 of this result $z \leq \forall b.(b a).x$. We use Lemma 4.1.3.
3. From part 2 of this result and condition 2 of distributivity (Definition 4.3.1). $\square$

Recall the notions of σ -algebra and termlike σ -algebra from Definitions 3.1.1 and 3.1.4:

DEFINITION 4.3.5. Suppose $\mathcal{X}$ and $\mathcal{X}'$ are σ -algebras. Call a pair of functions $f = (f_{\mathcal{X}}, f_{\mathcal{X}}^\partial)$ where $f_{\mathcal{X}} \in |\mathcal{X}| \rightarrow |\mathcal{X}'|$ and $f_{\mathcal{X}}^\partial \in |\mathcal{X}^\partial| \rightarrow |\mathcal{X}'^\partial|$ a **(σ -algebra) morphism** from $\mathcal{X}$ to $\mathcal{X}'$ when:

1. $f_{\mathcal{X}}^\partial(a_{\mathcal{X}^\partial}) = a_{\mathcal{X}'^\partial}$ (so f maps atoms to atoms).
2. $f_{\mathcal{X}}^\partial(\pi \cdot u) = \pi \cdot f_{\mathcal{X}}^\partial(u)$ and $f_{\mathcal{X}}(\pi \cdot x) = \pi \cdot f_{\mathcal{X}}(x)$ (so f is equivariant).
3. $f_{\mathcal{X}}^\partial(u'[a \mapsto u]) = f_{\mathcal{X}}^\partial(u')[a \mapsto f_{\mathcal{X}}^\partial(u)]$ and $f_{\mathcal{X}}(x[a \mapsto u]) = f_{\mathcal{X}}(x)[a \mapsto f_{\mathcal{X}}^\partial(u)]$ (so f commutes with the σ -action).

If $\mathcal{X}$ is termlike then we insist $\mathcal{X} = \mathcal{X}^\partial$ and we insist that $f_{\mathcal{X}} = f_{\mathcal{X}}^\partial$.

We may omit the subscripts, writing for instance f and f^∂ , or even $f = (f, f^\partial)$, where the meaning is clear.

DEFINITION 4.3.6. Suppose $\mathcal{D}$ and $\mathcal{D}'$ are nominal distributive lattices with $\forall$. Call a morphism $f = (f_{\mathcal{D}}, f_{\mathcal{D}}^\partial) : \mathcal{D} \rightarrow \mathcal{D}'$ of underlying σ -algebras (Definition 4.3.5) a **morphism** of nominal distributive lattices with $\forall$ when f commutes with fresh-finite limits and with finite colimits:

1. $f(\top) = \top$, and $f(x \wedge y) = f(x) \wedge f(y)$ and $f(\forall a.x) = \forall a.f(x)$, and
2. $f(\perp) = \perp$ and $f(x \vee y) = f(x) \vee f(y)$.

Write $\mathbf{nDiV}$ for the category of nominal distributive lattices with $\forall$ and morphisms between them.

4.4. Impredicative nominal distributive lattices

We are interested in modelling the λ -calculus, so we care about lattices where the substitution action is over *itself*. Therefore we introduce *impredicative* nominal distributive lattices with $\forall$: this is Definition 4.4.1.

Recall the notion of termlike σ -algebra from Definition 3.1.1, and the notion of a nominal distributive lattice with $\forall$ from Definition 4.3.3.

DEFINITION 4.4.1. An **impredicative** nominal distributive lattice with $\forall$ is a tuple $(\mathcal{D}, \partial_{\mathcal{D}})$ where:

1. $\mathcal{D} \in \mathbf{nDiV}$ is a nominal distributive lattice with $\forall$ (Definition 4.3.3).
2. $(\partial_{\mathcal{D}}, id) : \mathcal{D}^{\partial} \rightarrow \mathcal{D}$ is a morphism of σ -algebras (Definition 4.3.5).
3. $|\mathcal{D}|$ has cardinality no greater than $size(\mathbb{A})$ (Definition 2.1.1)—in other words, there are no more programs than there are names.

The interested reader can find Definition 4.3.3 extended with further structure in Definition 9.1.1.

REMARK 4.4.2. The dual definition to Definition 4.4.1 is Definition 7.7.1.

So $\mathcal{D}$ is impredicative when the $u \in |\mathcal{D}^{\partial}|$ can be viewed as a subset of the $x \in |\mathcal{D}|$. We use an explicit casting function $\partial_{\mathcal{D}}$ to do this.¹⁵

Thus given $u \in |\mathcal{D}^{\partial}|$, we can obtain $\partial_{\mathcal{D}}u \in |\mathcal{D}|$ and so write (for instance) $(\partial_{\mathcal{D}}u)[a \mapsto u]$. This is not quite λ -calculus self-application, but we are moving in that direction.

We use the size limit (condition 3 of Definition 4.4.1) in Theorem 6.1.23. The intuition for why is that when we come to build prime filters we will need to ‘name’ every element of $\mathcal{D}$ with an atom; so we need to make sure that we will not run out. This is not precisely true, but it captures the spirit of the proof.

More on this in Subsection 12.2.4.

NOTATION 4.4.3. We introduce some notation for Definition 4.4.1:

- We may write $\partial_{\mathcal{D}}$ for $(\partial_{\mathcal{D}}, id)$.
- We may drop subscripts and write ∂u for $\partial_{\mathcal{D}}u$ where $u \in |\mathcal{D}^{\partial}|$.
- We may write ∂a for $\partial_{\mathcal{D}}(a_{\mathcal{D}^{\partial}})$ where $a_{\mathcal{D}^{\partial}}$ is itself shorthand for $\text{atm}_{\mathcal{D}^{\partial}}(a)$ from Definition 3.1.1.¹⁶

¹⁵This costs notation; casting functions always do. Would it be simpler to take $\partial\mathcal{U} \subseteq \mathcal{U}$ as a literal subset inclusion? At this stage it probably would—but when we consider τ -algebras, and then dualities, an explicit casting function gives cleaner results, precisely because our constructions do not need to maintain a literal subset inclusion.

¹⁶Atoms get mapped into $\mathcal{D}^{\partial}$ by $\text{atm}_{\mathcal{D}^{\partial}}$, and $\mathcal{D}^{\partial}$ gets mapped into $\mathcal{D}$ by $\partial_{\mathcal{D}}$... so atoms get mapped into $\mathcal{D}$.

- We may write $\partial\mathcal{D}$ for $\{\partial u \mid u \in |\mathcal{D}^\partial|\} \subseteq |\mathcal{D}|$ and call this set the **programs** of $\mathcal{D}$.

REMARK 4.4.4. It might help to break down the notation a little:

- $\mathbb{A}$ injects into $\mathcal{D}^\partial$ via an injection $\text{atm}_{\mathcal{D}^\partial}$ (this is the equivariant injection specified in Definition 3.1.1).
- $\mathcal{D}^\partial$ maps to $\mathcal{D}$ via $\partial_{\mathcal{D}}$.
- Thus we obtain $\partial a \in \partial\mathcal{D}$ —an atom-as-a-program—living in a sub- σ -algebra of $\mathcal{D}$ which is an image of $\mathcal{D}^\partial$, and which we call the *programs* of $\mathcal{D}$.

Lemma 4.4.5 is a routine sanity check that the definitions match up sensibly. It will be useful later:

LEMMA 4.4.5. *Suppose $\mathcal{D}$ is impredicative and $u \in |\mathcal{D}^\partial|$. Then $(\partial_{\mathcal{D}} a_{\mathcal{D}^\partial})[a \mapsto u] = \partial_{\mathcal{D}} u$.*

Proof. By assumption $\partial_{\mathcal{D}}$ is a morphism of σ -algebras from $\mathcal{D}^\partial$ to $\mathcal{D}$. By Definition 4.3.5 $(\partial_{\mathcal{D}} a_{\mathcal{D}^\partial})[a \mapsto u] = \partial_{\mathcal{D}}(a_{\mathcal{D}^\partial}[a \mapsto u])$ (using the third condition, noting that $u = \text{id}(u)$). The result follows by (σa) from Figure 1 for $\mathcal{D}^\partial$. $\square$

REMARK 4.4.6. Definition 4.4.1 can be looked at in some interesting ways:

- $\mathcal{D}$ is impredicative when it has substitution $x[a \mapsto u]$ over a substructure of itself.
- $\mathcal{D}$ is impredicative when its quantifier $\forall a.x$ quantifies over a sub- σ -structure of $\mathcal{D}$. Thus, $|\partial\mathcal{D}| \subseteq |\mathcal{D}|$ is the set of things we quantify over when we write $\forall a.x$, if $\mathcal{D}$ is impredicative.

REMARK 4.4.7. The programs of $\mathcal{D}$ need not be closed under logical structure like $\wedge$, $\vee$, and $\forall$.

So for instance $x, x' \in \partial\mathcal{D}$ does not imply $x \vee x' \in \partial\mathcal{D}$ and it is not necessarily the case that $\perp \in \partial\mathcal{D}$, and so on. We do not forbid this either.

DEFINITION 4.4.8. Suppose $\mathcal{D}$ and $\mathcal{D}'$ are impredicative nominal distributive lattices with $\forall$.

Call $f = (f_{\mathcal{D}}, f_{\mathcal{D}}^\partial) : \mathcal{D} \rightarrow \mathcal{D}'$ a **morphism** in $\text{inDi}\forall$ when it is a morphism in $\text{nDi}\forall$ (Definition 4.3.6) and when in addition:

3. $f_{\mathcal{D}} \circ \partial_{\mathcal{D}} = \partial_{\mathcal{D}'} \circ f_{\mathcal{D}}^\partial$. That is,

$$f_{\mathcal{D}}(\partial_{\mathcal{D}} u) = \partial_{\mathcal{D}'}(f_{\mathcal{D}}^\partial(u)) \text{ for every } u \in |\mathcal{D}^\partial|.$$

Definition 4.4.9 extends Definition 4.3.6:

DEFINITION 4.4.9. Write $\text{inDi}\forall$ for the category of **impredicative nominal distributive lattices with $\forall$** , and morphisms between them.

As standard write $\mathcal{D} \in \text{inDi}\forall$ for “ $\mathcal{D}$ is an impredicative nominal distributive lattice with $\forall$ ” and $f : \mathcal{D} \rightarrow \mathcal{D}' \in \text{inDi}\forall$ for “ $\mathcal{D}, \mathcal{D}' \in \text{inDi}\forall$ and f is a morphism in $\text{inDi}\forall$ from $\mathcal{D}$ to $\mathcal{D}'$ ”.

REMARK 4.4.10. We continue the notation of Definition 4.4.9 and the discussion of Remark 4.4.4. Suppose $f : \mathcal{D} \rightarrow \mathcal{D}' \in \text{inD}\mathbb{V}$ is a morphism.

Note that $f_{\mathcal{D}}(\partial_{\mathcal{D}} a_{\mathcal{D}\partial}) = \partial_{\mathcal{D}'}(a_{\mathcal{D}'\partial})$. Informally we can say that f maps atoms-as-programs (Remark 4.4.4) in $\mathcal{D}$ to themselves in $\mathcal{D}'$. In symbols we can be even more brief:

$$f(\partial a) = \partial a.$$

We informally trace through how this happens. By condition 1 of Definition 4.3.5 f maps an atom in $\mathcal{D}^{\partial}$ to its incarnation in $\mathcal{D}'^{\partial}$. By condition 3 of Definition 4.4.8 these are mapped to atoms-as-programs in $\mathcal{D}^{\partial}$ and $\mathcal{D}'^{\partial}$ respectively.

5. The σ -powerset as a nominal distributive lattice with $\forall$

We saw in Proposition 3.4.11 how the nominal powerset of an $\mathfrak{o}$ -algebra $\mathcal{P}$ generates a σ -algebra $\text{Pow}_{\sigma}(\mathcal{P})$ (Definition 3.4.6). But powersets are also a lattice under subset inclusion, so perhaps $\text{Pow}_{\sigma}(\mathcal{P})$ has more structure?

In fact, $\text{Pow}_{\sigma}(\mathcal{P})$ is a nominal distributive lattice with $\forall$ (Definition 4.3.3). This is Theorem 5.2.6.

5.1. Basic sets operations

Suppose $\mathcal{P} = (|\mathcal{P}|, \cdot, \mathcal{P}^{\partial}, \text{amgis}_{\mathcal{P}})$ is an $\mathfrak{o}$ -algebra. Recall the nominal powerset $\text{Nom}(\mathcal{P})$ from Subsection 2.5.1.

LEMMA 5.1.1. Suppose $\mathcal{X}, \mathcal{Y} \subseteq |\text{Nom}(\mathcal{P})|$ and $X, Y \subseteq |\mathcal{P}|$. Then:

1. If $\mathcal{X}$ is strictly small-supported (Definition 2.5.2) then

$$\left(\bigcap_{X \in \mathcal{X}} X \right) [a \mapsto u] = \bigcap_{X \in \mathcal{X}} (X [a \mapsto u]).$$

In words: σ commutes with strictly small-supported sets intersections.

Note by Lemma 2.5.5 that this holds in particular if $\mathcal{X}$ is finite.

2. If $\mathcal{X}$ is strictly small-supported then

$$\left(\bigcup_{X \in \mathcal{X}} X \right) [a \mapsto u] = \bigcup_{X \in \mathcal{X}} (X [a \mapsto u]).$$

In words: σ commutes with strictly small-supported sets unions.

Note by Lemma 2.5.5 that this holds in particular if $\mathcal{X}$ is finite.

3. For any $\mathcal{X} \subseteq |\text{Nom}(\mathcal{P})|$,

$$\pi \cdot \bigcap_{X \in \mathcal{X}} X = \bigcap_{X \in \mathcal{X}} \pi \cdot X \quad \text{and} \quad \pi \cdot \bigcup_{X \in \mathcal{X}} X = \bigcup_{X \in \mathcal{X}} \pi \cdot X.$$

In words: intersections and unions are equivariant.

4. If $X \subseteq Y$ then $X [a \mapsto u] \subseteq Y [a \mapsto u]$. In words: σ is monotone (Definition 4.2.1).

Proof. For part 1 we reason as follows:

$$\begin{aligned}
 p \in (\bigcap_{X \in \mathcal{X}} X)[a \mapsto u] &\Leftrightarrow \forall c. p[u \leftarrow c] \in \bigcap_{X \in \mathcal{X}} (c a) \cdot X && \text{Prop 3.4.3, Thm 2.3.2} \\
 &\Leftrightarrow \forall c. \forall X \in \mathcal{X}. p[u \leftarrow c] \in (c a) \cdot X && \text{Fact} \\
 p \in \bigcap_{X \in \mathcal{X}} (X[a \mapsto u]) &\Leftrightarrow \forall X \in \mathcal{X}. p \in X[a \mapsto u] && \text{Fact} \\
 &\Leftrightarrow \forall X \in \mathcal{X}. \forall c. p[u \leftarrow c] \in (c a) \cdot X && \text{Proposition 3.4.3}
 \end{aligned}$$

We note that by Lemma 2.5.3(2), $c\#X$ if and only if $c\#X$ for every $X \in \mathcal{X}$. This allows us to swap the $\forall$ and the $\forall$ quantifiers, and the result follows.

The second and third parts are similar. Part 4 follows from part 1 as in the proof of Lemma 4.2.2. $\square$

Recall the definition of $Pow_\sigma(\mathcal{P})$ from Definition 3.4.6.

COROLLARY 5.1.2. *$Pow_\sigma(\mathcal{P})$ ordered under subset inclusion is a nominal poset with a monotone σ -action.*

Proof. Subset inclusion partially orders $Pow_\sigma(\mathcal{P})$, which is a σ -algebra by Proposition 3.4.11. This action is monotone by Lemma 5.1.1(4). $\square$

LEMMA 5.1.3. • $\emptyset$ and $|\mathcal{P}|$ are in $|Pow_\sigma(\mathcal{P})|$ and these are least and greatest elements in the subset inclusion ordering.

- If X and Y are in $|Pow_\sigma(\mathcal{P})|$ then so are $X \cap Y$ and $X \cup Y$ and these are greatest lower bounds and least upper bounds in the subset inclusion ordering.

Proof. We check the properties listed in Definition 3.4.6 for $X \cap Y$; the case of $X \cup Y$ is similar and the cases of $\emptyset$ and $|\mathcal{P}|$ are even easier. We check that $X \cap Y$ is a greatest lower bound for $\{X, Y\}$ just as for ordinary sets. $X \cap Y$ has small support by Theorem 2.3.2.

1. If a is fresh (so $a\#X, Y, u$) then $(X \cap Y)[a \mapsto u] = X \cap Y$. By Lemmas 5.1.1(1) and 3.4.7 $(X \cap Y)[a \mapsto u] = (X[a \mapsto u]) \cap (Y[a \mapsto u]) = X \cap Y$.
2. If b is fresh (so $b\#X, Y$) then $(X \cap Y)[a \mapsto b] = (b a) \cdot (X \cap Y)$. We reason as follows:

$$\begin{aligned}
 (X \cap Y)[a \mapsto b] &= (X[a \mapsto b]) \cap (Y[a \mapsto b]) && \text{Lemma 5.1.1(1)} \\
 &= ((b a) \cdot X) \cap ((b a) \cdot Y) && \text{Part 2 of Lemma 3.4.7} \\
 &= (b a) \cdot (X \cap Y) && \text{Theorem 2.3.2} \quad \square
 \end{aligned}$$

5.2. Sets quantification

We now explore quantification. This is where we part company from Boolean algebras.

Suppose $\mathcal{P} = (|\mathcal{P}|, \cdot, \mathcal{P}^\partial, \text{amgis}_{\mathcal{P}})$ is an σ -algebra. Recall the definitions of $Nom(\mathcal{P})$ from Subsection 2.5.1 and of $Pow_\sigma(\mathcal{P})$ from Definition 3.4.6.

DEFINITION 5.2.1. If $X \in |Nom(\mathcal{P})|$ then define¹⁷

$$\bigcap^{\#a} X = \bigcap \{X[a \mapsto u] \mid u \in |\mathcal{P}^\partial|\}.$$

¹⁷We do not use the dual $\bigcup^{\#a} X = \bigcup \{X[a \mapsto u] \mid u \in |\mathcal{P}^\partial|\}$ but investigating it would be interesting future work. See also Subsection 12.2.1 and Appendix B.2.

We need the key technical Lemma 5.2.2 for Proposition 5.2.3:

LEMMA 5.2.2. Suppose $X \in \text{Nom}(\mathcal{P})$ and $v \in |\mathcal{P}^\partial|$ and suppose $a \# v$. Suppose $p \in |\mathcal{P}|$. Then

$$\begin{aligned} \mathbb{I}b'. \forall u \in |\mathcal{P}^\partial|. \mathbb{I}a'. p[v \leftarrow b'] [u \leftarrow a'] \in (b' b) \cdot (a' a) \cdot X & \text{ if and only if} \\ \forall u \in |\mathcal{P}^\partial|. \mathbb{I}b'. \mathbb{I}a'. p[u \leftarrow a'] [v \leftarrow b'] \in (b' b) \cdot (a' a) \cdot X. \end{aligned}$$

Proof. We prove two implications:

- *The up-down implication.* Assume

$$\mathbb{I}b'. \forall u \in |\mathcal{P}^\partial|. \mathbb{I}a'. p[v \leftarrow b'] [u \leftarrow a'] \in (b' b) \cdot (a' a) \cdot X.$$

Choose $u \in |\mathcal{P}^\partial|$. Choose fresh b' and a' (so $b', a' \# X, v, u$). Then by assumption (since $b' \# X, v$ and $a' \# X, v, u$) $p[v \leftarrow b'] [u \leftarrow a'] \in (b' b) \cdot (a' a) \cdot X$ so that by $(\tau\sigma)$ of Figure 1 (since $a' \# v$) $p[u[b' \mapsto v] \leftarrow a'] [v \leftarrow b'] \in (b' b) \cdot (a' a) \cdot X$. Now by $(\sigma\#)$ $u[b' \mapsto v] = u$ (since $b' \# u$). Therefore $p[u \leftarrow a'] [v \leftarrow b'] \in (b' b) \cdot (a' a) \cdot X$.

- *The down-up implication.* Assume

$$\forall u \in |\mathcal{P}^\partial|. \mathbb{I}b'. \mathbb{I}a'. p[u \leftarrow a'] [v \leftarrow b'] \in (b' b) \cdot (a' a) \cdot X.$$

Choose fresh b' (so $b' \# X, v$), choose $u \in |\mathcal{P}^\partial|$ (for which b need not necessarily be fresh), and choose fresh a' (so $a' \# X, v, u$). By Lemma 3.2.6 $b' \# u[b' \mapsto v]$ (since $b' \# v$). Therefore $p[u[b' \mapsto v] \leftarrow a'] [v \leftarrow b'] \in (b' b) \cdot (a' a) \cdot X$ and by $(\tau\sigma)$ of Figure 1 (since $a' \# v$) $p[v \leftarrow b'] [u \leftarrow a'] \in (b' b) \cdot (a' a) \cdot X$. $\square$

We cannot use Lemma 5.1.1(1) to derive Proposition 5.2.3 because $\{X[a \mapsto u] \mid u \in |\mathcal{P}^\partial|\}$ is not necessarily strictly small-supported. The result still holds, by a proof using Lemma 5.2.2:

PROPOSITION 5.2.3. Suppose $X \in |\text{Nom}(\mathcal{P})|$ and $v \in |\mathcal{P}^\partial|$ and $a \# v$. Then

$$(\bigcap^{a} X)[b \mapsto v] = \bigcap^{a} (X[b \mapsto v]).$$

(Note by our permutative convention in Definition 2.1.1 that a and b are assumed distinct.)

Proof. Consider $p \in |\mathcal{P}|$. We reason as follows:

$$\begin{aligned} p \in (\bigcap^{a} X)[b \mapsto v] & \\ \Leftrightarrow \mathbb{I}b'. p[v \leftarrow b'] \in \bigcap^{a} (b' b) \cdot X & \text{ Prop 3.4.3, Thm 2.3.2} \\ \Leftrightarrow \mathbb{I}b'. p[v \leftarrow b'] \in \bigcap_{u \in |\mathcal{P}^\partial|} ((b' b) \cdot X)[a \mapsto u] & \text{ Definition 5.2.1} \\ \Leftrightarrow \mathbb{I}b'. \forall u \in |\mathcal{P}^\partial|. \mathbb{I}a'. p[v \leftarrow b'] [u \leftarrow a'] \in (b' b) \cdot (a' a) \cdot X & \text{ Proposition 3.4.3} \\ \Leftrightarrow \forall u \in |\mathcal{P}^\partial|. \mathbb{I}b'. \mathbb{I}a'. p[u \leftarrow a'] [v \leftarrow b'] \in (b' b) \cdot (a' a) \cdot X & \text{ Lemma 5.2.2} \\ \Leftrightarrow \forall u \in |\mathcal{P}^\partial|. \mathbb{I}a'. p[u \leftarrow a'] \in ((a' a) \cdot X)[b \mapsto v] & \text{ Proposition 3.4.3} \\ \Leftrightarrow \forall u \in |\mathcal{P}^\partial|. \mathbb{I}a'. p[u \leftarrow a'] \in (a' a) \cdot (X[b \mapsto v]) & \text{ T2.3.2, C2.3.5, } a', a \# v \\ \Leftrightarrow \forall u \in |\mathcal{P}^\partial|. p \in X[b \mapsto v][a \mapsto u] & \text{ Proposition 3.4.3} \\ \Leftrightarrow p \in \bigcap^{a} (X[b \mapsto v]) & \text{ Definition 5.2.1} \end{aligned}$$

$\square$

LEMMA 5.2.4. Suppose $X \in |\text{Nom}(\mathcal{P})|$. Then

$$b\#X \text{ implies } \bigcap^{\#a} X = \bigcap^{\#b}(b\ a) \cdot X.$$

As a corollary, $a\#\bigcap^{\#a} X$ and $\text{supp}(\bigcap^{\#a} X) \subseteq \text{supp}(X) \setminus \{a\}$.

Proof. The corollary follows by Corollary 2.3.5(3) and by Theorem 2.3.2. For the first part, we reason as follows:

$$\begin{aligned} \bigcap^{\#a} X &= \bigcap \{X[a \mapsto u] \mid u \in |\mathcal{P}^\partial|\} && \text{Definition 5.2.1} \\ &= \bigcap \{((b\ a) \cdot X)[b \mapsto u] \mid u \in |\mathcal{P}^\partial|\} && \text{Lemma 3.4.4} \\ &= \bigcap^{\#b}(b\ a) \cdot X && \text{Definition 5.2.1} \end{aligned} \quad \square$$

THEOREM 5.2.5. If $X \in |\text{Pow}_\sigma(\mathcal{P})|$ then

1. $\bigcap^{\#a} X \in |\text{Pow}_\sigma(\mathcal{P})|$, and as a corollary
2. $\bigcap^{\#a} X$ is equal to $\forall a.X$ (the a -fresh limit of X) in $\text{Pow}_\sigma(\mathcal{P})$ considered as a nominal poset with a monotone σ -action.

Proof. The corollary is from Corollary 5.1.2 and Proposition 4.2.5(2). We now prove part 1 of this Theorem.

Small support is from Theorem 2.3.2. It remains to check conditions 1 and 2 of Definition 3.4.6:

1. Suppose b is fresh (so $b\#X$) and suppose $v \in |\mathcal{P}^\partial|$. Using Lemma 5.2.4 suppose without loss of generality that $a\#v$. Then we reason as follows:

$$\begin{aligned} (\bigcap^{\#a} X)[b \mapsto v] &= \bigcap^{\#a}(X[b \mapsto v]) && \text{Proposition 5.2.3} \\ &= \bigcap^{\#a} X && \text{C1 of Def 3.4.6, } b\#X \end{aligned}$$

2. Suppose b' is fresh (so $b'\#X$). Then we reason as follows:

$$\begin{aligned} (\bigcap^{\#a} X)[b' \mapsto b'] &= \bigcap^{\#a}(X[b' \mapsto b']) && \text{Proposition 5.2.3} \\ &= \bigcap^{\#a}((b'\ b) \cdot X) && \text{C2 of Def 3.4.6, } b'\#X \\ &= (b'\ b) \cdot (\bigcap^{\#a} X) && \text{Theorem 2.3.2} \end{aligned} \quad \square$$

THEOREM 5.2.6. Suppose $\mathcal{P}$ is an $\mathfrak{v}$ -algebra. Then the σ -algebra $\text{Pow}_\sigma(\mathcal{P})$ from Definition 3.4.6 naturally becomes a nominal distributive lattice with $\forall$ where $\top$, $\wedge$, $\perp$, $\vee$, and $\forall$ are interpreted as $|\mathcal{P}|$, set intersection $\cap$, the empty set $\emptyset$, set union $\cup$, and $\bigcap^{\#a}$.

Proof. By Lemma 5.1.3 for most of the connectives, and by Theorem 5.2.5 for $\forall$. $\square$

This completes Part I. So far, we have defined nominal distributive lattices with $\forall$ and seen how to build them using nominal powersets. In Part II we show how to go from topologies (i.e. subsets of powersets subject to various sanity conditions, since powersets are usually very large) back to nominal distributive lattices with $\forall$.

Part II

Nominal spectral space representation

6. Completeness

The key definition of this section is that of *filter* in Definition 6.1.2. The main result is a representation result, Theorem 6.4.4, which represents an impredicative nominal distributive lattice with $\forall$ as a set of sets of prime filters.

The key technical results are in the sequence Proposition 6.1.13, Proposition 6.1.19, and Theorem 6.1.23, which use Zorn-style arguments to exhibit every filter as a subset of some prime filter.

This story is familiar: it is standard to represent a lattice using sets of prime filters. However, making this work is not trivial. It is interesting to highlight three reasons for this:

1. We must account for $\forall$, of course.
This is extra structure and our treatment uses properties of nominal sets and the $\forall$ -quantifier in subtle ways. See the discussion opening Subsection 6.1.
2. Zorn's Lemma is related to the Axiom of Choice, which can cause difficulties with nominal sets because it may lead to non-small support (our definition of a nominal set from Definition 2.1.6 requires small support).
We will find ourselves using elements which do have a permutation action—we are still within nominal techniques—but the elements do not necessarily have small support. This is unusual. See Remark 6.1.6.
3. Once these difficulties are navigated, we must still give points (prime filters) an $\neg$ -algebra structure.
There is no reason to expect prime filters to behave well and support an $\neg$ -algebra structure. 'By magic', it just works: see Lemma 6.2.1.

A final technical hurdle is generated by our intended application of giving semantics to the untyped λ -calculus. In effect, this means that we want to consider lattices with a substitution action over themselves, in a suitable sense, whence the notion of *impredicativity* developed in Subsection 4.4. As usual for impredicative definitions, care is needed. Yet, once these definitions and results are in place, the main result Theorem 6.4.4 becomes quite natural.

6.1. Filters and prime filters

For this subsection, fix $\mathcal{D} \in \mathbf{nDi}\forall$ a nominal distributive lattice with $\forall$ (Definition 4.3.3). Recall from Definition 4.3.3 that $\mathcal{D}$ is a set with a small-supported permutation action, a σ -action (like a substitution action but abstractly specified as a nominal algebra), finite joins, fresh-finite meets, and satisfying a generalisation of the usual distributivity properties for lattices.

We start by defining our notion of (prime) filter, and proving that every filter is included in some prime filter.

The main definition is Definition 6.1.2 and the main result is Theorem 6.1.23. The main technical result is Proposition 6.1.13.

REMARK 6.1.1 (Discussion of condition 4). Condition 4 of Definition 6.1.2 is specific to the *nominal* filters. See also its verification in Proposition 6.1.13, in which $\forall$ is decomposed into $\mathbb{V}$ and the permutation action π —echoing Proposition 4.2.6 and Proposition 6.1.10. This decomposition of $\forall$ into $\mathbb{V}$ and π is important for two reasons:

- it converts an infinite conjunction over the entire domain into a $\mathbb{V}$ -quantified assertion—the $\mathbb{V}$ -quantifier has some excellent properties, such as commuting with conjunction *and* disjunction—and
- it does not depend on $\mathcal{D}^\partial$.

So condition 4 of Definition 6.1.2 means that to check the universal quantifier $\forall a.x$ we do not need to know about all of the programs of $\mathcal{D}$. We just need to know about the atoms, and in particular, we just need to know about the fresh atoms.

This is familiar from proof-theory. To prove $\Gamma \vdash \forall x.\phi$ we do not need to check $\phi[a \mapsto t]$ for every term t ; we just check $\phi[x \mapsto y]$ for fresh y .

More on this in Remark 6.1.6.

6.1.1. Filters

DEFINITION 6.1.2. A **filter** in $\mathcal{D}$ is a nonempty subset $p \subseteq |\mathcal{D}|$ (which need not have small support) such that:

1. $\perp \notin p$ (we say p is **consistent**).
2. If $x \in p$ and $x \leq x'$ then $x' \in p$ (we call p **up-closed**).
3. If $x \in p$ and $x' \in p$ then $x \wedge x' \in p$.
4. If $\mathbb{V}b.(b\ a) \cdot x \in p$ then $\forall a.x \in p$.

The notion of *prime filter* is in Definition 6.1.16, and has no further surprises.

Ideals are dual to filters; Definition 6.1.3 is standard:

DEFINITION 6.1.3. An **ideal** in $\mathcal{D}$ is a nonempty subset $Z \subseteq |\mathcal{D}|$ (which need not have small support) such that:

1. $\top \notin Z$.
2. If $x \in Z$ and $x' \leq x$ then $x' \in Z$ (we call Z **down-closed**).
3. If $x \in Z$ and $x' \in Z$ then $x \vee x' \in Z$.

REMARK 6.1.4. Definition 6.1.3 is not a perfect dual to Definition 6.1.2: we do not have $\forall$. (Correspondingly, we assume that a universal quantifier exists in $\mathcal{D}$, but not an existential.) This will not be a problem.¹⁸ Indeed, the lack of a fourth condition in Definition 6.1.3 will be convenient in Lemma 6.1.21.

REMARK 6.1.5. Condition 4 of Definition 6.1.2 seems odd in view of, say, Proposition 4.2.5 or Definition 5.2.1. Should it not be

$$\forall u \in \mathcal{D}^\partial. x[a \mapsto u] \in p \text{ implies } \forall a.x \in p?$$

¹⁸We need $\wedge$ and $\vee$ to build filters. Later on when we model the untyped λ -calculus in Subsection 10.2, we will need $\forall$, $\neg$, and $\bullet$. We will not need an existential $\exists$. The existential may still exist; see Appendix B.2.

Or, in view of Proposition 4.2.6 should it not be at least

$$\forall n \in \mathbb{A}. x[a \mapsto n_{\mathcal{D}^\partial}] \in p \text{ implies } \forall a. x \in p?$$

In fact, we shall see in Proposition 6.1.10 that condition 4 as written, implies both of these.

REMARK 6.1.6. We continue Remark 6.1.1. We do not assume that p has small support, so the b bound by the $\mathbb{V}$ -quantifier in condition 4 of Definition 6.1.2 need not necessarily be fresh for p . So the reader familiar with nominal techniques should note that our use of $\mathbb{V}$ is atypical. The ‘standard’ decomposition of $\mathbb{V}$ into ‘ $\forall$ +freshness’ and ‘ $\exists$ +freshness’ familiar from e.g. Theorem 2.17 of [Gab14b], Theorem 6.5 of [Gab11b], or Theorem 9.4.6 of [Gab01] will not necessarily work unless p has small support, which in general is not the case. Nevertheless, we have enough structure to obtain the results we need.

What is the case, is that $x \in |\mathcal{D}|$ is assumed to have small support, and b will be fresh for x .

It will be important for the proof of Theorem 6.1.23 that we allow p to have non-small support.

REMARK 6.1.7. Condition 4 of Definition 6.1.2 seems not only elegant but necessary:

1. The design choice for condition 4 of Definition 6.1.2 to assume $\forall n \in \mathbb{A}. x[a \mapsto n_{\mathcal{D}^\partial}] \in p$ causes failure in case 4 of Lemma 6.2.1 where (in the notation of that case) $n=a$.
2. The design choice $\forall u \in \mathcal{D}^\partial. x[a \mapsto u] \in p$ causes failure in the final part of Proposition 8.2.2 (morphisms commute with quantification; see also Remark 8.2.3).

We will see a dual version of the condition in condition 2 of Definition 7.9.1. See also Remark 7.9.2 and Lemma 11.8.13.

Definition 6.1.8 and Lemma 6.1.9 give examples of filters and ideals. The definitions and proofs are standard, except we must verify condition 4 of Definition 6.1.2 in Lemma 6.1.9:

DEFINITION 6.1.8. If $x \in |\mathcal{D}|$ then define $x \uparrow$ and $x \downarrow$ by

$$x \uparrow = \{y \mid x \leq y\} \quad \text{and} \quad x \downarrow = \{y \mid y \leq x\}.$$

LEMMA 6.1.9. • If $x \neq \perp$ then $x \uparrow$ from Definition 6.1.8 is a small-supported filter.

• If $x \neq \top$ then $x \downarrow$ is a small-supported ideal.

Proof. It is routine to verify conditions 1 to 3 of Definition 6.1.2. Suppose $x \leq (b \ a) \cdot y$ for cosmall many b . We take one particular $b \# x, y$ and use part 2 of Lemma 4.3.4. Small support is direct from Theorem 2.3.2. The case of $x \downarrow$ is no harder. $\square$

PROPOSITION 6.1.10. Suppose p is a filter in $\mathcal{D}$ and a is an atom. Then:

1. The following conditions are equivalent (below, n ranges over all atoms, including a):

$$\begin{aligned} \forall a. x \in p &\Leftrightarrow \forall u \in |\mathcal{D}^\partial|. x[a \mapsto u] \in p \\ &\Leftrightarrow \forall n \in \mathbb{A}. x[a \mapsto n] \in p \\ &\Leftrightarrow \mathbb{V}b. (b \ a) \cdot x \in p \end{aligned}$$

2. If furthermore p is small-supported and $a \# p$ then the following conditions are equivalent:

$$\forall a. \phi \in p \Leftrightarrow \phi \in p$$

Proof. If $\forall a. x \in p$ then by Lemma 4.2.3 and condition 2 of Definition 6.1.2 also $x[a \mapsto u] \in p$ for every $u \in |\mathcal{D}^\partial|$. It follows in particular that $x[a \mapsto n_{\mathcal{D}^\partial}] \in p$ for every $n \in \mathbb{A}$, so $x[a \mapsto b] \in p$ for all $b \# x$ so by Lemma 3.2.4 also $\mathcal{U}b.(b a) \cdot x \in p$. We complete the cycle of implications using condition 4 of Definition 6.1.2.

Part 2 follows from part 1 using Theorem 2.3.9. $\square$

REMARK 6.1.11. Recall $\bigcap^{\#a}$ from Definition 5.2.1. If we are willing to borrow the notation $x^\bullet = \{p \mid x \in p\}$ from the future Definition 6.3.1, then we can rewrite Proposition 6.1.10(1) as follows:

$$(\forall a. x)^\bullet = \bigcap^{\#a} x^\bullet = \bigcap_{n \in \mathbb{A}} x^\bullet[a \mapsto n] = \{p \in x^\bullet \mid \mathcal{U}b.(b a) \cdot p \in x^\bullet\}.$$

6.1.2. Growing filters

We consider one useful way to build new (larger) filters out of old filters:

DEFINITION 6.1.12. Suppose $p \subseteq |\mathcal{D}|$ and $y \in |\mathcal{D}|$. Then define $p+y \subseteq |\mathcal{D}|$ by

$$p+y = \{x' \mid \exists x \in p. x \wedge y \leq x'\}.$$

PROPOSITION 6.1.13. Suppose p is a small-supported filter in $\mathcal{D}$ and suppose $y \in |\mathcal{D}|$. Then:

- $p \subseteq p+y$.
- $y \in p+y$.
- $p+y$ is closed under conditions 2 to 4 of Definition 6.1.2 (so if $p+y$ is consistent then it is a filter).

As a corollary, if Z is an ideal and $(p+y) \cap Z = \emptyset$ then $p+y$ is a filter.

Proof. The corollary follows from the body of this result because since from condition 2 of Definition 6.1.3 $\perp \in Z$, so $\perp \notin p+y$.

We now consider the body of this result. It is clear from the construction that $p \subseteq p+y$ and $y \in p+y$. We now check that $p+y$ satisfies conditions 2 to 4 of Definition 6.1.2:

2. If $z \in p+y$ and $z \leq z'$ then $z' \in p+y$. By construction.
3. If $z \in p+y$ and $z' \in p+y$ then $z \wedge z' \in p+y$. Suppose $z \geq x \wedge y$ and $z' \geq x' \wedge y$ for $x, x' \in p$. Then by condition 3 of Definition 6.1.2 $x \wedge x' \in p$, and it is a fact that $z \wedge z' \geq (x \wedge x') \wedge y$.
4. If $\mathcal{U}b.(b a) \cdot z \in p+y$ then $\forall a. z \in p+y$. Suppose for cosmall many b there exists an $x_b \in p$ such that $(b a) \cdot z \geq x_b \wedge y$. Then certainly there exists some b such that $b \# y, z, p$ and $(b a) \cdot z \geq x_b \wedge y$. We apply $\forall b$ to both sides and use distributivity (Definition 4.3.1), and we conclude that

$$\forall b. (b a) \cdot z \geq (\forall b. x_b) \wedge y.$$

We assumed p has small support so by Proposition 6.1.10(2) (since $x_b \in p$ and $b \# p$) $\forall b. x_b \in p$. By Lemma 4.1.3 and condition 2 of Definition 6.1.2 we conclude that $\forall a. z \in p+y$ as required. $\square$

6.1.3. Growing ideals

We consider one useful way to build new (larger) ideals out of old ideals:

DEFINITION 6.1.14. Suppose $Z, Y \subseteq |\mathcal{D}|$. Then define $Z+Y \subseteq |\mathcal{D}|$ by

$$Z+Y = \{z' \mid \exists z \in Z, n \in \mathbb{N}, y_1, \dots, y_n \in Y. z' \leq z \vee y_1 \vee \dots \vee y_n\}.$$

Lemma 6.1.15 is a version of Proposition 6.1.13 for ideals. It is the simpler result, because Definition 6.1.3 has nothing corresponding to condition 2 of Definition 6.1.2:

LEMMA 6.1.15. Suppose $Z \subseteq |\mathcal{D}|$ is an ideal and $Y \subseteq |\mathcal{D}|$. Then:

- $Z \subseteq Z+Y$ and $Y \subseteq Z+Y$.
- $Z+Y$ is closed under conditions 2 and 3 of Definition 6.1.3 (so that if $\top \notin Z+Y$ then it is an ideal).

Proof. By routine calculations. $\square$

6.1.4. Maximal and prime filters

The results of this subsection are Lemma 6.1.18 and Proposition 6.1.19. From a great distance they follow a familiar pattern:

- Lemma 6.1.18 expresses “if a filter is not maximal, then we can extend it”.
- Proposition 6.1.19 expresses “if a filter is maximal, then it is prime”.

But in Lemma 6.1.18 we work not with filters but with filter-ideal pairs, and we express something quite subtle:

“If a filter-ideal pair is not maximal but is small-supported, then we can extend *either* the filter with a universal quantification $\forall a. y$, *or* we can extend the ideal with an equivalence class—really an α -equivalence class—of $(b \ a) \cdot y$ for fresh atoms b .”

To see why this is so, see Remark 6.1.22 and Theorem 6.1.23.

DEFINITION 6.1.16. • Call a filter $p \subseteq |\mathcal{D}|$ **prime** when $x_1 \vee x_2 \in p$ implies either $x_1 \in p$ or $x_2 \in p$.

• Suppose p is a filter and $Z \subseteq |\mathcal{D}|$ is an ideal. Call p **maximal with respect to Z** when $p \cap Z = \emptyset$ and for every filter p' with $p' \cap Z = \emptyset$, if $p \subseteq p'$ then $p = p'$.

• Call p **maximal** when it is maximal with respect to the ideal $\{\perp\}$.

LEMMA 6.1.17. Suppose p is a filter in $\mathcal{D}$ (Definition 6.1.2). Then:

1. $\perp \notin p$ and $\top \in p$.

2. $x \wedge y \in p$ if and only if $x \in p$ and $y \in p$.
3. $\forall a. x \in p$ if and only if $\forall b. (b \ a) \cdot x \in p$.¹⁹
4. If p is prime (Definition 6.1.16) then $x \vee y \in p$ if and only if $x \in p$ or $y \in p$.

Proof. 1. The first part is condition 1 of Definition 6.1.2. For the second part, by assumption in Definition 6.1.2 p is nonempty, so there exists some $x \in p$. Now $\mathcal{D}$ has a top element $\top$, so $x \leq \top$ and by condition 2 of Definition 6.1.2 $\top \in p$.

2. From conditions 2 and 3 of Definition 6.1.2.

3. This is Proposition 6.1.10(1).

4. Routine, again using condition 2 of Definition 6.1.2. $\square$

We use Lemma 6.1.18 and Proposition 6.1.19 to prove Theorem 6.1.23. We are most interested in Lemma 6.1.18 for the case that Z is small-supported (as well as p), but the proof does not depend on it.

LEMMA 6.1.18. *Suppose*

- $p \subseteq |\mathcal{D}|$ is a small-supported filter and $Z \subseteq |\mathcal{D}|$ is an ideal, and suppose
- $\perp \in p + \forall a. y$ and $p \cap Z = \emptyset$.

Write $Y = \{(b \ a) \cdot y \mid b \in \mathbb{A} \setminus (\text{supp}(p) \cup \text{supp}(y) \cup \{a\})\}$. Then:

1. $p \cap (Z + Y) = \emptyset$.
2. As a corollary, $Z + Y$ is an ideal (and by part 1 is disjoint from p).

Proof. Suppose $\perp \in p + \forall a. y$ and $p \cap Z = \emptyset$ and $p \cap (Z + Y) \neq \emptyset$. We note the following:

- Since $p \cap (Z + Y) \neq \emptyset$, there exist $b_1, \dots, b_n \# p, y$ and $z \in Z$ with
$$z \vee (b_1 \ a) \cdot y \vee \dots \vee (b_n \ a) \cdot y \in p.$$
- Since $\perp \in p + \forall a. y$ there exists $x \in p$ with $x \wedge \forall a. y = \perp$.
We assumed p has small support so by Proposition 6.1.10(2) $\forall b_1 \dots \forall b_n. x \in p$, and we see that we may assume without loss of generality that $b_1, \dots, b_n \# x$.
- Since $p \cap Z = \emptyset$ we have $\forall x' \in p, z' \in Z. x' \wedge z' = \perp$.

By condition 3 of Definition 6.1.2

$$x \wedge (z \vee (b_1 \ a) \cdot y \vee \dots \vee (b_n \ a) \cdot y) \in p$$

and therefore by distributivity (Definition 4.3.1)

$$(x \wedge z) \vee (x \wedge (b_1 \ a) \cdot y) \vee \dots \vee (x \wedge (b_n \ a) \cdot y) \stackrel{x \wedge z = \perp}{=} (x \wedge (b_1 \ a) \cdot y) \vee \dots \vee (x \wedge (b_n \ a) \cdot y) \in p.$$

¹⁹We assume that $x \in |\mathcal{D}|$ is small-supported because in Definition 4.1.1 we assume $(|\mathcal{D}|, \cdot)$ is a nominal set. We do not assume that a filter p is small-supported in Definition 6.1.2.

This means that we may assume that b is fresh for x under the $\forall b$ quantifier, but we do not know that b is fresh for p . This will not be a problem.

By assumption $b_1, \dots, b_n \# p$ so by Proposition 6.1.10(2) we deduce that

$$\forall b_1 \dots b_n. ((x \wedge (b_1 a) \cdot y) \vee \dots \vee (x \wedge (b_n a) \cdot y)) \in p.$$

Recall that by assumption $b_1, \dots, b_n \# x, y$; by calculations using Proposition 2.3.4, (distrib $\forall$) from Definition 4.3.1, and Lemma 4.1.3 we conclude that

$$x \wedge \forall a. y = \perp \in p.$$

This contradicts condition 1 of Definition 6.1.2.

For the corollary, we note by Lemma 6.1.17(1) that $\top \in p$ so that $\top \notin Z + Y$. We use Lemma 6.1.15. $\square$

PROPOSITION 6.1.19. *Suppose $p \subseteq |\mathcal{D}|$ is a filter and $Z \subseteq |\mathcal{D}|$ is an ideal, and suppose $p \cap Z = \emptyset$. If p is a maximal filter with respect to Z then it is prime.*

Proof. Suppose $y_1 \vee y_2 \in p$ and $y_1, y_2 \notin p$. By Proposition 6.1.13 and maximality we have that $(p + y_1) \cap Z \neq \emptyset$ and $(p + y_2) \cap Z \neq \emptyset$. It follows that there exist $x_1, x_2 \in p$ with $x_1 \wedge y_1, x_2 \wedge y_2 \in Z$. Since Z is an ideal, by condition 3 of Definition 6.1.3

$$(x_1 \wedge y_1) \vee (x_2 \wedge y_2) \in Z.$$

Now we rearrange the left-hand-side to deduce that

$$u = (x_1 \vee x_2) \wedge (x_1 \vee y_2) \wedge (y_1 \vee x_2) \wedge (y_1 \vee y_2) \in Z. \quad (1)$$

We now note that $x_1 \vee x_2 \in p$ (since $x_1 \in p$, and indeed also $x_2 \in p$) and $x_1 \vee y_2 \in p$ (since $x_1 \in p$) and $y_1 \vee x_2 \in p$ (since $x_2 \in p$) and $y_1 \vee y_2 \in p$ by assumption. But then $u \in p$, contradicting our assumption that $p \cap Z = \emptyset$. $\square$

REMARK 6.1.20. Experts who have read [GLP11] might wish to compare Proposition 6.1.19 with Lemma 6.13 of [GLP11]. We cannot limit the support of p by wrapping y_1 and y_2 in universal quantifiers—like we did for the duality result for the $\mathbb{I}$ -quantifier in Lemma 6.13 and indirectly in Definition 6.5 of [GLP11]—because $\forall$ does not distribute over $\vee$, whereas $\mathbb{I}$ does.

This property of $\mathbb{I}$ is one precise technical reason that the proofs of [GLP11] can be simpler than the proofs here.

6.1.5. The Zorn argument

We start off with an easy technical result:

LEMMA 6.1.21. *If $(Z_\alpha)_{\alpha \leq \beta}$ is an ascending chain of ideals (for some ordinal β) then $\bigcup_{\alpha \leq \beta} Z_\alpha$ is an ideal.*

Proof. By standard calculations on the three conditions of Definition 6.1.3. $\square$

REMARK 6.1.22. The reader might now expect us to prove a version of Lemma 6.1.21 for filters, and using Lemma 6.1.18 and Proposition 6.1.19 along with Zorn's Lemma [End77, page 153] deduce the existence of prime filters.

This does not work because condition 4 of Definition 6.1.2 is not closed under ascending chains of filters.²⁰ We might ‘accidentally’ insert elements $(b\ a) \cdot y$ for cosmall many b yet ‘forget’ to insert $\forall a.y$ (or worse, $\forall a.y$ might be in Z). Thus, we cannot use Zorn’s lemma directly because the sets union of a countably ascending chain of filters is not necessarily a filter.

However, we do not really need closure under all chains; we only need the existence of some chain. So we build a ‘bespoke’ chain—unusually, this is not just a chain of filters but a chain of filter-ideal pairs (p, Z) —in such a way as to preserve condition 4 of Definition 6.1.2. Intuitively, Lemma 6.1.18 has the form that it does because we grow the pair (p, Z) at each stage such that

- either we put $\forall a.y$ into p ,
- or we make sure that $\forall b.(b\ a) \cdot y \in p$ can never hold—not even after infinitely many stages—by putting $(b\ a) \cdot y$ into the ideal Z immediately for most b ; recall that p must remain disjoint from Z .

THEOREM 6.1.23. *Suppose $p \subseteq |\mathcal{D}|$ is a small-supported filter and $Z \subseteq |\mathcal{D}|$ is an ideal and suppose $p \cap Z = \emptyset$. Then there exists a prime filter q with $p \subseteq q$ and $q \cap Z = \emptyset$. As a corollary, if p is a filter then there exists a prime filter q containing p .*

Proof. The corollary follows taking Z to be $\{\perp\}$, which we can easily verify is an ideal. We now consider the main result.

Let α be the least ordinal of cardinality $\text{size}(\mathbb{A})$ from Definition 2.1.1. Note that $\mathbb{A}$ is assumed at least countably large, so $\alpha \times \alpha$ has the same cardinality as α . Note also that α is necessarily a limit ordinal, because it is the least ordinal with cardinality $\text{size}(\mathbb{A})$.²¹

We enumerate $\mathbb{A} \times \mathcal{D}$ as a list of pairs

$$(a_{i+1}, x_{i+1})_{i < \alpha}. \quad (2)$$

We write $(a_{i+1}, x_{i+1})_{i < \alpha}$ instead of $(a_i, x_i)_{i < \alpha}$ to ensure that every pair (a, x) is indexed by a successor ordinal and not a limit ordinal (so for instance we have (a_1, x_1) but not (a_0, x_0)). This is just for convenience in the base and limit cases of the inductive construction which now follows.

Recall the notations ‘ $p+y$ ’ from Definition 6.1.12 and ‘ $Z+Y$ ’ from Definition 6.1.14. We define a sequence of (by Theorem 2.3.2) small-supported disjoint filter-ideal pairs $(p_i, Z_i)_{i \leq \alpha}$ inductively as follows:

1. *Base case.* We take $(p_0, Z_0) = (p, Z)$. By assumption $p \cap Z = \emptyset$.
2. *Successor ordinal (first case).* If $p_i + \forall a_{i+1}.x_{i+1} \cap Z_i = \emptyset$ then take

$$(p_{i+1}, Z_{i+1}) = (p_i + \forall a_{i+1}.x_{i+1}, Z_i).$$

²⁰Many thanks to an anonymous referee for pointing this out.

²¹There are two natural foundational definitions of ‘cardinal’: (1) A cardinal is an ordinal that cannot be bijected with any lesser ordinal; or (2) A cardinal is an equivalence class of bijectable sets. It is routine to biject an infinite ordinal α' with $\alpha' + 1$, so in either (1) or (2) we see that the least ordinal with cardinality $\text{size}(\mathbb{A})$ must be a limit. Furthermore, in case (1) $\text{size}(\mathbb{A})$ is that limit; in case (2) $\alpha \in \text{size}(\mathbb{A})$ is the least ordinal in $\text{size}(\mathbb{A})$.

It follows from Proposition 6.1.13 (since p_i is small-supported) that p_{i+1} is a filter and from Theorem 2.3.2 that p_{i+1} is small-supported.

3. *Successor ordinal (second case).* If $p_i + \forall a_{i+1}.x_{i+1} \cap Z_i \neq \emptyset$ then take

$$Y = \{(b a_{i+1}).x_{i+1} \mid b \in \mathbb{A} \setminus (\text{supp}(p_i) \cup \text{supp}(x_{i+1}) \cup \{a_{i+1}\})\} \quad \text{and} \\ (p_{i+1}, Z_{i+1}) = (p_i, Z_i + Y).$$

It follows from Lemma 6.1.18 that Z_{i+1} is an ideal and $p_{i+1} \cap Z_{i+1} = \emptyset$.

4. *Nonzero limit ordinal.* If i is a nonzero limit ordinal no greater than α then we define

$$(p_i, Z_i) = \left(\bigcup_{j < i} p_j, \bigcup_{j < i} Z_j \right).$$

By construction $p_i \cap Z_i = \emptyset$. By Lemma 6.1.21 Z_i is an ideal.

We note that p_i is a filter: Conditions 1 to 3 of Definition 6.1.2 are routine. To check condition 4, suppose $\forall b.(b a).x \in p_i$ meaning that $(b a).x \in p_i$ for cosmall many b ; we need to prove $\forall a.x \in p_i$.

So let $j < i$ be that index, which by construction in equation 2 above must be a successor ordinal, such that $(a, x) = (a_j, x_j)$. At stage j , when we built (p_j, Z_j) , there were two possibilities:

- If $p_{j-1} + \forall a.x \cap Z = \emptyset$ then we must have put $\forall a.x$ into p_j , so that $\forall a.x \in \bigcup_{j < i} p_j$ and we are done.
- If $p_{j-1} + \forall a.x \cap Z \neq \emptyset$ then we must have put $(b a).x$ into Z_j for cosmall many b , so that $\forall b.(b a).x \in \bigcup_{j < i} Z_j$. But this is impossible because we assumed that cosmall many $(b a).x$ were in p_i , which is disjoint from Z_i .

If $i \prec \alpha$ then it follows using Theorem 2.3.2(3) that p_i is small-supported.²²

We note that p_α is a maximal filter disjoint from the ideal Z_α : For consider any x and choose some fresh a (so $a \# x$). Let j be that index in the enumeration above such that $(a, x) = (a_j, x_j)$. Note by Definition 4.1.2 that $\forall a.x = x$, since $a \# x$. It follows from the structure of the algorithm above that precisely one of $\forall a.x = x \in p_j$ or $x \in Z_j$ will hold. Maximality follows.

By Proposition 6.1.19 p_α is prime, and by construction $p \subseteq p_\alpha$ and $p_\alpha \cap Z = \emptyset$. $\square$

REMARK 6.1.24. It might be helpful to trace how Theorem 6.1.23 will be used in the rest of this paper:

- we use it in Lemma 6.3.2 to prove Corollary 6.3.3 (injectivity of $x \mapsto x^*$);
- we use it in Lemma 6.3.5;
- we use it in Propositions 7.3.4 and 7.3.5; and

²²If $i = \alpha$ then p_α need not have small support, but this will not be a problem because if $i = \alpha$ then we are finished with the induction.

- we use it in Lemmas 9.3.5 and 9.3.6, which are needed for Theorem 9.4.7 (compositionality of $x \mapsto x^\bullet$ with respect to $\bullet$ and $\neg\bullet$).

REMARK 6.1.25. Theorem 6.1.23 assumes a filter p and an ideal Z . This is more asymmetric than it might seem:

1. The notion of filter in Definitions 6.1.2 has a condition for $\forall$ (condition 4) whereas the notion of ideal in Definition 6.1.3 does not. This is important; it gives us a closure property Lemma 6.1.21 for ideals, which we use to prove Theorem 6.1.23. As noted in Remark 6.1.22 we cannot expect a corresponding closure property for filters; perhaps we should call the ideals we need for this paper *proposition-like*, because ideals only interact with propositional structure, and the filters *predicate-like*, because filters must also interact well with the predicate quantifier $\forall$.
2. We assume p is small-supported and we do not assume Z is small-supported. We use small support of p in the proof of Theorem 6.1.23 and, later, we will require the full generality of allowing non-small-supported Z to prove Lemmas 9.3.5 and 9.3.6, which handle applicative structure $\bullet$. See Remark 9.3.7.

Asymmetry 2 is related to asymmetry 1: we require small support for p because this helps handle its interaction with $\forall$; ideals are ‘proposition-like’, so for ideals small support does not matter.

REMARK 6.1.26. Theorem 6.1.23 is set up to work for $\mathbb{A}$ and $\mathcal{D}$ of equal cardinality—as per condition 3 of Definition 4.4.1 (see also condition 3 of Definition 7.7.1). We need the cardinalities to be equal so that we do not run out of fresh atoms in the construction of the chain of small-supported filters in the proof of Theorem 6.1.23.

Note that the canonical model (see Example 10.3.2 and subsequent proofs) is countable by construction and the soundness proof in Theorem 10.4.7 holds for any size $\mathcal{D}$ regardless of the size of $\mathbb{A}$. So the reader interested only in duality for countable models,²³ or only interested in soundness and completeness but not in duality, can ignore the generality in Theorem 6.1.23 and take atoms and $\mathcal{D}$ to be countable.

If we want a full duality result also for *uncountable* $\mathcal{D}$, then we require the potent generality of Theorem 6.1.23 and the size conditions in Definitions 4.4.1 and 7.7.1. More on this in Subsection 12.2.4.

6.2. The *amgis*-action on (prime) filters

For this subsection, fix $\mathcal{D} \in \mathbf{nDiV}$ a nominal distributive lattice with $\forall$ (Definition 4.3.3).

Recall from Definition 3.3.1 the pointwise $\neg$ -action $p[u \leftarrow a] = \{x \mid x[a \rightarrow u] \in p\}$ where $u \in |\mathcal{D}^\partial|$. In this subsection we check that this action preserves the property of being a (prime) filter (Definitions 6.1.2 and 6.1.16).

The work happens in the key technical result Lemma 6.2.1; Proposition 6.2.4 then puts the result in a some nice packaging.

²³This is a reasonable special case for this particular paper, since one could argue that λ -calculus models should be countable since they are supposed to represent computation. However, the fully general version of Theorem 6.1.23 is no harder to write out—the mathematics does not really care.

LEMMA 6.2.1. *If p is a filter in $\mathcal{D}$ then so is $p[u \leftarrow a]$. Furthermore, if p is prime then so is $p[u \leftarrow a]$.*

Proof. We check the conditions of Definition 6.1.2 We use Proposition 3.3.2 without comment:

1. $\perp \notin p[u \leftarrow a]$. Since by $(\sigma\#)$ $\perp[a \mapsto u] = \perp$.
2. If $x \in p[u \leftarrow a]$ and $x \leq x'$ then $x' \in p[u \leftarrow a]$. From Lemma 4.2.2.
3. If $x \in p[u \leftarrow a]$ and $x' \in p[u \leftarrow a]$ then $x \wedge x' \in p[u \leftarrow a]$. Since by assumption the σ -action is compatible, so $(x \wedge x')[a \mapsto u] = x[a \mapsto u] \wedge (x'[a \mapsto u])$ (Definition 4.2.1).
4. If $\forall b'. ((b' b) \cdot x \in p[u \leftarrow a])$ then $\forall b. x \in p[u \leftarrow a]$.

Choose some fresh c (so $c\#x, u$). It is a fact that $x = (c b) \cdot ((c b) \cdot x)$ and by $(\forall\alpha)$ also $\forall b. x = \forall c. (c b) \cdot x$. Thus, we may rename to assume without loss of generality that $b\#u$.

Now suppose $((b' b) \cdot x)[a \mapsto u] \in p$ for cosmall many b' ; so suppose $b'\#u$. By Corollary 2.3.5 $(b' b) \cdot u = u$. Thus by Remark 3.1.2 and Corollary 2.3.5 we have that $(b' b) \cdot (x[a \mapsto u]) \in p$ for cosmall many b' . Therefore $\forall b. (x[a \mapsto u]) \in p$ and by compatibility (Definition 4.2.1) $(\forall b. x)[a \mapsto u] \in p$.

Now suppose p is prime and suppose $(y_1 \vee y_2)[a \mapsto u] \in p$. Then by compatibility (Definition 4.2.1) $y_1[a \mapsto u] \vee (y_2[a \mapsto u]) \in p$. Therefore either $y_1[a \mapsto u] \in p$ or $y_2[a \mapsto u] \in p$. $\square$

DEFINITION 6.2.2. If $\mathcal{D} \in \mathbf{nDiV}$ write $Pnt(\mathcal{D})$ for the $\mathfrak{o}$ -algebra determined by prime filters and the pointwise actions from Definition 3.3.1. That is:

- $|Pnt(\mathcal{D})| = \{p \subseteq |\mathcal{D}| \mid p \text{ is a prime filter}\}$.
- $Pnt(\mathcal{D})^\partial = \mathcal{D}^\partial$.
- $\pi \cdot p = \{x \in p \mid x \in p\}$ and $p[u \leftarrow a] = \{x \mid x[a \mapsto u] \in p\}$ for $u \in |Pnt(\mathcal{D})^\partial|$.

NOTATION 6.2.3. We will use **points** and *prime filters* synonymously henceforth.

PROPOSITION 6.2.4. $Pnt(\mathcal{D})$ is indeed an $\mathfrak{o}$ -algebra.

Proof. This is just Lemma 6.2.1 combined with Proposition 3.3.5. $\square$

6.3. Injecting $\mathcal{D}$ into the set of sets of prime filters

Recall from Definition 3.4.6 the notion of σ -powerset algebra Pow_σ , and from Definition 6.1.16 the notion of a *prime filter*.

In this subsection we consider how to embed $\mathcal{D}$ a nominal distributive lattice with $\forall$ in the σ -powerset of its prime filters. The main definition is Definition 6.3.1. The main results are Lemma 6.4.2, Corollary 6.3.3, and Lemma 6.3.5.

It is standard to embed a lattice into sets of prime filters, and Definition 6.3.1 has the form one would expect. There is extra structure; for instance Lemma 6.4.1 and part 3 of Lemma 6.4.2. With the results we have proven so far, we can deal with this extra structure.

Recall from Definition 6.1.16 the notion of prime filter:

DEFINITION 6.3.1. Suppose $\mathcal{D}$ is a nominal distributive nominal lattice with $\forall$. Define

$$x^\bullet = \{p \text{ a prime filter} \mid x \in p\} \subseteq |\text{Pnt}(\mathcal{D})|.$$

LEMMA 6.3.2. $x \leq y$ if and only if $x^\bullet \subseteq y^\bullet$.

Proof. Suppose $x \leq y$. By condition 2 of Definition 6.1.2 if $x \in p$ then $y \in p$. It follows that $x^\bullet \subseteq y^\bullet$.

Suppose $x \not\leq y$. There are three easy cases:

- If $x = \perp$ then using Lemma 6.1.17(1) $x^\bullet$ is the empty set and $x^\bullet \subseteq y^\bullet$.
- If $y = \top$ then using Lemma 6.1.17(1) $y^\bullet$ is the set of all prime filters and $x^\bullet \subseteq y^\bullet$.
- Otherwise, by Lemma 6.1.9 $x\downarrow$ is a (small-supported) ideal and $y\uparrow$ is a small-supported filter. Also since $x \not\leq y$ we have that $x\downarrow \cap y\uparrow = \emptyset$ and so by Theorem 6.1.23 there exists a prime filter p containing $y\uparrow$ (so containing y) and disjoint from $x\downarrow$ thus not containing x . Then $p \in y^\bullet$ and $p \notin x^\bullet$. $\square$

COROLLARY 6.3.3. The assignment $x \mapsto x^\bullet$ is injective.

Proof. Direct from Lemma 6.3.2. $\square$

COROLLARY 6.3.4. $\text{supp}(x^\bullet) = \text{supp}(x)$.

Proof. Using part 3 of Theorem 2.3.2 and Corollary 6.3.3; see [Gab11b, Theorem 4.7]. $\square$

We never use Lemma 6.3.5 or Corollary 6.3.6 later but we mention them anyway because they express that the set of all filters is (intuitively) a conservative extension of the set of all prime filters:

LEMMA 6.3.5. Suppose $p \subseteq |\mathcal{D}|$ is a (not necessarily prime) filter and $x \in |\mathcal{D}|$, and suppose for every prime filter r , if $p \subseteq r$ then $x \in r$. Then $x \in p$.

Proof. Suppose $x \notin p$. By Lemma 6.1.9 $x\downarrow$ is an ideal and by Theorem 6.1.23 there exists a prime filter r such that $p \subseteq r$ and $x \notin r$. The result follows. $\square$

COROLLARY 6.3.6. A nice rephrasing of Lemma 6.3.5 is possible using Definition 6.3.1. If p is a (not necessarily prime) filter then

$$p = \bigcap \{x^\bullet \mid x \in p\}.$$

6.4. The map from x , to prime filters containing x , as a morphism

For this subsection, fix $\mathcal{D} \in \mathbf{nDiV}$ a nominal distributive lattice with $\forall$ (Definition 4.3.3). Recall from Definition 6.3.1 that if $x \in |\mathcal{D}|$ then $x^\bullet$ is the set of prime filters in $\mathcal{D}$ that contain x .

By Proposition 6.2.4 $Pnt(\mathcal{D})$ is an $\mathfrak{o}$ -algebra. So following Definition 3.4.1, sets of points $X \subseteq |Pnt(\mathcal{D})|$ inherit an action $X[a \mapsto u] = \{p \mid p[u \leftarrow a] \in X\}$. With this action we have the following:

LEMMA 6.4.1. *Suppose $x \in |\mathcal{D}|$ and $u \in |\mathcal{D}^\partial|$. Then:*

1. $\pi \cdot (x^\bullet) = (\pi \cdot x)^\bullet$
2. $x^\bullet[a \mapsto u] = (x[a \mapsto u])^\bullet$

Proof. The case of $\pi \cdot (x^\bullet)$ is direct from Theorem 2.3.2 (a proof by concrete calculations similar to the case of $x^\bullet[a \mapsto u]$ is also possible). For the case of $x^\bullet[a \mapsto u]$, we reason as follows:

$$\begin{aligned}
 p \in (x^\bullet)[a \mapsto u] &\Leftrightarrow \forall c. p[u \leftarrow c] \in ((c a) \cdot x)^\bullet && \text{Prop 3.4.3, pt 1 this result} \\
 &\Leftrightarrow \forall c. (c a) \cdot x \in p[u \leftarrow c] && \text{Definition 6.3.1} \\
 &\Leftrightarrow \forall c. ((c a) \cdot x)[c \mapsto u] \in p && \text{Proposition 3.3.2} \\
 &\Leftrightarrow p \in (x[a \mapsto u])^\bullet && (\sigma\alpha), \text{ Definition 6.3.1} \quad \square
 \end{aligned}$$

LEMMA 6.4.2. 1. $\perp^\bullet = \emptyset$ and $\top^\bullet = |Pnt(\mathcal{D})|$

2. $(x \wedge y)^\bullet = x^\bullet \cap y^\bullet$
3. $(\forall a. x)^\bullet = \bigcap^{\#a} (x^\bullet)$ (Definition 5.2.1)
4. $(x \vee y)^\bullet = x^\bullet \cup y^\bullet$

Proof. Parts 1, 2, and 4 just reformulate parts 1, 2, and 4 of Lemma 6.1.17.

For part 3, suppose $p \in (\forall a. x)^\bullet$. By Definition 6.3.1 this is if and only if $\forall a. x \in p$. By Proposition 6.1.10(1) this is if and only if $x[a \mapsto u] \in p$ for every $u \in |\mathcal{D}^\partial|$. Using Definition 6.3.1 and Lemma 6.4.1 this is if and only if $p \in x^\bullet[a \mapsto u]$ for every $u \in |\mathcal{D}^\partial|$. The result follows from Definition 5.2.1. $\square$

DEFINITION 6.4.3. Define $\mathcal{D}^\bullet \in \mathbf{nDiV}$ a nominal distributive lattice with $\forall$ by the following data:

1. $|\mathcal{D}^\bullet| = \{x^\bullet \mid x \in |\mathcal{D}|\}$.
2. $(\mathcal{D}^\bullet)^\partial = \mathcal{D}^\partial$.
3. $\mathcal{D}^\bullet$ has permutation and $\mathfrak{o}$ -actions following Definition 3.4.1 (so $\pi \cdot x^\bullet = \{\pi \cdot p \mid p \in x^\bullet\}$ and $x^\bullet[a \mapsto u] = \{p \mid p[u \leftarrow a] \in x^\bullet\}$).

If $\mathcal{D}$ is impredicative (Definition 4.4.1), so we assume a σ -algebra morphism $\partial : \mathcal{D}^\partial \rightarrow \mathcal{D}$, then $\mathcal{D}^\bullet$ naturally becomes impredicative where:

- $\partial_{\mathcal{D}^\bullet} u = (\partial_{\mathcal{D}} u)^\bullet$ for $u \in |\mathcal{D}^\partial| = |(\mathcal{D}^\bullet)^\partial|$.

It is now easy to state and prove a nominal sets representation theorem, representing an abstract $\mathcal{D} \in \text{nDiV}$ concretely as the nominal sets-based structure $\mathcal{D}^*$:

THEOREM 6.4.4 (First representation theorem). *If $\mathcal{D}$ is in nDiV then so is $\mathcal{D}^*$, and the pair of maps $(x \mapsto x^*, u \mapsto u)$ is an isomorphism from $\mathcal{D}$ to $\mathcal{D}^*$ in nDiV . If furthermore $\mathcal{D}$ is in inDiV (is impredicative) then so is $\mathcal{D}^*$ and $x \mapsto x^*$ is an isomorphism from $\mathcal{D}$ to $\mathcal{D}^*$.*

Proof. We unpack Definition 6.4.3 and use Lemmas 6.4.1 and 6.4.2 to check the conditions on morphisms from Definition 4.3.6. Surjectivity is by construction and injectivity by Corollary 6.3.3. $\square$

Theorem 9.4.11 extends Theorem 6.4.4 with $\bullet$ and $\multimap$.

7. Nominal σ -topological spaces

7.1. The basic definition

DEFINITION 7.1.1. A **nominal σ -topological space** $\mathcal{T}$ is a tuple $(|\mathcal{T}|, \cdot, \mathcal{T}^\partial, \text{amgis}, \text{Opn}(\mathcal{T}))$ where

- $(|\mathcal{T}|, \cdot, \mathcal{T}^\partial, \text{amgis})$ forms an $\mathfrak{v}$ -algebra (Definition 3.2.1) and
- $\text{Opn}(\mathcal{T}) \subseteq |\text{Nom}(\mathcal{T})|$ (i.e. a set of small-supported sets; see Subsection 2.5.1) is an equivariant (see Lemma 2.2.4) set of **open sets**.

Furthermore we impose the following conditions on $\text{Opn}(\mathcal{T})$:

1. $\emptyset \in \text{Opn}(\mathcal{T})$ and $|\mathcal{T}| \in \text{Opn}(\mathcal{T})$
2. If $X \in \text{Opn}(\mathcal{T})$ and $Y \in \text{Opn}(\mathcal{T})$ then $X \cap Y \in \text{Opn}(\mathcal{T})$.
3. If $\mathcal{X} \subseteq \text{Opn}(\mathcal{T})$ is small-supported then $\bigcup \mathcal{X} \in \text{Opn}(\mathcal{T})$; we call this a **small-supported union** of open sets.

REMARK 7.1.2. Topological spaces over Zermelo-Fraenkel (**ZF**) sets—that is, over ‘ordinary’ sets—are such that an arbitrary union of open sets is open.

Condition 3 of Definition 7.1.1 generalises that condition, because any ZF set is also naturally an FM set with the trivial permutation action, and with empty support; so arbitrary sets of ZF sets are already small-supported, by $\emptyset$.

As a design choice in Definition 7.1.1 we do not insist that $U \in \text{Opn}(\mathcal{T})$ implies $\bigcap^{\#a} U \in \text{Opn}(\mathcal{T})$. See the later notion of *coherence* in Definition 7.5.1.

7.2. The map F from distributive lattices to nominal σ -topological spaces

DEFINITION 7.2.1. Suppose $\mathcal{D} \in \mathbf{nDiV}$ is a nominal distributive lattice with $\forall$.

Define $F(\mathcal{D})$ a nominal σ -topological space (Definition 7.1.1) by (technical references follow):

1. $F(\mathcal{D})$ has as underlying $\mathfrak{o}$ -algebra $Pnt(\mathcal{D})$ from Definition 6.2.2, so that
 - $|F(\mathcal{D})| = |Pnt(\mathcal{D})|$ and $F(\mathcal{D})^\partial = \mathcal{D}^\partial$, and
 - $\pi \cdot p = \{\pi \cdot x \mid x \in p\}$ and $p[u \leftarrow a] = \{x \mid x[a \mapsto u] \in p\}$ for $u \in |\mathcal{D}^\partial|$.
2. The topology $Opn(F(\mathcal{D}))$ is generated under small-supported unions by $\{x^\bullet \mid x \in |\mathcal{D}|\}$.

REMARK 7.2.2. For the reader's convenience we give references for technical definitions above:

- $Pnt(\mathcal{D})$ is from Definition 6.2.2.
- The pointwise actions are from Definition 3.4.1.
- Small support is from Definition 2.1.5.
- $x^\bullet$ is from Definition 6.3.1.

So $X \in Opn(F(\mathcal{D}))$ when $X = \bigcup \mathcal{X}$ where $\mathcal{X} = \{x_i^\bullet \mid i \in I\}$ is a small-supported set of sets of points of the form $x^\bullet$.

THEOREM 7.2.3. If $\mathcal{D} \in \mathbf{nDiV}$ is a nominal distributive lattice (Definition 4.4.1) then $F(\mathcal{D})$ is a σ -topological space.

Proof. By Theorem 2.3.2 $Opn(F(\mathcal{D}))$ is equivariant. We consider the conditions in Definition 7.1.1 in turn:

1. $\emptyset$ is open by construction and $\mathbf{T}^\bullet = |Pnt(\mathcal{D})|$.
2. Suppose $X, Y \in Opn(F(\mathcal{D}))$. So $X = \bigcup x_i^\bullet$ and $Y = \bigcup y_j^\bullet$ for some small-supported sets $\{x_i \mid i \in I\}, \{y_j \mid j \in J\} \subseteq |\mathcal{D}|$. Thus using part 2 of Lemma 6.4.2 and some elementary sets calculations $X \cap Y = \bigcup_{i \in I, j \in J} (x_i \wedge y_j)^\bullet$ and by Theorem 2.3.2 this is a small-supported union.
3. Suppose $\mathcal{X} \subseteq Opn(F(\mathcal{D}))$ is small-supported. Using Theorem 2.3.2 so is $\bigcup \mathcal{X} \in Opn(F(\mathcal{D}))$. $\square$

7.3. Technical interlude: two important propositions

For this subsection, fix $\mathcal{D} \in \mathbf{nDiV}$ a nominal distributive lattice with $\forall$.

This subsection proves Proposition 7.3.4, and uses that to prove Proposition 7.3.5, which is an important technical lemma for Theorem 7.4.3.

Intuitively Proposition 7.3.5 says that the set of all prime filters containing some $z \in \mathcal{D}$ is compact (so any cover of $z^\bullet$ has a subcover with a very strict bound on its size; see Theorem 7.4.3 for the full result). The reader familiar with duality results should recognise the overall argument, and we just need to do some extra work to account for the extra ‘nominal’ structure.

Proposition 7.3.4 is new, though the statement and proof have a similar outline; intuitively it says that any cover of $z^\bullet$ has a subcover with a very strict bound on its support.

$$\begin{aligned}
 \mathcal{X}^\bullet &= \{x^\bullet \mid x \in \mathcal{X}\} \\
 \mathcal{X}_\wedge &= \{\top\} \cup \{x_1 \wedge \dots \wedge x_n \mid \{x_1, \dots, x_n\} \subseteq \mathcal{X}\} \\
 \mathcal{X}_\vee &= \{\perp\} \cup \{x_1 \vee \dots \vee x_n \mid \{x_1, \dots, x_n\} \subseteq \mathcal{X}\} \\
 \mathcal{X}_\downarrow &= \{x' \mid \exists x \in \mathcal{X}. x' \leq x\} \\
 \mathcal{X}_\uparrow &= \{x' \mid \exists x \in \mathcal{X}. x \leq x'\} \\
 \mathcal{X}_{\vee\downarrow} &= (\mathcal{X}_\vee)_\downarrow \\
 \mathcal{X}_{\wedge\uparrow} &= (\mathcal{X}_\wedge)_\uparrow
 \end{aligned}$$

Figure 2: Some useful notation

Some notation will be useful:

DEFINITION 7.3.1. Given $\mathcal{X} \subseteq |\mathcal{D}|$, we define $\mathcal{X}^\bullet$, $\mathcal{X}_\wedge$, $\mathcal{X}_\vee$, $\mathcal{X}_\downarrow$, $\mathcal{X}_\uparrow$, $\mathcal{X}_{\vee\downarrow}$, and $\mathcal{X}_{\wedge\uparrow}$ as in Figure 2.

REMARK 7.3.2. • The notation $\mathcal{X}_\downarrow$ and $\mathcal{X}_\uparrow$ echoes $x_\downarrow$ and $x_\uparrow$ from Definition 6.1.8, and indeed $x_\downarrow$ from Definition 6.1.8 is equal to $\{x\}_\downarrow$ from Definition 7.3.1.

- Later, we will encounter Definition 7.9.8, which is a kind of inverse to $\mathcal{X}^\bullet$.
- By convention, we take $\top \in \mathcal{X}_\wedge$ and $\perp \in \mathcal{X}_\vee$. Think of this as “the case that $n=0$ ”.

REMARK 7.3.3. Intuitively, $\mathcal{X}_{\vee\downarrow}$ is trying to be the least ideal (Definition 6.1.3) containing $\mathcal{X}$; it may fail to be an ideal if $\top \in \mathcal{X}_{\vee\downarrow}$.

Similarly, $\mathcal{X}_{\wedge\uparrow}$ is moving in the direction of being a filter (Definition 6.1.2) containing $\mathcal{X}$, though it may fail to be a filter if either $\perp \in \mathcal{X}_{\wedge\uparrow}$, or $\forall b.(b \wedge a) \cdot x \in \mathcal{X}_{\wedge\uparrow}$ and $\forall a.x \notin \mathcal{X}_{\wedge\uparrow}$.

If we look ahead to Definition 7.4.1 then we can state Proposition 7.3.4 intuitively as follows: a small-supported cover of $z^\bullet$ naturally generates a *strictly* small-supported cover.

PROPOSITION 7.3.4. Suppose $z \in |\mathcal{D}|$ and $\mathcal{X} \subseteq |\mathcal{D}|$ is small-supported, and suppose $z^\bullet \subseteq \bigcup \mathcal{X}^\bullet$. Then there exists $\mathcal{X}' \subseteq \mathcal{X}_\vee$ (Figure 2) such that:

1. $\mathcal{X}' \subseteq \mathcal{X}_\vee$ is strictly small-supported, and
2. $z^\bullet \subseteq \bigcup \mathcal{X}'^\bullet$.

Proof. If $z = \perp$ then by Lemma 6.4.2(1) $z^\bullet = \emptyset$ and we take $\mathcal{X}' = \emptyset$ (by convention, $\bigcup \emptyset = \emptyset$). So we assume henceforth that $z \neq \perp$.

If for every $x \in \mathcal{X}$ there exists $x' \in \mathcal{X}_\vee$ with $\text{supp}(x \vee x') \subseteq \text{supp}(z)$, then we build $\mathcal{X}'$ to contain $x \vee x'$ for each $x \in \mathcal{X}$ and some corresponding choice of $x' \in \mathcal{X}_\vee$. By construction $\mathcal{X}'$ is strictly supported by $\text{supp}(z)$ and $z^\bullet \subseteq \bigcup \mathcal{X}'^\bullet = \bigcup \mathcal{X}^\bullet$.

So now suppose there exists $x \in \mathcal{X}$ such that for any $x' \in \mathcal{X}_\vee$ it is the case that $\text{supp}(x \vee x') \not\subseteq \text{supp}(z)$. Recall $\neg\downarrow$ from Figure 2 and define $\mathcal{Y}_x$ by

$$\mathcal{Y}_x = \{x \vee x' \mid x' \in \mathcal{X}_\vee\}_\downarrow.$$

We note some properties of $\mathcal{Y}_x$:

1. $x \in \mathcal{Y}_x$ by construction (note that $\perp \in \mathcal{X}_{\mathcal{V}}$).
2. $\top \notin \mathcal{Y}_x$, since $\text{supp}(\top) = \emptyset \subseteq \text{supp}(z)$.
3. $\mathcal{Y}_x$ is down-closed, by the use of $\downarrow$ (Definition 6.1.8).
4. If $y \in \mathcal{Y}_x$ and $y' \in \mathcal{Y}_x$ then $y \vee y' \in \mathcal{Y}_x$, by the use of $\mathcal{X}_{\mathcal{V}}$ and $\downarrow$ in the construction of $\mathcal{Y}_x$.
5. Thus $\mathcal{Y}_x$ is an ideal (Definition 6.1.3).²⁴
6. $\bigcup \mathcal{Y}_x^* = \bigcup \mathcal{X}^*$, by construction.

There are now two possibilities: $z \in \mathcal{Y}_x$ or $z \notin \mathcal{Y}_x$. We treat each in turn:

- Suppose $z \in \mathcal{Y}_x$. Then $z \leq x \vee x'$ for some $x' \in \mathcal{X}_{\mathcal{V}}$ and we can take $\mathcal{X}' = \{x \vee x'\} \subseteq \mathcal{X}_{\mathcal{V}}$.²⁵
- Suppose $z \notin \mathcal{Y}_x$. We noted above that $\mathcal{Y}_x$ is a (small-supported) ideal, and by Lemma 6.1.9 $z \uparrow$ is a small-supported filter (recall we assumed above that $z \neq \perp$). By Theorem 6.1.23 there exists a prime filter p such that $z \in p$ and $p \cap \mathcal{Y}_x = \emptyset$. By Definition 6.3.1 $p \in z^*$ and by assumption $z^* \subseteq \bigcup \mathcal{X}^*$, so there exists $x' \in \mathcal{X}$ with $p \in x'^*$, thus $x' \in p$, thus by condition 2 of Definition 6.1.2 $x \vee x' \in p$. However by assumption $x \vee x' \in \mathcal{Y}_x$, a contradiction. So this case is impossible. $\square$

PROPOSITION 7.3.5. *Suppose $z \in |\mathcal{D}|$ and suppose $\mathcal{Y} \subseteq |\mathcal{D}|$ is small-supported. Then if $z^* \subseteq \bigcup \mathcal{Y}^*$ then $z \leq y$ for some $y \in \mathcal{Y}_{\mathcal{V}}$.*

Proof. If $z = \perp$ then by Lemma 6.4.2(1) $z^* = \emptyset$ and we may take $y = \perp \in \mathcal{Y}_{\mathcal{V}}$. So suppose $z \neq \perp$; it follows using Corollary 6.3.3 that $z^* \neq \emptyset$ and therefore that $\mathcal{Y}$ is nonempty.

Using Proposition 7.3.4 we may assume without loss of generality that $\mathcal{Y}$ is strictly small-supported. Write

$$\mathcal{X} = \{x \mid \exists y \in \mathcal{Y}. (z \leq y \vee x)\} \uparrow_{\mathcal{X}} \subseteq |\mathcal{D}|. \quad (3)$$

If $\perp \in \mathcal{X}$ then $\exists y \in \mathcal{Y}. z \leq y$ and we are done. So suppose $\perp \notin \mathcal{X}$ so that $\mathcal{X}$ satisfies condition 1 of Definition 6.1.2.

Now $\mathcal{Y}$ is nonempty, and since $z \leq y \vee \top$ for any $y \in \mathcal{Y}$, also $\top \in \mathcal{X}$ so that $\mathcal{X}$ is nonempty. We now observe that $\mathcal{X}$ also satisfies conditions 2, 3, and 4 of Definition 6.1.2 and so is a filter:

2. $\mathcal{X}$ is up-closed. By the use of $\uparrow_{\mathcal{X}}$.
3. $x \in \mathcal{X}$ and $x' \in \mathcal{X}$ imply $x \wedge x' \in \mathcal{X}$. By the use of $\uparrow_{\mathcal{X}}$.
4. If $\bigvee b. ((b \ a) \cdot x \in \mathcal{X})$ then $\forall a. x \in \mathcal{X}$. Choose fresh b (so $b \# z, x, \mathcal{X}, \mathcal{Y}$), so there exist $y_i \in \mathcal{Y}$ for $1 \leq i \leq n$ and $x_1, \dots, x_n$ such that $z \leq y_i \vee x_i$ for $1 \leq i \leq n$, and $(b \ a) \cdot x \geq x_1 \wedge \dots \wedge x_n$.

²⁴ $\mathcal{Y}_x$ is also small-supported by Theorem 2.3.2, since $\mathcal{X}$ and x are small-supported.

²⁵It may not be that $\text{supp}(x \vee x') \subseteq \text{supp}(z)$, but we do not care; we only need $\mathcal{X}'$ to be strictly supported.

Now $b\# \mathcal{Y}$ and by assumption above $\mathcal{Y}$ is strictly small-supported, so that by Lemma 2.5.3(4) and Corollary 6.3.4 also $b\# y_i$ for $1 \leq i \leq n$. We then have by part 3 of Lemma 4.3.4 that $z \leq y_i \vee \forall b. x_i$ for $1 \leq i \leq n$. Also, using Lemmas 4.1.5 and 4.1.7 $\forall b. (b a) \cdot x \geq (\forall b. x_1) \wedge \dots \wedge \forall b. x_n$. It follows using Lemma 4.1.3 that $\forall a. x \in \mathcal{X}$.

It is now useful to consider two distinct possibilities: $\mathcal{X} \cap \mathcal{Y}_{\downarrow} \neq \emptyset$ or $\mathcal{X} \cap \mathcal{Y}_{\downarrow} = \emptyset$. We treat each in turn:

- Suppose $\mathcal{X} \cap \mathcal{Y}_{\downarrow} \neq \emptyset$. So take any $x \in \mathcal{X} \cap \mathcal{Y}_{\downarrow}$. Since $x \in \mathcal{Y}_{\downarrow}$, there exist $y'_1, \dots, y'_m \in \mathcal{Y}$ with $x \leq y'_1 \vee \dots \vee y'_m$. Since $x \in \mathcal{X}$, there exist $y_1, \dots, y_n \in \mathcal{Y}$ and $x_1, \dots, x_n \in \mathcal{X}$ such that $z \leq y_i \vee x_i$ for $1 \leq i \leq n$ and $x \geq x_1 \wedge \dots \wedge x_n$. We note that $z \leq (y_1 \vee x_1) \wedge \dots \wedge (y_n \vee x_n)$ and by some easy calculations that

$$z \leq x \vee y_1 \vee \dots \vee y_n \leq y'_1 \vee \dots \vee y'_m \vee y_1 \vee \dots \vee y_n,$$

and we are done.

- Now suppose $\mathcal{X} \cap \mathcal{Y}_{\downarrow} = \emptyset$. We noted above that $\top \in \mathcal{X}$, so $\top \notin \mathcal{Y}_{\downarrow}$ and it follows that $\mathcal{Y}_{\downarrow}$ is an ideal. Also, $\mathcal{Y}$ is small-supported, and since z is also small-supported we can apply Theorem 2.3.2 to equation 3 to see that $\mathcal{X}$ is small-supported. Therefore by Theorem 6.1.23 $\mathcal{X} \subseteq p$ for some prime filter p such that $p \cap \mathcal{Y}_{\downarrow} = \emptyset$. It is easy to check that $z \in \mathcal{X}$, so that $z \in p$ and by Definition 6.3.1 $p \in z^*$. By assumption $z^* \subseteq \bigcup \mathcal{Y}^*$ so there exists some $y \in \mathcal{Y}$ with $p \in y^*$, that is, $y \in p$. But our assumption that $p \cap \mathcal{Y}_{\downarrow} = \emptyset$ implies that $p \cap \mathcal{Y} = \emptyset$, contradicting this. So this case is impossible. $\square$

7.4. Compactness

Fix $\mathcal{T}$ a nominal σ -topological space.

DEFINITION 7.4.1. Suppose $\mathcal{U} \subseteq \text{Opn}(\mathcal{T})$ and $U \in \text{Opn}(\mathcal{T})$.

- Say $\mathcal{U}$ **covers** U when $\mathcal{U}$ is small-supported and $U \subseteq \bigcup \mathcal{U}$. Call $\mathcal{U}$ a **cover** when it covers $|\mathcal{T}|$.
- Call U **compact** when every cover of U has a finite subcover. Write $\text{Cpct}(\mathcal{T})$ for the set of compact open sets of $\mathcal{T}$:

$$\text{Cpct}(\mathcal{T}) = \{U \in \text{Opn}(\mathcal{T}) \mid U \text{ is compact}\}$$

Proposition 7.4.2 looks familiar enough, but we are in a nominal context so we have to check facts about support. It all works:

PROPOSITION 7.4.2. Suppose $U, V \in \text{Cpct}(\mathcal{T})$. Then:

1. $\emptyset$ is compact.
2. $U \cup V$ is compact.
3. $\pi \cdot U$ is compact.

- Proof.* 1. There are two covers of $\emptyset$: the empty set of open sets and the set containing the empty set of points. Both are finite.
2. Suppose U and V are compact and $\mathcal{W}$ covers $U \cup V$.
By Theorem 2.3.2 $\text{supp}(\{W \cap U \mid W \in \mathcal{W}\}) \subseteq \text{supp}(\mathcal{W}) \cup \text{supp}(U)$ so it is small-supported, and it follows that $\{W \cap U \mid W \in \mathcal{W}\}$ covers U , and we obtain a finite subcover of U .
Reasoning similarly for $\{W \cap V \mid W \in \mathcal{W}\}$ we obtain a finite subcover of V .
Putting these two finite subcovers together, we obtain one of $U \cup V$.
3. Using Theorem 2.3.2 and Proposition 2.3.4. $\square$

In a sense, Definition 7.5.1 is a continuation of Proposition 7.4.2.

THEOREM 7.4.3. Suppose $\mathcal{D} \in \text{nDiV}$ and recall $F(\mathcal{D})$ from Definition 7.2.1. Then:

1. If $x \in |\mathcal{D}|$ then $x^\bullet$ is open and compact in $F(\mathcal{D})$.
2. If U is open and compact in $F(\mathcal{D})$ then $U = x^\bullet$ for some unique $x \in |\mathcal{D}|$.

Proof. We consider each part in turn:

1. $x^\bullet$ is open by construction in Definition 7.2.1. Now consider a cover $\mathcal{U}$ of $x^\bullet$.
By Definitions 7.4.1 and 7.2.1(2), $\mathcal{U}$ is a small-supported set of unions of small-supported sets of elements, and all these elements have the form $y^\bullet$ for $y \in |\mathcal{D}|$.
So we can write $\bigcup \mathcal{U} = \bigcup \mathcal{Y}^\bullet$ for some (by Theorem 2.3.2) small-supported $\mathcal{Y} \subseteq |\mathcal{D}|$. To find a finite subcover of $\mathcal{U}$, it would suffice to find a finite subset $\mathcal{X}' \subseteq \mathcal{Y}$ such that $x^\bullet \subseteq \bigcup \mathcal{X}'^\bullet$.
We use Proposition 7.3.5.
2. By construction in Definition 7.2.1 the open sets of $F(\mathcal{D})$ are unions of small-supported sets of sets the form $x^\bullet$. We assumed that U is compact so it has a finite subcover $x_1^\bullet \cup \dots \cup x_n^\bullet$. We use part 4 of Lemma 6.4.2. Uniqueness is Corollary 6.3.3. $\square$

7.5. Coherent spaces: closure under σ , $\cap$ and $\bigcap^{#a}$

Coherence usually means that the compact open sets are closed under lattice operations and generate all open sets via sets unions. Our lattices have more structure, notably: a σ -action, and $\bigcap^{#a}$ from Definition 5.2.1. Also, our notion of ‘generating’ open sets has nominal aspects to it; see condition 3 of Definition 7.1.1.

Definition 7.5.1 is how we extend the notion of coherence to account for this structure. Proposition 7.5.4 then checks that F from Definition 7.2.1 does indeed generate coherent spaces.

DEFINITION 7.5.1. Call a nominal σ -topological space $\mathcal{T}$ **coherent** when:

1. If U is open and compact then so is $U[a \mapsto u]$ for every $u \in |\mathcal{T}^\partial|$.
2. $|\mathcal{T}|$ is (open and) compact, and if U and V are open and compact then so is $U \cap V$.
3. If U is open and compact then so is $\bigcap^{#a} U$.
4. Every open $U \in \text{Opn}(\mathcal{T})$ is equal to $\bigcup \mathcal{U}$ for some small-supported $\mathcal{U} \subseteq \text{Cpct}(\mathcal{T})$.

REMARK 7.5.2. The $U[a \mapsto u]$ mentioned in Definition 7.5.1 is the *pointwise* action from Definition 3.4.1

$$U[a \mapsto u] = \{p \mid \forall c. p[u \leftarrow c] \in (c \ a) \cdot X\}.$$

It is applicable to any set of points (not just a compact set of points) and is inherited from the $\triangleright$ -action $p[u \leftarrow c]$ which we assumed on the underlying points.

REMARK 7.5.3. We rewrite Definition 7.5.1 in less precise, but more intuitive language:

1. Compactness is closed under the σ -action, so the compact sets form a σ -algebra over $\mathcal{T}^\partial$.
2. Compactness is closed under (possibly empty) sets intersection.
3. Compactness is closed under universal quantification.
4. Compact open sets are a small-supported (Subsection 2.5.2) basis for all open sets.

PROPOSITION 7.5.4. Suppose $\mathcal{D} \in \mathbf{nDiV}$. Then $F(\mathcal{D})$ (Definition 7.2.1) is coherent.

Proof. By Theorem 7.4.3 we can identify the compact open sets of $F(\mathcal{D})$ with sets of the form $x^\bullet$ for $x \in |\mathcal{D}|$. We now reason as follows:

1. By Lemma 6.4.1 $x^\bullet[a \mapsto u] = (x[a \mapsto u])^\bullet$.
2. By part 1 of Lemma 6.1.17 $\top^\bullet = |F(\mathcal{D})|$ (every point contains $\top$), and by part 2 of Lemma 6.4.2 $x^\bullet \cap y^\bullet = (x \wedge y)^\bullet$.
3. By part 3 of Lemma 6.4.2 $(\forall a. x)^\bullet = \bigcap^{\#a} x^\bullet$.
4. By construction in Definition 7.2.1 open sets are small-supported unions of the $x^\bullet$. $\square$

7.6. Completely prime filters in a coherent space

Recall the notion of *prime* filter from Definition 6.1.16. A stronger property will also be of interest; this definition is standard:

DEFINITION 7.6.1. Suppose $\mathcal{T}$ is a nominal σ -topological space and suppose $\mathcal{U} \subseteq \text{Opn}(\mathcal{T})$ is a filter (Definition 6.1.2) of open sets in $\mathcal{T}$.

Call a filter $\mathcal{U} \subseteq \text{Opn}(\mathcal{T})$ **completely prime** when:

- if $\mathcal{V} \subseteq \text{Opn}(\mathcal{T})$ is small-supported and $\bigcup \mathcal{V} \in \mathcal{U}$,
- then $V \in \mathcal{U}$ for some $V \in \mathcal{V}$.²⁶

For coherent spaces, a more economical characterisation will be useful:

LEMMA 7.6.2. If $\mathcal{T}$ is coherent then the completely prime filters of open sets are in a natural bijection with the prime filters of compact open sets, with the bijection given by:

- A completely prime filter $\mathcal{U}' \subseteq \text{Opn}(\mathcal{T})$ corresponds to $\mathcal{U} = \mathcal{U}' \cap \text{Cpct}(\mathcal{T})$.
- A prime filter $\mathcal{U} \subseteq \text{Cpct}(\mathcal{T})$ corresponds to $\mathcal{U}' = \{U' \in \text{Opn}(\mathcal{T}) \mid \exists U \in \mathcal{U}. U \subseteq U'\}$, the **up-closure** of $\mathcal{U}$ in $\text{Opn}(\mathcal{T})$.

²⁶A prime filter satisfies this property—for finite $\mathcal{V}$.

Proof. Suppose $\mathcal{U}'$ is a completely prime filter in $\text{Opn}(\mathcal{T})$. We will show that $\mathcal{U} = \mathcal{U}' \cap \text{Cpct}(\mathcal{T})$ is a filter (Definition 6.1.2) and is prime (Definition 6.1.16).

1. $\emptyset \notin \mathcal{U}$ since $\emptyset \notin \mathcal{U}'$.
2. If $U \in \mathcal{U}$ and $U' \in \text{Cpct}(\mathcal{T})$ and $U \subseteq U'$ then $U' \in \mathcal{U}'$ so $U' \in \mathcal{U}$.
3. It follows similarly that $U, U' \in \mathcal{U}$ implies $U \cap U' \in \mathcal{U}$.
4. Suppose $U \in \text{Cpct}(\mathcal{T})$ and suppose $\mathcal{U}b.(b a) \cdot U \in \mathcal{U}$. Then also $\mathcal{U}b.(b a) \cdot U \in \mathcal{U}'$ so by condition 4 of Definition 6.1.2 (since $\mathcal{U}'$ is a filter) the filter condition on $\mathcal{U}'$, $\bigcap^{\#a} U \in \mathcal{U}'$. By coherence (Definition 7.5.1 condition 3) $\bigcap^{\#a} U \in \text{Cpct}(\mathcal{T})$, therefore $\bigcap^{\#a} U \in \mathcal{U}$ as required.

By part 2 of Proposition 7.4.2 compactness is closed under finite unions and it follows that $\mathcal{U}$ is prime.

Conversely suppose $\mathcal{U} \subseteq \text{Cpct}(\mathcal{T})$ is a prime filter. First we will show that $\mathcal{U}' = \{U' \in \text{Opn}(\mathcal{T}) \mid \exists U \in \mathcal{U}. U \subseteq U'\}$ is a filter, then we will show it is completely prime.

1. $\emptyset \notin \mathcal{U}'$ since $\emptyset \notin \mathcal{U}$.
2. If $U \in \mathcal{U}'$ and $U' \in \text{Opn}(\mathcal{T})$ and $U \subseteq U'$ then by construction $U' \in \mathcal{U}'$.
3. It follows similarly that $U, U' \in \mathcal{U}'$ implies $U \cap U' \in \mathcal{U}'$.
4. Suppose $U' \in \text{Opn}(\mathcal{T})$ and suppose $\mathcal{U}b.(b a) \cdot U' \in \mathcal{U}'$. We need to show that $\bigcap^{\#a} U' \in \mathcal{U}'$.

Choose fresh b (so $b \# U', \mathcal{U}, \mathcal{U}'$). By Theorem 2.3.9 $(b a) \cdot U' \in \mathcal{U}'$ and this means that there exists $U \in \mathcal{U}$ such that $U \subseteq (b a) \cdot U'$. It follows by Theorem 2.3.2 and Corollary 2.3.5 that $\mathcal{U}b'.(b' b) \cdot U \in \mathcal{U}$. By condition 4 of Definition 6.1.2 (since $\mathcal{U}$ is a filter) $\bigcap^{\#b'} (b' b) \cdot U \in \mathcal{U}$ so that by Lemma 5.2.4 (since $b' \# U$) $\bigcap^{\#b} U \in \mathcal{U}$.

By assumption $U \subseteq (b a) \cdot U'$ so by Lemma 4.1.7 and Theorem 5.2.5 also $\bigcap^{\#b} (b a) \cdot U' \in \mathcal{U}'$. By Lemma 5.2.4 again (since $b \# U'$) $\bigcap^{\#a} U' \in \mathcal{U}'$, as required.

Now suppose $\mathcal{V} \subseteq \text{Opn}(\mathcal{T})$ is small-supported and suppose $\bigcup \mathcal{V} \in \mathcal{U}'$, so that $U \subseteq \bigcup \mathcal{V}$ for some $U \in \mathcal{U}$. But this just states that $\mathcal{V}$ covers U , and by compactness $\mathcal{V}$ has a finite subcover $\{V_1, \dots, V_n\} \subseteq \mathcal{V}$. It follows that $\bigcup \{V_1 \cap U, \dots, V_n \cap U\} = U \in \mathcal{U}$. Since $\mathcal{U}$ is prime it follows that $V_i \cap U \in \mathcal{U}$ for some i , and therefore that $V_i \in \mathcal{U}'$.

It is routine to verify that the correspondences between $\mathcal{U}$ and $\mathcal{U}'$ defined above are bijective. $\square$

7.7. Impredicativity

We saw in Subsection 4.4 and Definition 4.4.1 a notion of *impredicativity*, based on the idea that the things we substitute for should map to the things we substitute in. In the context of a topological space, this means that $\mathcal{T}^\partial$ should map to open sets. This is Definition 7.7.1, and Theorem 7.8.3 shows how $F(\mathcal{D})$ inherits any impredicative structure of $\mathcal{D}$.

We can think of Definition 7.7.1 as a dual to Definition 4.4.1, for the nominal σ -topological spaces from Definition 7.1.1:

DEFINITION 7.7.1. An **impredicative nominal σ -topological space** is a pair $(\mathcal{T}, \partial_{\mathcal{T}})$ where:

1. $\mathcal{T}$ is a nominal σ -topological space (Definition 7.1.1).
2. $\partial_{\mathcal{T}} : \mathcal{T}^{\partial} \rightarrow \text{Cpct}(\mathcal{T})$ is a morphism of σ -algebras (Definition 4.3.5).
3. $\text{Cpct}(\mathcal{T})$ has cardinality no greater than $\text{size}(\mathbb{A})$ (Definition 2.1.1).

NOTATION 7.7.2. Following Notation 4.4.3 we introduce some notation for Definition 7.7.1:

- We may drop subscripts and write ∂u for $\partial_{\mathcal{T}} u$ where $u \in |\mathcal{T}^{\partial}|$.
- We may write ∂a for $\partial_{\mathcal{T}}(a_{\mathcal{T}^{\partial}})$ where $a_{\mathcal{T}^{\partial}} = \text{atm}_{\mathcal{D}^{\partial}}(a)$ (Definition 3.1.1).
- We may write $\partial \mathcal{T}$ for $\{\partial u \mid u \in |\mathcal{T}^{\partial}|\} \subseteq \text{Cpct}(\mathcal{T})$ and call this set the **programs** of $\mathcal{T}$.

The exposition in and following Notation 4.4.3 is also valid here, so we do not repeat it.

THEOREM 7.7.3. If $\mathcal{D} \in \text{inDiV}$ (so $\mathcal{D}$ is impredicative) then $F(\mathcal{D})$ is also naturally impredicative.

Proof. We take $F(\mathcal{D})^{\partial} = \mathcal{D}^{\partial}$ and $\partial_{F(\mathcal{D})} u = (\partial_{\mathcal{D}} u)^*$. (In fact, this is an injection by Theorem 6.4.4.) $\square$

7.8. The map G from coherent spaces to distributive lattices

DEFINITION 7.8.1. Suppose $\mathcal{T}$ is a coherent (Definition 7.5.1) nominal σ -topological space.

Define $G(\mathcal{T})$ as follows:

- $|G(\mathcal{T})| = \text{Cpct}(\mathcal{T})$ (compact opens) and $G(\mathcal{T})^{\partial} = \mathcal{T}^{\partial}$.
- Given $U \in |G(\mathcal{T})|$ and $u \in |G(\mathcal{T})^{\partial}| = |\mathcal{T}^{\partial}|$ define $\pi \cdot U$ and $U[a \mapsto u]$ following Definition 3.4.1.
- $\top, \wedge, \perp, \vee$, and $\forall$ are interpreted as the whole underlying set $\mathcal{T}$, set intersection $\cap$, the empty set $\emptyset$, set union $\cup$, and $\bigcap^{\#a}$ from Definition 5.2.1.

REMARK 7.8.2. We unpack Definition 3.4.1 for $U \in \text{Cpct}(\mathcal{T})$ for the reader's convenience:

$$\pi \cdot U = \{\pi \cdot t \mid t \in U\} \quad U[a \mapsto u] = \{t \mid \forall c. t[u \leftarrow c] \in (c \cdot a) \cdot U\}.$$

THEOREM 7.8.3. Continuing Definition 7.8.1, if $\mathcal{T}$ is coherent then $G(\mathcal{T})$ is a nominal distributive lattice with $\forall$.

Furthermore, if $\mathcal{T}$ is impredicative then so naturally is $G(\mathcal{T})$.

Proof. By Proposition 7.4.2 $\perp, \top, \vee$, and the permutation action give results in $|G(\mathcal{T})|$. By our assumption that $\mathcal{T}$ is coherent, so do $\wedge$, the σ -action and $\forall$. We use Theorem 5.2.6.

Now suppose $\mathcal{T}$ is impredicative, so it is equipped with a σ -algebra morphism $\partial_{\mathcal{T}} : \mathcal{T}^{\partial} \rightarrow \text{Cpct}(\mathcal{T})$ (Definition 4.3.5). We take $\partial_{G(\mathcal{T})} = \partial_{\mathcal{T}}$. $\square$

PROPOSITION 7.8.4. If $\mathcal{D} \in \mathbf{nDiV}/\mathbf{inDiV}$ then $GF(\mathcal{D})$ is equal to $\mathcal{D}^\bullet$ from Definition 6.4.3, and the map $x \mapsto x^\bullet$ is an isomorphism in $\mathbf{nDiV}/\mathbf{inDiV}$.

Proof. By Theorem 7.4.3 $|GF(\mathcal{D})| = |\mathcal{D}^\bullet|$. We use Theorem 6.4.4. $\square$

7.9. Sober spaces

DEFINITION 7.9.1. Call a nominal σ -topological space $\mathcal{T}$ **sober** when:

1. If $\mathcal{U} \subseteq \text{Opn}(\mathcal{T})$ is a completely prime filter then there exists a unique $t_{\mathcal{U}} \in |\mathcal{T}|$ such that

$$\forall U \in \text{Opn}(\mathcal{T}). U \in \mathcal{U} \quad \text{if and only if} \quad t_{\mathcal{U}} \in U.$$

2. If $t \in |\mathcal{T}|$ and $U \in \text{Cpct}(\mathcal{T})$ then

$$\forall b. t \in (b \ a) \cdot U \quad \text{implies} \quad t \in \bigcap^{#a} U.$$

REMARK 7.9.2. Condition 1 of Definition 7.9.1 is the standard notion of sobriety. Intuitively, it states that completely prime filters characterise the underlying points of the space.

Condition 2 is clearly related to condition 4 of Definition 6.1.2, and carries the intuition that universal quantification is determined by the behaviour at a fresh atom (see Remark 6.1.1 and the discussion following Definition 6.1.2). We use it for just that in Proposition 7.9.5. More on this in Remark 8.2.3.

Lemma 7.9.3 is related to Proposition 6.1.10. Technically, it will be useful later in Proposition 8.2.2. Intuitively, it corresponds to the quantifier intro- and elim-rules in logic that if a is not free in Γ then $\Gamma \vdash \phi$ is derivable if and only if $\Gamma \vdash \forall a. \phi$ is derivable (see the discussion in Remark 6.1.1):

LEMMA 7.9.3. Suppose $\mathcal{T}$ is a sober nominal σ -topological space and $U \in \text{Cpct}(\mathcal{T})$. Then $t \in \bigcap^{#a} U$ if and only if $\forall b. t \in (b \ a) \cdot U$.

Proof. The right-to-left implication is condition 2 of Definition 7.9.1. For the left-to-right implication, suppose $t \in \bigcap^{#a} U$. By Definition 5.2.1 $t \in U[a \mapsto u]$ for all $u \in |\mathcal{T}^\partial|$ and (just as in the proof of Proposition 6.1.10(1)) $t \in U[a \mapsto b]$ for fresh b so using Lemma 3.2.4 $\forall b. t \in (b \ a) \cdot U$. $\square$

DEFINITION 7.9.4. Suppose $\mathcal{T}$ is a nominal σ -topological space and $t \in |\mathcal{T}|$. Define

$$t^* = \{U \in \text{Cpct}(\mathcal{T}) \mid t \in U\},$$

so that

$$U \in t^* \Leftrightarrow t \in U.$$

Recall from Definition 7.2.1 that if $\mathcal{D} \in \mathbf{inDiV}$ then $|F(\mathcal{D})|$ is the set of prime filters in $\mathcal{D}$.

PROPOSITION 7.9.5. Suppose a nominal σ -topological space $\mathcal{T}$ is coherent (Definition 7.5.1) and suppose $t \in |\mathcal{T}|$. Then t^* is a prime filter in $G(\mathcal{T})$, so that t^* is an element of $|FG(\mathcal{T})|$.

Proof. First we check that t^* is a filter (Definition 6.1.2). Conditions 1, 2, and 3 of Definition 6.1.2 are easy to check. For condition 4 it suffices to show that if U is open and compact in $\mathcal{T}$ and $\forall b.t \in (b\ a) \cdot U$ then $t \in \bigcap^{#a} U$. This is condition 2 of Definition 7.9.1.

It is a fact that t^* is prime, since if $t \in U \cup V$ then $t \in U$ or $t \in V$. $\square$

COROLLARY 7.9.6. *Suppose a nominal σ -topological space $\mathcal{T}$ is coherent and sober. Then $\mathcal{U}$ is a prime filter in $G(\mathcal{T})$ if and only if $\mathcal{U} = t^*$ for some $t \in |\mathcal{T}|$, and that t is unique.*

As a corollary, the map $t \mapsto t^*$ is a bijection between $|\mathcal{T}|$ and $|FG(\mathcal{T})|$.

Proof. Suppose we are given $t \in |\mathcal{T}|$. By Proposition 7.9.5 t^* is a prime filter in $G(\mathcal{T})$, so we map t to $t^* \in |FG(\mathcal{T})|$.

Conversely suppose we are given $\mathcal{U} \in |FG(\mathcal{T})|$, so $\mathcal{U}$ is a prime filter in $G(\mathcal{T})$. By Lemma 7.6.2 $\mathcal{U}' = \{U' \in \text{Opn}(\mathcal{T}) \mid \exists U \in \mathcal{U}. U \subseteq U'\}$ is a completely prime filter. Since $\mathcal{T}$ is sober we can uniquely map $\mathcal{U}'$ to an element $t_{\mathcal{U}'} \in \mathcal{T}$ such that $\mathcal{U}' = t_{\mathcal{U}'}^*$.

The bijection follows. $\square$

LEMMA 7.9.7. *Suppose a nominal σ -topological space $\mathcal{T}$ is coherent and assume:*

1. *The map $t \in |\mathcal{T}| \mapsto t^* \in |FG(\mathcal{T})|$ (a point maps to the prime filter of compact open sets containing it) is a bijection.*
2. *If $t \in |\mathcal{T}|$ and $U \in \text{Cpct}(\mathcal{T})$ then $\forall b.t \in (b\ a) \cdot U$ implies $t \in \bigcap^{#a} U$.*

Then $\mathcal{T}$ is sober.

Proof. We examine Definition 7.9.1 and see that we must verify two conditions:

1. Suppose $\mathcal{U}' \subseteq \text{Opn}(\mathcal{T})$ is a completely prime filter in $\mathcal{T}$. We use the correspondence of Lemma 7.6.2 to map to $\mathcal{U} = \mathcal{U}' \cap \text{Cpct}(\mathcal{T})$ and then assumption 1 above to obtain a unique $t_{\mathcal{U}}$ such that $U \in \mathcal{U}$ if and only if $t_{\mathcal{U}} \in U$. It follows by construction of $\mathcal{U}$ that $U' \in \mathcal{U}'$ if and only if $t_{\mathcal{U}} \in U'$.
2. Condition 2 is immediate. $\square$

Definition 7.9.8 is in some sense dual to Definition 7.9.4, and will be useful:

DEFINITION 7.9.8. If $\mathcal{Y} \subseteq \text{Cpct}(F(\mathcal{D}))$ then define $\mathcal{Y}^{\bullet-1} \subseteq |\mathcal{D}|$ by

$$\mathcal{Y}^{\bullet-1} = \{y \in |\mathcal{D}| \mid y^{\bullet} \in \mathcal{Y}\},$$

so that $y \in \mathcal{Y}^{\bullet-1} \Leftrightarrow y^{\bullet} \in \mathcal{Y}$.

REMARK 7.9.9. By Theorem 7.4.3 an element of $\mathcal{Y}$ in Definition 7.9.8 has the form $y^{\bullet}$ for some $y \in |\mathcal{D}|$. So intuitively, Definition 7.9.8 just ‘strips the $\bullet$ s’ from inside $\mathcal{Y}$.

Recall the definitions of $x^{\bullet}$ and $p^{\bullet}$ from Definitions 6.3.1 and 7.9.4 respectively.

COROLLARY 7.9.10. *Suppose $\mathcal{D} \in \text{nDiV}$. Then:*

1. *If $p \in |F(\mathcal{D})|$ then $(p^{\bullet})^{\bullet-1} = p$.*
2. *If $\mathcal{U} \in |FGF(\mathcal{D})|$ then $(\mathcal{U}^{\bullet-1})^{\bullet} = \mathcal{U}$.*
3. *If $p \in |F(\mathcal{D})|$ and $U \in \text{Cpct}(F(\mathcal{D}))$ then $\forall b.p \in (b\ a) \cdot U$ implies $p \in \bigcap^{#a} U$.*

As a corollary, $F(\mathcal{D})$ is sober (Definition 7.9.1).

Proof. We unravel definitions and see that:

1. $x \in p$ if and only if $p \in x^\bullet$ if and only if $x^\bullet \in p^*$ if and only if $x \in (p^*)^{\bullet-1}$.
2. $x^\bullet \in \mathcal{U}$ if and only if $x \in \mathcal{U}^{\bullet-1}$ if and only if $\mathcal{U}^{\bullet-1} \in x^\bullet$ if and only if $x^\bullet \in (\mathcal{U}^{\bullet-1})^*$.
3. Suppose $p \in |F(\mathcal{D})|$ (so p is a prime filter in $\mathcal{D}$) and suppose $U \in \text{Cpct}(F(\mathcal{D}))$ (so U is a compact set of points). By Theorem 7.4.3 $U = x^\bullet$ for some $x \in |\mathcal{D}|$.

We reason as follows:

$$\begin{aligned}
 \text{Mb}.p \in (b\ a) \cdot x^\bullet &\Leftrightarrow p \in ((b\ a) \cdot x)^\bullet && \text{Lemma 6.4.1} \\
 &\Leftrightarrow \text{Mb}.(b\ a) \cdot x \in p && \text{Definition 6.3.1} \\
 &\Rightarrow \forall a.x \in p && \text{Definition 6.1.2 condition 4} \\
 &\Leftrightarrow p \in (\forall a.x)^\bullet && \text{Definition 6.3.1} \\
 &\Leftrightarrow p \in \bigcap^{#a} x^\bullet && \text{Lemma 6.4.2(3)}
 \end{aligned}$$

The corollary follows by combining parts 1 to 3 of this result with Lemma 7.9.7. $\square$

7.10. Nominal spectral spaces

DEFINITION 7.10.1. A **nominal spectral space** is a coherent (Definition 7.5.1) sober (Definition 7.9.1) impredicative nominal σ -topological space $\mathcal{T}$ (Definition 7.7.1).

PROPOSITION 7.10.2. If $\mathcal{D} \in \text{inD}\forall$ then $F(\mathcal{D})$ (Definition 7.2.1) is a nominal spectral space.

Proof. By Theorem 7.2.3 $F(\mathcal{D})$ is a σ -topological space. By Theorem 7.7.3 it is impredicative. By Proposition 7.5.4 it is coherent. By Corollary 7.9.10 it is sober. $\square$

8. Morphisms of nominal spectral spaces

8.1. The definition of $\text{inSpect}\forall$, and F viewed as a functor to it

We see from Definition 7.8.1 that we obtain a nominal distributive lattice with $\forall$ from an impredicative nominal spectral space by taking the lattice of compact open sets.

A *spectral* morphism is usually taken to be a map of points whose inverse preserves the property of being compact. Our compact sets have permutation and σ -actions (and our points have permutation and τ -actions) so we need morphisms to interact appropriately with this extra structure. This is Definition 8.1.1.

Then, we extend F from Definition 7.2.1 to act on morphisms, and check that this does indeed yield a functor. This is Definition 8.1.5 and Proposition 8.1.9. Theorem 8.1.10 packages this all up into a theorem.

DEFINITION 8.1.1. Suppose $\mathcal{S}$ and $\mathcal{T}$ are nominal spectral spaces (Definition 7.10.1).

Suppose $g_{\mathcal{T}} \in |\mathcal{T}| \rightarrow |\mathcal{S}|$. Then:

- Call $g_{\mathcal{T}}$ **continuous** when $X \in \text{Opn}(\mathcal{S})$ implies $(g_{\mathcal{T}})^{-1}(X) \in \text{Opn}(\mathcal{T})$ (inverse image of an open is open).

- Call $g_{\mathcal{T}}$ **spectral** when $X \in \text{Cpct}(\mathcal{S})$ implies $(g_{\mathcal{T}})^{-1}(X) \in \text{Cpct}(\mathcal{T})$ (inverse image of a compact open is compact open).

Call $g = (g_{\mathcal{T}}, g_{\mathcal{S}}^{\partial})$ a **morphism** from $\mathcal{T}$ to $\mathcal{S}$ when:

1. $g_{\mathcal{T}}$ is equivariant from $|\mathcal{T}|$ to $|\mathcal{S}|$, meaning that $\pi \cdot g_{\mathcal{T}}(t) = g_{\mathcal{T}}(\pi \cdot t)$.
2. $g_{\mathcal{T}} \in |\mathcal{T}| \rightarrow |\mathcal{S}|$ is continuous and spectral, and $g_{\mathcal{S}}^{\partial}$ is a σ -algebra morphism (Definition 4.3.5) from $\mathcal{S}^{\partial}$ to $\mathcal{T}^{\partial}$.
There is a ‘braided’ structure here, that $g_{\mathcal{T}}$ goes from $\mathcal{T}$ to $\mathcal{S}$ but $g_{\mathcal{S}}^{\partial}$ goes from $\mathcal{S}^{\partial}$ to $\mathcal{T}^{\partial}$. Of course, the inverse image $(g_{\mathcal{T}})^{-1}$ maps from $\text{Opn}(\mathcal{S})$ to $\text{Opn}(\mathcal{T})$ and so points in the same direction as $g_{\mathcal{S}}^{\partial}$. This is used in Lemma 8.1.3.
3. The inverse image $(g_{\mathcal{T}})^{-1}$ maps atoms-as-programs in $\mathcal{S}$ to atoms-as-programs in $\mathcal{T}$, meaning that $(g_{\mathcal{T}})^{-1}(\partial_{\mathcal{S}}a) = \partial_{\mathcal{T}}a$ for every atom a .
4. $g_{\mathcal{T}}$ commutes with the τ -action, meaning that for $u \in |\mathcal{S}^{\partial}|$ and $p \in |\mathcal{T}|$

$$g_{\mathcal{T}}(p)[u \leftarrow a] = g_{\mathcal{T}}(p[g_{\mathcal{S}}^{\partial}(u) \leftarrow a]).$$

See Lemma 8.1.3 for a view of this as “the inverse image g^{-1} commutes with the σ -action”.

Write $\text{inSpect}\mathcal{V}$ for the category of nominal spectral spaces, and morphisms between them.

We may drop subscripts and write $g = (g, g^{\partial}) : \mathcal{T} \rightarrow \mathcal{S}$, so that for example the equation above becomes

$$g(p)[u \leftarrow a] = g(p[g^{\partial}(u) \leftarrow a]).$$

Lemmas 8.1.2 and 8.1.3 are direct derivatives of condition 4 of Definition 8.1.1:

LEMMA 8.1.2. If $g : \mathcal{T} \rightarrow \mathcal{S}$ then for $p \in |\mathcal{T}|$ and $n \in \mathbb{A}$, $g(p)[n_{\mathcal{S}^{\partial}} \leftarrow a] = g(p[n_{\mathcal{T}^{\partial}} \leftarrow a])$.

Proof. The proof is just a special case of condition 4 of Definition 8.1.1, noting that g^{∂} is assumed to be a σ -algebra morphism, and by condition 1 of Definition 4.3.5 $g^{\partial}(n_{\mathcal{S}^{\partial}}) = n_{\mathcal{T}^{\partial}}$. $\square$

LEMMA 8.1.3. Suppose $g : \mathcal{T} \rightarrow \mathcal{S}$ and suppose $U \in \text{Cpct}(\mathcal{S})$ and $u \in |\mathcal{S}^{\partial}|$. Then

$$g^{-1}(U[a \mapsto u]) = g^{-1}(U)[a \mapsto g^{\partial}(u)].$$

(The σ -action $[a \mapsto u]$ and $[a \mapsto g^{\partial}(u)]$ is from Definition 3.4.1.)

Proof. Suppose $p \in |\mathcal{T}|$. Then:

$p \in g^{-1}(U[a \mapsto u]) \Leftrightarrow g(p) \in U[a \mapsto u]$	Pointwise action
$\Leftrightarrow \forall c. g(p)[u \leftarrow c] \in (c \ a) \cdot U$	Proposition 3.4.3
$\Leftrightarrow \forall c. g(p[g^{\partial}(u) \leftarrow c]) \in (c \ a) \cdot U$	C4 of Def 8.1.1
$\Leftrightarrow \forall c. p[g^{\partial}(u) \leftarrow c] \in (c \ a) \cdot g^{-1}(U)$	Pointwise action
$\Leftrightarrow p \in g^{-1}(U)[a \mapsto g^{\partial}(u)]$	Proposition 3.4.3 $\square$

We take a moment to note an interaction of Lemma 8.1.3 with condition 3 of Definition 8.1.1:

COROLLARY 8.1.4. g^{-1} maps programs to programs, meaning that $g^{-1}(\partial_S u) = \partial_T u$ for every $u \in |\mathcal{S}^\partial|$.

Proof. By condition 3 of Definition 8.1.1 $g^{-1}(\partial_S a) = \partial_T a$. We apply $[a \mapsto u]$ to both sides and use Lemma 8.1.3 to deduce that $g^{-1}(\partial_S u) = \partial_T u$. $\square$

Definition 8.1.5 extends Definition 7.2.1 from objects to morphisms (and is extended further to $\bullet$ and $\dashv$ in Definition 9.5.1):

DEFINITION 8.1.5. Given a morphism $f = (f_{\mathcal{D}}, f_{\mathcal{D}}^\partial) : \mathcal{D} \rightarrow \mathcal{D}'$ in $\text{inDi}\mathcal{V}$ (Definitions 4.4.8, 4.3.6, and 4.3.5) define $F(f) : F(\mathcal{D}') \rightarrow F(\mathcal{D})$ by

- $F(f)_{\mathcal{D}'}(p') = (f_{\mathcal{D}})^{-1}(p')$ where $p' \in |F(\mathcal{D}')|$
(so p' is a point—a prime filter—in $\mathcal{D}'$), and
- $F(f)_{\mathcal{D}}^\partial(u) = f_{\mathcal{D}^\partial}(u)$ where $u \in |\mathcal{D}^\partial|$.

That is, without subscripts:

$x \in F(f)(p) \Leftrightarrow f(x) \in p \quad \text{and} \quad F(f)^\partial(u) = f^\partial(u)$
--

REMARK 8.1.6 (A word on subscripts). Our convention is that a subscript on a function/morphism indicates domain/source. This information may elided, but the authors find some explicit bookkeeping helpful.

So consider $f = (f_{\mathcal{D}}, f_{\mathcal{D}}^\partial) : \mathcal{D} \rightarrow \mathcal{D}'$. Both components are subscripted with $\mathcal{D}$. So far, so normal.

Now consider Theorem 8.1.10, the culminating result of this Subsection. It proves that F is a functor from $\text{inDi}\mathcal{V}$ to $\text{inSpect}\mathcal{V}^{op}$ —that is, it is a contravariant functor from $\text{inDi}\mathcal{V}$ to $\text{inSpect}\mathcal{V}$. Thus

$$\text{if } f : \mathcal{D} \rightarrow \mathcal{D}' \in \text{inDi}\mathcal{V} \quad \text{then} \quad F(f) : F(\mathcal{D}') \rightarrow F(\mathcal{D}) \in \text{inSpect}\mathcal{V}.$$

Unpacking Definition 8.1.5 we see that $F(f)$ has two components called $F(f)_{\mathcal{D}'}$ and $F(f)_{\mathcal{D}}^\partial$, where $F(f)_{\mathcal{D}'}$ is subscripted with $\mathcal{D}'$ and $F(f)_{\mathcal{D}}^\partial$ is subscripted with $\mathcal{D}$.

- The first component $F(f)_{\mathcal{D}'}$ maps $|F(\mathcal{D}')|$ to $|F(\mathcal{D})|$ (by functional preimage $(f_{\mathcal{D}})^{-1}$, as standard), whereas
- the second component $F(f)_{\mathcal{D}}^\partial$ maps $|\mathcal{D}^\partial|$ to $|\mathcal{D}'^\partial|$.

F is contravariant and the ‘braided’ structure of $\text{inSpect}\mathcal{V}$ (clause 2 of Definition 8.1.1) causes subscripts on $F(f)$ to be $\mathcal{D}'$ for the first component ... and $\mathcal{D}$ for the second.

We now work towards proving that $F(f)$ maps points to points in Proposition 8.1.9.

LEMMA 8.1.7. Suppose $f = (f_{\mathcal{D}}, f_{\mathcal{D}}^\partial) : \mathcal{D} \rightarrow \mathcal{D}'$ is a morphism in $\text{inDi}\mathcal{V}$ and suppose $x \in |\mathcal{D}|$. Then $F(f)^{-1}(x^\bullet) = (f(x))^\bullet$.

Proof. We reason as follows, where $p \in |\text{Pnt}(\mathcal{D}')|$:

$p \in F(f)^{-1}(x^\bullet) \Leftrightarrow F(f)(p) \in x^\bullet$	Inverse image	
$\Leftrightarrow x \in F(f)(p)$	Definition 6.3.1	
$\Leftrightarrow f(x) \in p$	Definition 8.1.5	
$\Leftrightarrow p \in (f(x))^\bullet$	Definition 6.3.1	$\square$

LEMMA 8.1.8. Suppose $f : \mathcal{D} \rightarrow \mathcal{D}'$ is a morphism (Definition 4.4.8). Then if $p \subseteq |\mathcal{D}'|$ is a filter then so is $F(f)(p) = f^{-1}(p)$, and if p is prime then so is $f^{-1}(p)$.

Proof. Suppose p is a filter. We check the conditions of Definition 6.1.2 for $f^{-1}(p)$, freely using the fact that $x \in f^{-1}(p)$ if and only if $f(x) \in p$:

1. $\perp \notin f^{-1}(p)$. By assumption in Definition 4.3.6 $f(\perp) = \perp$, and by condition 1 of Definition 6.1.2 $\perp \notin p$.
2. $x \in f^{-1}(p)$ and $x \leq y$ implies $y \in f^{-1}(p)$. Immediate.
3. $x \in f^{-1}(p)$ and $y \in f^{-1}(p)$ implies $x \wedge y \in f^{-1}(p)$. Since $f(x \wedge y) = f(x) \wedge f(y)$.
4. If $\forall b. (b \cdot a \cdot x \in f^{-1}(p))$ then $\forall a. x \in f^{-1}(p)$. Suppose $\forall b. (b \cdot a \cdot x \in f^{-1}(p))$. It follows from Definition 4.3.6 that $\forall b. (b \cdot a) \cdot f(x) \in p$, so by condition 3 of Definition 6.1.2 $\forall a. f(x) \in p$. By Definition 4.3.6 again $\forall a. f(x) = f(\forall a. x)$. The result follows.

Now suppose p is prime and suppose $x \vee y \in f^{-1}(p)$, so that $f(x \vee y) \in p$. From Definition 4.3.6 we have $f(x) \vee f(y) \in p$. Since p is prime, either $f(x) \in p$ or $f(y) \in p$. $\square$

PROPOSITION 8.1.9. If $f : \mathcal{D} \rightarrow \mathcal{D}'$ is a morphism (Definition 4.4.8) then $F(f)$ from Definition 8.1.5 is a morphism from $F(\mathcal{D}')$ to $F(\mathcal{D})$ (Definition 8.1.1).

Proof. By Lemma 8.1.8 $F(f)$ maps prime filters of $\mathcal{D}'$ to prime filters of $\mathcal{D}$ —that is, $F(f)_{\mathcal{D}'} \in |F(\mathcal{D}')| \rightarrow |F(\mathcal{D})|$.

Now we show that $F(f)$ is a morphism. We verify the properties of Definition 8.1.1. By definition $F(f)^{\partial} = f^{\partial}$ which by construction is a σ -algebra morphism from $\mathcal{D}^{\partial}$ to $\mathcal{D}'^{\partial}$. Also:

1. $F(f)$ is equivariant. We briefly sketch the reasoning; in step (*) we use condition 1 of Definition 8.1.1 for f :

$$\begin{aligned} x \in \pi \cdot (F(f)(p)) &\Leftrightarrow \pi^{-1} \cdot x \in F(f)(p) \Leftrightarrow f(\pi^{-1} \cdot x) \in p \stackrel{(*)}{\Leftrightarrow} \pi^{-1} \cdot f(x) \in p \\ &\Leftrightarrow f(x) \in \pi \cdot p \Leftrightarrow x \in F(f)(\pi \cdot p) \end{aligned}$$

2. $F(f)^{-1}$ is continuous and spectral. We must prove two things:

- $F(f)^{-1}$ maps open sets to open sets. By construction $F(f)^{-1}$ preserves unions, and by construction in Definition 7.2.1(2) every $X \in \text{Opn}(F(\mathcal{D}))$ has the form of some small-supported sets union $\bigcup \mathcal{X}$ for some small-supported set $\mathcal{X} = \{x_i^* \mid i \in I\}$. Note by Corollary 6.3.3 that the assignment $x \mapsto x^*$ is injective. Therefore the map Q mapping $\mathcal{X} = \{x_i^* \mid i \in I\}$ to $Q\mathcal{X} = \{f(x_i)^* \mid i \in I\}$ is well-defined, and by Theorem 2.3.2 $Q\mathcal{X}$ has small support if $\mathcal{X}$ does. By Lemma 8.1.7 $F(f)^{-1}(x_i^*) = f(x_i)^*$. It follows that $F(f)^{-1}(X) = \bigcup_{i \in I} f(x_i)^*$ which by the note in the previous paragraph is a small-supported union and so is open in $F(\mathcal{D}')$.
- $F(f)^{-1}$ maps compact sets to compact sets. By Lemma 8.1.7 $F(f)^{-1}(x^*) = f(x)^*$. We use Theorem 7.4.3.

3. $F(f)^{-1}$ maps atoms-as-programs to themselves. By Lemma 8.1.7 $F(f)^{-1}((\partial_{\mathcal{D}}a)^{\bullet}) = (f(\partial_{\mathcal{D}}a))^{\bullet}$ and by assumption in Definition 4.4.8 $f(\partial_{\mathcal{D}}a) = \partial_{\mathcal{D}'}a$.
4. $F(f)$ commutes with the $\triangleright$ -action. Suppose $p' \in |F(\mathcal{D}')|$ and $u \in |F(\mathcal{D})^{\partial}|$ and $x \in |\mathcal{D}|$. Following Theorem 7.7.3 $|F(\mathcal{D})^{\partial}| = |\mathcal{D}^{\partial}|$ so $u \in |\mathcal{D}^{\partial}|$. We reason as follows:

$$\begin{aligned}
 x \in F(f)(p'[f^{\partial}(u) \leftarrow a]) &\Leftrightarrow f(x) \in p'[f^{\partial}(u) \leftarrow a] && \text{Definition 8.1.5} \\
 &\Leftrightarrow f(x)[a \mapsto f^{\partial}(u)] \in p' && \text{Proposition 3.3.2} \\
 &\Leftrightarrow f(x[a \mapsto u]) \in p' && \text{C3 of Def 4.3.5} \\
 &\Leftrightarrow x[a \mapsto u] \in F(f)(p') && \text{Definition 8.1.5} \\
 &\Leftrightarrow x \in (F(f)(p'))[u \leftarrow a] && \text{Proposition 3.3.2}
 \end{aligned}$$

□

THEOREM 8.1.10. F from Definitions 7.2.1 and 8.1.5 is a functor from inDiV to inSpectV^{op} .

Proof. By Proposition 7.10.2 F maps objects of inDiV to objects of inSpectV^{op} . Furthermore if $f : \mathcal{D} \rightarrow \mathcal{D}'$ in inDiV then by Proposition 8.1.9 $F(f)$ is a morphism from $F(\mathcal{D}')$ to $F(\mathcal{D})$. The result follows by some easy calculations. □

8.2. The action of G on morphisms in inSpectV

In Subsection 8.1 we went from inDiV (Definition 4.4.8) to inSpectV (Definition 8.1.1). Now we go back.

So Definition 8.2.1 mapping inSpectV to inDiV is the dual to Definition 8.1.5 mapping inDiV to inSpectV :

DEFINITION 8.2.1. Given $g : \mathcal{T}' \rightarrow \mathcal{T}$ in inSpectV define $G(g) : G(\mathcal{T}) \rightarrow G(\mathcal{T}')$ by $G(g)_{\mathcal{T}}(U) = (g_{\mathcal{T}'})^{-1}(U)$ and $G(g)_{\mathcal{T}^{\partial}}(u) = u$, that is (without subscripts):

$$t' \in G(g)(U) \Leftrightarrow g(t') \in U \quad \text{and} \quad G(g)^{\partial}(u) = g^{\partial}(u)$$

PROPOSITION 8.2.2. G from Definitions 7.8.1 and 8.2.1 is a functor from inSpectV^{op} to inDiV .

Proof. By Theorem 7.8.3 G maps objects of inSpectV^{op} to objects of inDiV .

Now consider a morphism $g : \mathcal{T}' \rightarrow \mathcal{T}$ in the sense of Definition 8.1.1; the interesting part is to check that $G(g)_{\mathcal{T}}$ —that is, $(g_{\mathcal{T}'})^{-1}$ —is a morphism in the sense of Definition 4.4.8.

We may drop subscripts henceforth. If $U \in \text{Cpct}(\mathcal{T})$, so U is compact, then since g is assumed spectral in Definition 8.1.1 also $g^{-1}(U)$ is compact. It is routine to check that g^{-1} preserves the top $\top$ and bottom elements ($|\mathcal{T}|$ and $\emptyset$ respectively) and interacts correctly with intersections and unions.

It remains to show that g^{-1} is equivariant, commutes with the σ -action, and commutes with $\bigcap^{\#a}$.

Equivariance, meaning that $\pi \cdot g^{-1}(U) = g^{-1}(\pi \cdot U)$, is immediate by Theorem 2.3.2 (a proof by concrete calculations using Proposition 3.4.3 and condition 1 of Definition 8.1.1 is also possible). Commuting with the σ -action, meaning that $(g_{\mathcal{T}'})^{-1}(U)[a \mapsto g_{\mathcal{T}}^{\partial}(u)] = (g_{\mathcal{T}'})^{-1}(U[a \mapsto u])$, is Lemma 8.1.3.

Finally we check that g^{-1} commutes with $\bigcap^{\#a}$. First, note that by assumption $\mathcal{T}, \mathcal{T}' \in \mathbf{inSpectV}$ so by Definition 8.1.1 they are sober and so satisfy Lemma 7.9.3. Now suppose $t' \in |\mathcal{T}'|$ and $U \in |G(\mathcal{T})| = Cpct(\mathcal{T})$. We reason as follows:

$$\begin{aligned} t \in g^{-1}(\bigcap^{\#a} U) &\Leftrightarrow g(t) \in \bigcap^{\#a} U && \text{Pointwise action} \\ &\Leftrightarrow \forall b. g(t) \in (b \ a) \cdot U && \text{Lemma 7.9.3} \\ &\Leftrightarrow \forall b. t \in (b \ a) \cdot g^{-1}(U) && \text{Pointwise action, Thm 2.3.2} \\ &\Leftrightarrow t \in \bigcap^{\#a} g^{-1}(U) && \text{Lemma 7.9.3} \end{aligned}$$

Thus $G(g)$ is a morphism in $\mathbf{inDiV}$. $\square$

REMARK 8.2.3. We continue Remark 7.9.2. Note in the proof of Lemma 7.9.3 that we prove a property of an infinite intersection using its characterisation using the $\forall$ -quantifier. We can do this thanks to sobriety; specifically, thanks to condition 2 of Definition 7.9.1 (which reflects condition 4 of Definition 6.1.2).

We want g^{-1} to commute with $\bigcap^{\#a}$ because in the duality g^{-1} should correspond to a morphism of nominal distributive lattices *with* $\forall$ (the formal statement and proof are in Subsection 8.3, below).

When we designed Definition 8.1.1 we could imagine just insisting on $g^{-1}(\bigcap^{\#a} U) = \bigcap^{\#a} g^{-1}(U)$. But this would have been unsatisfactory: we did not insist on $g^{-1}(U \cap V) = g^{-1}(U) \cap g^{-1}(V)$ or $g^{-1}(|\mathcal{T}| \setminus U) = |\mathcal{T}| \setminus g^{-1}(U)$ because these emerge from the structure of $\mathcal{T}$, by properties of sets of points and the inverse image. Imposing $g^{-1}(\bigcap^{\#a} U) = \bigcap^{\#a} g^{-1}(U)$ as a condition *on* g is not the best or most informative design choice; better to impose conditions *on* $\mathcal{T}$.

Thus, one way to read condition 2 of Definition 7.9.1 is that it gives $\mathcal{T}$ the structure to ensure that g^{-1} will commute with $\bigcap^{\#a}$ as well.

8.3. The equivalence

In Subsections 8.1 and 8.2 we considered two functors $F : \mathbf{inDiV} \rightarrow \mathbf{inSpectV}$ and $G : \mathbf{inSpectV} \rightarrow \mathbf{inDiV}$. They are dual; the key is to observe that $FG(\mathcal{T})$ is isomorphic to $\mathcal{T}$. This is Lemma 8.3.1 and Proposition 8.3.2. Theorem 8.3.3 puts it all together.

LEMMA 8.3.1. *Suppose $\mathcal{T} \in \mathbf{inSpectV}$ and $t \in |\mathcal{T}|$. If we give $t^* \in |FG(\mathcal{T})|$ from Definition 7.9.4 the pointwise $\triangleright$ -action from Definition 3.3.1 then for $U \in Cpct(\mathcal{T})$*

$$U \in t^*[u \leftarrow a] \Leftrightarrow \forall c. (c \ a) \cdot U \in (t[u \leftarrow c])^*.$$

Proof. We reason as follows:

$$\begin{aligned} U \in t^*[u \leftarrow a] &\Leftrightarrow U[a \rightarrow u] \in t^* && \text{Proposition 3.3.2} \\ &\Leftrightarrow t \in U[a \rightarrow u] && \text{Definition 7.9.4} \\ &\Leftrightarrow \forall c. t[u \leftarrow c] \in (c \ a) \cdot U && \text{Proposition 3.4.3} \\ &\Leftrightarrow \forall c. (c \ a) \cdot U \in (t[u \leftarrow c])^* && \text{Definition 7.9.4} \end{aligned} \quad \square$$

Recall from Definitions 7.2.1 and 7.8.1 that $FG(\mathcal{T})$ is a topological space whose points are prime filters of compact opens in $\mathcal{T}$.

PROPOSITION 8.3.2. *If $\mathcal{T} \in \mathbf{inSpectV}$ then $\alpha_{\mathcal{T}}$ mapping $t \in |\mathcal{T}|$ to $t^* \in |FG(\mathcal{T})|$ defines an isomorphism in $\mathbf{inSpectV}$ between $\mathcal{T}$ and $FG(\mathcal{T})$.*

Proof. Injectivity and surjectivity are Corollary 7.9.6. Commutativity with the $\triangleright$ -action is Lemma 8.3.1, as can be checked by unravelling definitions.

We also need to show that α is continuous. The reasoning is standard [BS81, Section 4] so we just sketch it. First, if $U \in G(\mathcal{T})$ (so U is a compact open set of $\mathcal{T}$) consider the inverse image under $-^*$ of $U^\bullet$.²⁷

$$\begin{aligned} t \in \alpha_{\mathcal{T}}^{-1}(U^\bullet) &\Leftrightarrow \alpha_{\mathcal{T}}(t) \in U^\bullet \\ &\Leftrightarrow \{U' \mid t \in U'\} \in U^\bullet \\ &\Leftrightarrow U \in \{U' \mid t \in U'\} \\ &\Leftrightarrow t \in U \end{aligned}$$

Thus, $\alpha_{\mathcal{T}}^{-1}(U^\bullet) = U$.

Now by construction any open set in $FG(\mathcal{T})$ is a union of $U^\bullet$, and it is a fact that the inverse image function $\alpha_{\mathcal{T}}^{-1}$ preserves these unions. It follows that the inverse image of an open set is open. $\square$

THEOREM 8.3.3. $G : \text{inSpect}\mathbf{V}^{\text{op}} \rightarrow \text{inDi}\mathbf{V}$ defines an equivalence between $\text{inDi}\mathbf{V}$ and $\text{inSpect}\mathbf{V}^{\text{op}}$.

Proof. We use [Mac71, Theorem 1, Chapter IV, Section 4].

- G is essentially surjective on objects. This is Proposition 7.8.4.
- G is faithful. Suppose $g_1, g_2 : \mathcal{T} \rightarrow \mathcal{S} \in \text{inSpect}\mathbf{V}$ and $g_1 \neq g_2$; the interesting case here is then that there exists $p \in |\mathcal{T}|$ such that $g_1(p) \neq g_2(p)$. (F and G leave programs unchanged, so we can elide g_1^∂ and g_2^∂ .)
By assumption $\mathcal{S}$ is coherent and sober, so that by Corollary 7.9.6 $g_1(p)^*$ and $g_2(p)^*$ —these are the sets of compact open sets in $\mathcal{S}$ containing $g_1(p)$ and $g_2(p)$ respectively—are distinct.
Thus there exists a compact open set $U \in \text{Opn}(\mathcal{S})$ with $g_1(p) \in U$ and $g_2(p) \notin U$. Examining Definition 8.2.1 we see that $p \in G(g_1)(U)$ and $p \notin G(g_2)(U)$. Thus, $G(g_1) \neq G(g_2)$.
- G is full. Given $\mathcal{S}, \mathcal{T}$ in $\text{inSpect}\mathbf{V}$ and $f : G(\mathcal{S}) \rightarrow G(\mathcal{T})$ in $\text{inDi}\mathbf{V}$ we construct a morphism $g : \mathcal{T} \rightarrow \mathcal{S}$ in $\text{inSpect}\mathbf{V}$ such that $G(g) = f$.
By Proposition 8.3.2 $\alpha_{\mathcal{T}} : \mathcal{T} \rightarrow FG(\mathcal{T})$ mapping t to t^* is an isomorphism in $\text{inSpect}\mathbf{V}$. Set $g = \alpha_{\mathcal{S}}^{-1} \circ F(f) \circ \alpha_{\mathcal{T}}$. By routine calculations we can check that $G(g)(U') = f(U')$ for every $U' \in |G(\mathcal{S})|$. $\square$

²⁷Unpacking Definitions 6.3.1 and 7.8.1, $U^\bullet$ is the set of prime filters of compact opens of $\mathcal{T}$ of which U is an element.

Part III

Adding application and its topological dual the combination operator

So far we have seen $\text{inDi}\forall$ and $\text{inSpect}\forall$, and Theorem 8.3.3 is a topological duality theorem relating them. This is in itself an interesting result: duality for an impredicative propositional logic (propositional logic with quantifiers over propositions; the reader might be familiar with this kind of logical system in the form of the type system of System F [GTL89]).

However, to model the λ -calculus we need more structure.

This is developed in Section 9, and our results so far are extended accordingly—culminating in Subsection 9.6 with Theorem 9.6.6.

9. $\text{inDi}\forall_\bullet$ and $\text{inSpect}\forall_\bullet$

9.1. Adding $\bullet$ and $\multimap$ to $\text{inDi}\forall$ to get $\text{inDi}\forall_\bullet$

DEFINITION 9.1.1. We extend the notion of an impredicative nominal distributive lattice with $\forall$ from Definition 4.4.1 with two equivariant operators $\bullet : (\mathcal{X} \times \mathcal{X}) \Rightarrow \mathcal{X}$ and $\multimap : (\mathcal{X} \times \mathcal{X}) \Rightarrow \mathcal{X}$, written infix as $x \bullet y$ and $y \multimap x$. They must be **adjoint** as described in Figure 3, and they must be **compatible** as described in Figure 4 (the notation ∂b is from Notation 4.4.3).

$$\begin{array}{ll} (\bullet\epsilon) & (u \multimap x) \bullet u \leq x \\ (\bullet\eta) & x \leq u \multimap (x \bullet u) \end{array}$$

Figure 3: Adjointness properties for $\bullet$ and $\multimap$

$$\begin{array}{lll} (\sigma\bullet) & (x \bullet y)[b \mapsto v] = (x[b \mapsto v]) \bullet (y[b \mapsto v]) & \\ (\sigma\multimap) & (\partial b \multimap x)[a \mapsto v] = \partial b \multimap (x[a \mapsto v]) & \\ (\bullet\perp) & \perp \bullet u = \perp & x \bullet \perp = \perp \\ (\bullet\wedge) & (x \wedge y) \bullet u \leq (x \bullet u) \wedge (y \bullet u) & x \bullet (u \wedge v) \leq (x \bullet u) \wedge (x \bullet v) \\ (\bullet\vee) & (x \vee y) \bullet u = (x \bullet u) \vee (y \bullet u) & x \bullet (u \vee v) = (x \bullet u) \vee (x \bullet v) \\ (\bullet\forall) & b \# u \Rightarrow (\forall b. x) \bullet u \leq \forall b. (x \bullet u) & \\ (\multimap\wedge) & u \multimap (x \wedge y) = (u \multimap x) \wedge (u \multimap y) & \\ (\multimap\vee) & u \multimap (x \vee y) \geq (u \multimap x) \vee (u \multimap y) & \\ (\multimap\forall) & b \# u \Rightarrow \forall b. (u \multimap x) \leq u \multimap (\forall b. x) & \end{array}$$

Figure 4: Compatibility properties for $\bullet$ and $\multimap$

REMARK 9.1.2. In categorical terminology, axiom $(\bullet\epsilon)$ is a *counit* and $(\bullet\eta)$ is a *unit*. In Proposition 10.2.4 we will derive β -reduction from $(\bullet\epsilon)$ and η -expansion from $(\bullet\eta)$. Later on in Remark 11.5.8 we will examine how $\dashv\bullet$ behaves in a concrete model.

REMARK 9.1.3. In the presence of $(\sigma\#)$ there is redundancy in $(\sigma\dashv\bullet)$; we could take $(\sigma\dashv\bullet)$ to be $\partial b\dashv\bullet(x[a\mapsto v]) \leq (\partial b\dashv\bullet x)[a\mapsto v]$ and get the reverse inequality from Lemma 9.1.9 (below). The form given in Definition 9.1.1 is slightly more convenient to work with.

REMARK 9.1.4. $(\dashv\bullet\forall)$ is an inequality, not an equality. This seems odd, given that $(\dashv\bullet\wedge)$ is an equality; is not $\forall$ intuitively an infinite conjunction or a fresh-finite limit? The reason is that this reflects the *inequality* in Lemma 9.1.9 below. To see how this works, we refer the interested reader to the case of $(\dashv\bullet\forall)$ in Lemma 9.6.3. The even more interested reader is referred to Proposition 9.3.3, where $(\dashv\bullet\forall)$ is just what we need for the final stages of the proof.

REMARK 9.1.5. $(\sigma\dashv\bullet)$ might seem odd: why $\partial b\dashv\bullet x$ and not $y\dashv\bullet x$? The precise technical reason is in Proposition 11.6.4; the $\partial b\dashv\bullet x$ form is what we prove of our canonical syntactic model Pnt_{Π} .

We package our definitions up as a category:

DEFINITION 9.1.6. Continuing Definition 9.1.1, write $\text{inDiV}_{\bullet}$ for the category with objects impredicative nominal distributive lattices with $\forall$, $\bullet$, and $\dashv\bullet$, and morphisms are morphisms in inDiV (Definition 4.4.9) that commute with $\bullet$ and $\dashv\bullet$ in the following sense:

$$\begin{aligned} f(x\bullet y) &= f(x)\bullet f(y) \quad \text{and} \\ f(\partial b\dashv\bullet x) &= \partial b\dashv\bullet f(x). \end{aligned}$$

For more on ∂b see Remarks 4.4.4 and 4.4.10.

We conclude with technical lemmas concerning the interaction of $\bullet$ and $\dashv\bullet$ with $\leq$ and the σ -action. For the rest of this subsection we fix $\mathcal{D} \in \text{inDiV}_{\bullet}$ and $x, y, x', y' \in |\mathcal{D}|$ and $u \in |\mathcal{D}^{\partial}|$.

LEMMA 9.1.7. 1. If $x \leq x'$ then $x\bullet y \leq x'\bullet y$ and $y\bullet x \leq y\bullet x'$.
2. If $x \leq x'$ then $y\dashv\bullet x \leq y\dashv\bullet x'$.

Proof. It is a fact that $x \leq x'$ if and only if $x\forall x' = x'$, if and only if $x = x\wedge x'$. We reason as follows:

1. $x'\bullet y = (x\forall x')\bullet y \stackrel{(\bullet\forall)}{=} (x\bullet y)\forall (x'\bullet y)$. The proof that $y\bullet x \leq y\bullet x'$ is similar.
2. $y\dashv\bullet x = y\dashv\bullet (x\wedge x') \stackrel{(\dashv\bullet\wedge)}{=} (y\dashv\bullet x)\wedge (y\dashv\bullet x') \leq y\dashv\bullet x'$ □

LEMMA 9.1.8. $x\bullet y \leq z$ if and only if $x \leq y\dashv\bullet z$.

Proof. The reasoning is standard:

- If $x\bullet y \leq z$ then $x \stackrel{(\bullet\eta)}{\leq} y\dashv\bullet(x\bullet y) \stackrel{\text{L9.1.7(2)}}{\leq} y\dashv\bullet z$.
- If $x \leq y\dashv\bullet z$ then $x\bullet y \stackrel{\text{L9.1.7(1)}}{\leq} (y\dashv\bullet z)\bullet y \stackrel{(\bullet\epsilon)}{\leq} z$. □

LEMMA 9.1.9. $(y\dashv\bullet x)[a\mapsto u] \leq y[a\mapsto u]\dashv\bullet x[a\mapsto u]$.

Proof. We reason as follows:

$$\begin{aligned}
 x[a \mapsto u] &\leq x[a \mapsto u] \\
 &\Rightarrow^{(\dagger)} ((y \multimap x) \bullet y)[a \mapsto u] \leq x[a \mapsto u] && \text{Lem 4.2.2, } (\bullet\eta) \\
 &\Rightarrow^{(*)} (y \multimap x)[a \mapsto u] \bullet y[a \mapsto u] \leq x[a \mapsto u] && \text{From } (\sigma\bullet) \\
 &\Leftrightarrow (y \multimap x)[a \mapsto u] \leq y[a \mapsto u] \multimap x[a \mapsto u] && \text{Lemma 9.1.8} \quad \square
 \end{aligned}$$

REMARK 9.1.10 (An aside on explicit substitutions). Note the implications $\Rightarrow^{(\dagger)}$ and $\Rightarrow^{(*)}$ in the proof of Lemma 9.1.9 above. The $\Rightarrow^{(*)}$ could be replaced by a stronger if-and-only-if $\Leftrightarrow$. This would make no difference to the final statement of the Lemma, because of the $\Rightarrow^{(\dagger)}$ immediately above. We have a reason for writing the proof above as we have done, which we now describe:

Recall from Figure 4 that $(\sigma\bullet)$ is an equality (that $(x\bullet y)[b \mapsto v] = (x[b \mapsto v])\bullet(y[b \mapsto v])$ for all x, y , and v) and so can be viewed as two inequalities. The precise inequality required to derive $\Rightarrow^{(*)}$ in the proof of Lemma 9.1.9 above is that

$$(y \multimap x)[a \mapsto u] \bullet y[a \mapsto u] \leq ((y \multimap x) \bullet y)[a \mapsto u].$$

This paper is based on λ -calculus without explicit substitutions (see the absence of an explicit substitution [ACCL91] in Definitions 10.1.1 or 10.3.1 below). So substitution is not a reduction step, and this is why $(\sigma\bullet)$ is an equality.

To model explicit substitutions we would naturally weaken $(\sigma\bullet)$ from Figure 4 to an inequality that $(x\bullet y)[b \mapsto v] \leq (x[b \mapsto v])\bullet(y[b \mapsto v])$ and condition 6 of Definition 9.2.7 to a subset inclusion, reflecting that substitution becomes a distinct reduction step.²⁸ We already noted in Remark 9.1.3 that $(\sigma\multimap)$ could already be taken as an inequality at some small cost in complexity. So this seems very natural.

Therefore, it seems worth noting that the inequality we need for the proof above is $((y \multimap x) \bullet y)[a \mapsto u] \leq (y \multimap x)[a \mapsto u] \bullet y[a \mapsto u]$, which is not the inequality we would expect to get from the reduction rule for an explicit substitution distributing over an application. We would lose Lemma 9.1.9 in its current form.

This is probably not fatal: the uses of Lemma 9.1.9 required for this paper to work are for the special case that $a \# y$ (Proposition 10.2.6 uses the full result but is not itself needed for the rest of the paper). This suggests that a suitable generalisation would include an inequality $(x\bullet y)[b \mapsto v] \leq (x[b \mapsto v])\bullet(y[b \mapsto v])$, along with an equality $x[b \mapsto v] \bullet y = (x\bullet y)[b \mapsto v]$ if $a \# y$ (which is the special case for which we need Lemma 9.1.9), and this for a calculus for explicit substitutions which includes a reduction $(st)[a \mapsto u] \rightarrow (s[a \mapsto u])(t[a \mapsto u])$, along with a syntactic equivalence $s[a \mapsto u]t = (st)[a \mapsto u]$ (or just a reduction $s[a \mapsto u]t \rightarrow (st)[a \mapsto u]$) if $a \# t$.

Investigating further which explicit substitution calculi this paper would naturally generalise to, if any, is future research.

9.2. The combination operator $\circ$: a topological dual to $\bullet$ and $\multimap$

In Subsection 9.1 we extended inDiV with extra structure $\bullet$ and $\multimap$.

²⁸This is nice also semantically: it is easier to build models of the inequality and subset inclusion than of the equality.

We can expect this to be reflected in the topologies by some kind of extension with a structure dual to $\bullet$ and $\dashv$.²⁹

What should dually correspond to $\bullet$ and $\dashv$? Remarkably, this requires only a little more structure on points: see the combination operator $\circ$ in Definition 9.2.1.

9.2.1. The basic definition

Recall the small-supported powerset $Nom(-)$ from Subsection 2.5.1.

DEFINITION 9.2.1. We define an $\mathfrak{o}$ -**algebra with $\circ$** or $\mathfrak{o}$ -**algebra with combination** by extending the notion of $\mathfrak{o}$ -algebra $\mathcal{P}$ from Definition 3.2.1 with an equivariant **combination operator**

$$\circ : (\mathcal{P} \times \mathcal{P}) \Rightarrow Nom(\mathcal{P}),$$

written infix as $p \circ q$.

An $\mathfrak{o}$ -algebra with combination is just an $\mathfrak{o}$ -algebra with an equivariant *combination function* $\circ$ mapping pairs of points to sets of points. We now outline how the notion of spectral space is enriched by assuming $\circ$ (Definition 9.2.1) on its points.

We extend Definition 7.1.1:

DEFINITION 9.2.2. A **nominal σ -topological space with $\circ$** $\mathcal{T}$ (or just ‘ $\sigma\circ$ -topological space’) is a nominal σ -topological space (Definition 7.1.1) whose points have the additional structure of a combination operator $\circ$ (Definition 9.2.1).

REMARK 9.2.3. What concrete models of Definition 9.2.2 look like is a very interesting question—especially what the *combination operator* $\circ$ looks like.

Concrete models of $\circ$ arise from Kripke-style models with a *ternary* accessibility relation. We refer the interested reader to [GG10, Definition 2.1] and the discussion in [GG10, Subsection 5.2], which notes the similarity (and differences) of the Kripke model of $\circ$ compared with multiplicative conjunction, phase spaces, and certain models of relevance logic.

Combination $\circ$ will be dual to λ -calculus application $\bullet$. Since the dual to sigma is *amgis*, perhaps $\circ$ should be called *ppa*, pronounced ‘pah’ to rhyme with ‘bah’. *Combination* is a more serious name for it.

9.2.2. $\bullet$ and $\dashv$, and spectral spaces

Extend Definition 3.4.1 as follows:

DEFINITION 9.2.4. Suppose $\mathcal{P} = (|\mathcal{P}|, \cdot, \mathcal{P}^\partial, \text{amgis}, \circ)$ is an $\mathfrak{o}$ -algebra with $\circ$. Suppose $X, Y \subseteq |\mathcal{P}|$ and suppose $p \in |\mathcal{P}|$. Then the following notation will be useful:

$$\begin{aligned} X \bullet Y &= \bigcup \{p \circ q \mid p \in X, q \in Y\} \\ Y \dashv X &= \{p \mid \forall q \in Y. p \circ q \subseteq X\} \\ p \circ Y &= \bigcup \{p \circ q \mid q \in Y\}. \end{aligned}$$

²⁹For instance, $\mathcal{D} \in \text{inDiV}$ has a σ -action, and this is reflected dually as an $\mathfrak{o}$ -action on prime filters.

Similarly, $\mathcal{D}$ has a permutation action; permutations are invertible, so the dual structure on prime filters is ...another permutation action.

LEMMA 9.2.5. $Y \multimap X$ can be conveniently rewritten as

$$Y \multimap X = \{p \mid p \circ Y \subseteq X\}.$$

Proof. By routine calculations. $\square$

We can extend Proposition 3.4.3 to reflect the structure created by the combination action $\circ$:

PROPOSITION 9.2.6. We have the following:

$$\begin{aligned} r \in X \bullet Y &\Leftrightarrow \exists p \in X, q \in Y. r \in p \circ q & r \in Y \multimap X &\Leftrightarrow r \circ Y \subseteq X \\ & & &\Leftrightarrow \forall q \in Y. r \circ q \subseteq X \end{aligned}$$

We extend Definition 7.5.1 to account for the extra structure:

DEFINITION 9.2.7. Call a nominal σ -topological space $\mathcal{T}$ with $\circ$ **coherent** when it is coherent in the sense of Definition 7.5.1 and in addition, for all X and Y open and compact (so $X, Y \in \text{Cpct}(\mathcal{T})$):

5. $X \bullet Y$ and $Y \multimap X$ are open and compact.
6. $(X \bullet Y)[a \mapsto u] = X[a \mapsto u] \bullet Y[a \mapsto u]$ (so $\text{Cpct}(\mathcal{T})$ satisfies $(\sigma \bullet)$).
7. $(\partial_{\mathcal{T}} b \multimap X)[a \mapsto u] = \partial_{\mathcal{T}} b \multimap (X[a \mapsto u])$ (so $\text{Cpct}(\mathcal{T})$ satisfies $(\sigma \multimap)$).

LEMMA 9.2.8. Suppose $\mathcal{T}$ is a nominal σ -topological space with $\circ$ (Definition 9.2.2) and $X, Y \subseteq |\mathcal{T}|$. Then

$$\forall W \in \text{Opn}(\mathcal{T}). (W \subseteq Y \multimap X \Leftrightarrow W \bullet Y \subseteq X).$$

If furthermore $\mathcal{T}$ is coherent and $X, Y \in \text{Cpct}(\mathcal{T})$, then

$$Y \multimap X = \bigcup \{W \in \text{Cpct}(\mathcal{T}) \mid W \bullet Y \subseteq X\}.$$

Proof. Consider $W \in \text{Opn}(\mathcal{T})$ and $X, Y \subseteq |\mathcal{T}|$.

By Lemma 9.2.5 $q \circ Y \subseteq X$ for any $q \in Y$ ($q \circ Y$ is from Definition 9.2.4). It follows from Definition 9.2.4 that $W \bullet Y \subseteq X$. Conversely if $W \bullet Y \subseteq X$ then by Definition 9.2.4, $p \circ Y \subseteq X$ for every $p \in W$, and thus by Lemma 9.2.5 also $p \in Y \multimap X$ for every $p \in W$. So $W \subseteq Y \multimap X$.

By the first part, $\bigcup \{W \in \text{Opn}(\mathcal{T}) \mid W \bullet Y \subseteq X\} \subseteq Y \multimap X$. If in addition $\mathcal{T}$ is coherent then if $X, Y \in \text{Cpct}(\mathcal{T})$ then $Y \multimap X \in \text{Cpct}(\mathcal{T})$ and so $Y \multimap X$ is one of the W such that $W \bullet Y \subseteq X$, so $Y \multimap X \subseteq \bigcup \{W \in \text{Opn}(\mathcal{T}) \mid W \bullet Y \subseteq X\}$. $\square$

Definition 9.2.9 extends Definition 7.10.1:

DEFINITION 9.2.9. A **nominal spectral space with** $\circ$ is a nominal $\sigma \circ$ -topological space $\mathcal{T}$ (Definition 9.2.2) that is impredicative (7.7.1), coherent (Definition 9.2.7), and sober (Definition 7.9.1).

DEFINITION 9.2.10. We extend the notion of **morphism** $g : \mathcal{T}' \rightarrow \mathcal{T}$ of nominal spectral spaces from Definition 8.1.1 to insist that the inverse image g^{-1} should commute with $\bullet$ and $\multimap$ in the following sense:

- If $X, Y \in \text{Cpct}(\mathcal{T})$ then $g^{-1}(X \bullet Y) = g^{-1}(X) \bullet g^{-1}(Y)$.
- If $X \in \text{Cpct}(\mathcal{T})$ then $g^{-1}(\partial b \dashv X) = \partial b \dashv g^{-1}(X)$.

Write $\text{inSpect}\mathbb{V}_\bullet$ for the category of nominal spectral spaces with $\circ$ (Definition 9.2.9), and morphisms between them whose inverse image functions commute with $\bullet$ and $\dashv$, as described above.

9.2.3. Useful technical lemmas

We conclude with a pair of useful technical lemmas:

- LEMMA 9.2.11. $\bullet (\bigcup_i X_i) \bullet (\bigcup_j Y_j) = \bigcup_{i,j} (X_i \bullet Y_j)$.
- $\emptyset \bullet Y = \emptyset = X \bullet \emptyset$.
 - If $X \subseteq X'$ and $Y \subseteq Y'$ then $X \bullet Y \subseteq X' \bullet Y'$.

Proof. By elementary sets calculations on Definition 9.2.4. □

LEMMA 9.2.12. The adjoint axioms $(\bullet \epsilon)$ and $(\bullet \eta)$ from Definition 9.1.1 hold. That is, for $X, Y \subseteq |\mathcal{P}|$:

$$(U \dashv X) \bullet U \subseteq X \quad \text{and} \quad X \subseteq U \dashv (X \bullet U)$$

Proof. Suppose $r \in (U \dashv X) \bullet U$. By Proposition 9.2.6 there exist $p \in U \dashv X$ and $q \in U$ such that $r \in p \circ q$, and by Proposition 9.2.6 again, $p \circ U \subseteq X$. Therefore by Definition 9.2.4 $p \circ q \subseteq X$ and in particular $r \in X$ as required.

Now consider $p \in X$. By Proposition 9.2.6 it suffices to show that $p \circ U \subseteq X \bullet U$. But this is immediate from Definition 9.2.4. □

9.3. Filters in the presence of $\bullet$ and $\dashv$

The notions of filter and ideal from Definitions 6.1.2 and 6.1.3 do not change with the addition of $\bullet$ and $\dashv$. This is very convenient, because it leaves unaffected the ‘logical’ structure studied previously to Section 9, and the theorems we proved so far still hold.

However, the addition of $\bullet$ and $\dashv$ adds structure, and this gives us three useful ways to build new filters out of old ones, which we now consider. Fix some $\mathcal{D} \in \text{inDi}\mathbb{V}_\bullet$ (Definition 9.1.6).

9.3.1. Combining filters $I(q \circ y)$

DEFINITION 9.3.1. Suppose q is a filter in $\mathcal{D}$ and $y \in |\mathcal{D}|$. Define $q \circ y \subseteq |\mathcal{D}|$ by

$$q \circ y = \{x \in |\mathcal{D}| \mid y \dashv x \in q\}$$

A justification for the notation $q \circ y$ is Lemma 9.3.2, which exhibits $q \circ y$ as a kind of dual to $y \dashv x$:

LEMMA 9.3.2. $x \in q \circ y$ if and only if $y \dashv x \in q$.

Proof. Routine from Definition 9.3.1. □

PROPOSITION 9.3.3. Suppose q is a filter in $\mathcal{D}$ and $y \in |\mathcal{D}|$. Then:

1. $q \circ y$ satisfies conditions 2, 3, and 4 of Definition 6.1.2.

2. If q is prime then $q \circ y$ in addition satisfies the primeness condition (Definition 6.1.16).
3. If q is a prime filter in $\mathcal{D}$ and $\perp \notin q \circ y$ then $q \circ y$ is a prime filter.

Proof. Part 3 follows from parts 1 and 2 since the only remaining condition from the conditions of Definitions 6.1.2 and 6.1.16 that $q \circ y$ can fail to satisfy, in order to avoid being a prime filter, is condition 1 of Definition 6.1.2.

We now prove parts 1 and 2 of this result by checking the conditions of Definitions 6.1.2 and 6.1.16 for $q \circ x$, freely using Lemma 9.3.2:

2. If $x \in q \circ y$ and $x \leq x'$ then $x' \in q \circ y$. If $x \in q \circ y$ then $y \multimap x \in q$. By part 2 of Lemma 9.1.7 and condition 2 of Definition 6.1.2 for q we have $y \multimap x' \in q$, and so $x' \in q \circ y$.
3. If $x \in q \circ y$ and $x' \in q \circ y$ then $x \wedge x' \in q \circ y$. Suppose $x, x' \in q \circ y$, so that $y \multimap x, y \multimap x' \in q$. Therefore $(y \multimap x) \wedge (y \multimap x') \in q$ and by $(\multimap \wedge) y \multimap (x \wedge x') \in q$. Thus $x \wedge x' \in q \circ y$.
4. If $\forall b.(b \ a) \cdot x \in q \circ y$ then $\forall a.x \in q \circ y$. We note that for fresh a' (so $a' \# x, y$)³⁰ if $b \# x$ then by Corollary 2.3.5 $(b \ a) \cdot x = (b \ a') \cdot ((a' \ a) \cdot x)$ and by Lemma 4.1.3 $\forall a.x = \forall a'.(a' \ a) \cdot x$. So using Proposition 2.3.4 we may assume without loss of generality that $a \# y$.

Now suppose $\forall b.(b \ a) \cdot x \in q \circ y$. By Lemma 9.3.2 $\forall b.(y \multimap (b \ a) \cdot x \in q)$, and by Lemma 2.4.2 (for the equivariant function $\multimap$) $\forall b.(b \ a) \cdot (y \multimap x) \in q$. By assumption q is prime, so that by condition 4 of Definition 6.1.2 $\forall a.(y \multimap x) \in q$, thus by $(\multimap \forall)$ and condition 2 of Definition 6.1.2 $y \multimap \forall a.x \in q$ and so by Lemma 9.3.2 $\forall a.x \in q \circ y$.

5. If $x \vee x' \in q \circ y$ then either $x \in q \circ y$ or $x' \in q \circ y$. Suppose $x \vee x' \in q \circ y$. By Lemma 9.3.2 $y \multimap (x \vee x') \in q$, so that (since q is prime; Definition 6.1.16) $y \multimap x \in q$ or $y \multimap x' \in q$. We use Lemma 9.3.2 again. $\square$

9.3.2. Combining filters II ($q \bullet x$)

We prove Lemmas 9.3.5 and 9.3.6, which will help us prove Theorem 9.4.7. These lemmas are versions of (and corollaries of) Theorem 6.1.23, but for the applicative structure.

Definition 9.3.4 is essentially a special case of Definition 9.4.3 for the case of $q \bullet (x \uparrow)$.³¹

DEFINITION 9.3.4. If q is a filter in $\mathcal{D}$ and $x \in |\mathcal{D}|$ then define $q \bullet x \subseteq |\mathcal{D}|$ by

$$q \bullet x = \{y \bullet x \mid y \in q\}.$$

$q \bullet x$ is not necessarily a filter, but the notation will be useful.

LEMMA 9.3.5. Suppose $r \subseteq |\mathcal{D}|$ is a prime filter and $y \in |\mathcal{D}|$.

³⁰We do not assume that q has small support so we do not know that $a' \# q$, but that will not be a problem.

³¹It is *not* literally true that $q \bullet x = q \bullet (x \uparrow)$, just because $q \bullet x$ is not up-closed (neither is $q \bullet (x \uparrow)$). This will not matter.

1. If $\top \bullet y \in r$ then

$$I = \{x \in |\mathcal{D}| \mid x \bullet y \notin r\}$$

is an ideal (Definition 6.1.3).

2. For any $x \in |\mathcal{D}|$, if $x \bullet y \in r$ then there exists a prime filter p such that $x \in p$ and $p \bullet y \subseteq r$.

Proof. 1. By $(\bullet \perp)$ of Figure 4 $\perp \bullet y = \perp$ and by condition 1 of Definition 6.1.2 $\perp \notin r$. Thus $\perp \in I$ and I is nonempty.

We now verify that I satisfies the other conditions of Definition 6.1.3:

- (a) We assumed $\top \bullet y \in r$, so $\top \notin I$.
 - (b) By condition 2 of Definition 6.1.2 r is up-closed so that by Lemma 9.1.7(1) I is down-closed.
 - (c) We use $(\bullet \vee)$ of Figure 4. If $x \bullet y \notin r$ and $x' \bullet y \notin r$ then by primeness of r (Definition 6.1.16) it follows that $(x \bullet y) \vee (x' \bullet y) \stackrel{(\bullet \vee)}{=} (x \vee x') \bullet y \notin r$.
2. Suppose $x \bullet y \in r$; then by Lemma 9.1.7(1) $\top \bullet y \in r$ and we use Theorem 6.1.23 for the (by Lemma 6.1.9(1)) small-supported filter $x \uparrow$ and the (by part 1 of this result) ideal I . $\square$

LEMMA 9.3.6. Suppose $p, r \subseteq |\mathcal{D}|$ are prime filters and $y \in |\mathcal{D}|$. Then:

1. If $p \bullet \top \subseteq r$ then

$$J = \{y \in |\mathcal{D}| \mid p \bullet y \not\subseteq r\}$$

is an ideal (Definition 6.1.3).

2. For any $y \in |\mathcal{D}|$, if $p \bullet y \subseteq r$ then there exists a prime filter q such that $y \in q$ and $p \bullet q \subseteq r$.

Proof. 1. Using $(\bullet \perp)$ of Figure 4 $p \bullet \perp = \{\perp\}$ and by condition 1 of Definition 6.1.2 $\{\perp\} \not\subseteq r$. Thus $\perp \in J$ and J is nonempty.

We now verify that J satisfies the other conditions of Definition 6.1.3:

- (a) We assumed $p \bullet \top \subseteq r$, so $\top \notin J$.
 - (b) By condition 2 of Definition 6.1.2 r is up-closed so that by Lemma 9.1.7(1) J is down-closed.
 - (c) Suppose $p \bullet y \not\subseteq r$ and $p \bullet y' \not\subseteq r$; then there exist $x \in p$ and $x' \in p$ such that $x \bullet y \notin r$ and $x' \bullet y' \notin r$. By up-closure of r (condition 2 of Definition 6.1.2) and Lemma 9.1.7(1) $(x \wedge x') \bullet y \notin r$ and $(x \wedge x') \bullet y' \notin r$ so that by primeness of r (Definition 6.1.16) $((x \wedge x') \bullet y) \vee ((x \wedge x') \bullet y') \stackrel{(\bullet \vee)}{=} (x \wedge x') \bullet (y \vee y') \notin r$. Thus $p \bullet (y \vee y') \not\subseteq r$.
2. Suppose $p \bullet y \subseteq r$; then by Lemma 9.1.7(1) $p \bullet \top \subseteq r$ and we use Theorem 6.1.23 for the (by Lemma 6.1.9(1)) small-supported filter $y \uparrow$ and the (by part 1 of this result) ideal J . $\square$

REMARK 9.3.7. I and J from Lemmas 9.3.5 and 9.3.6 are ideals, but they need not be small-supported, because the prime filter r need not be small-supported.

9.4. The second representation theorem

For the rest of this subsection fix some $\mathcal{D} \in \text{inDiV}_\bullet$ (Definition 9.1.6). We now show how to build an $\circ$ -algebra with $\circ$ using $\mathcal{D}$. This is reasonable; recall from Subsection 9.2 that $\circ$ is the topological dual to application $\bullet$.

9.4.1. The basic actions

The first two clauses of Definition 9.4.1 echo Definition 3.3.1; the third gives prime filters a $\circ$ action in the sense of Definition 9.2.1. Recall that $x^\bullet$ is from Definition 6.3.1:

DEFINITION 9.4.1. Give prime filters p and q in $\mathcal{D}$ actions as follows:

$$\begin{aligned} \pi \cdot p &= \{\pi \cdot x \mid x \in p\} & p[u \leftarrow a] &= \{x \mid x[a \mapsto u] \in p\} \\ p \circ q &= \bigcap \{(x \bullet y)^\bullet \mid x \in p, y \in q\} \end{aligned}$$

PROPOSITION 9.4.2. Prime filters of $\mathcal{D}$ form an $\circ$ -algebra with $\circ$ in the sense of Definition 9.2.1.

Proof. Equivariance of $\circ$ is immediate from Theorem 2.3.2. We just use Proposition 6.2.4. $\square$

We now work towards Proposition 9.4.6, which will be needed for Theorem 9.4.7.

9.4.2. Combining filters III ($p \bullet q$)

We can extend the applicative structure $x \bullet y$ from Definition 9.1.1, to points $p \bullet q$:

DEFINITION 9.4.3. Suppose p and q are prime filters in $\mathcal{D}$ and $y \in |\mathcal{D}|$. Define $p \bullet q$ by

$$p \bullet q = \{x \bullet y \mid x \in p, y \in q\}$$

$p \bullet q$ is not necessarily a filter; for instance, it is not necessarily up-closed (condition 2 of Definition 6.1.2). However, we have Lemmas 9.4.4 and 9.4.5:

LEMMA 9.4.4. Suppose p , q , and r are prime filters in $\mathcal{D}$. Then $r \in p \circ q$ if and only if $p \bullet q \subseteq r$.

Proof. By construction in Definition 9.4.1. $\square$

LEMMA 9.4.5. Suppose p and r are prime filters in $\mathcal{D}$ and suppose $y \in |\mathcal{D}|$. Then $r \in p \circ (y^\bullet)$ if and only if there exists a prime filter q in $\mathcal{D}$ such that $y \in q \wedge p \bullet q \subseteq r$.

Proof. Recall from Definition 6.3.1 that if $y \in |\mathcal{D}|$ then $y^\bullet$ is the set of prime filters in $\mathcal{D}$ containing y . Recall from Definition 9.2.4 that $r \in p \circ y^\bullet$ if and only if $p \bullet q \subseteq r$ for some prime filter $q \in y^\bullet$. The result follows. $\square$

9.4.3. The representation theorem

PROPOSITION 9.4.6. *Suppose r is a prime filter in $\mathcal{D}$ and $x, y \in |\mathcal{D}|$. Then $r \circ (y^\bullet) \subseteq x^\bullet$ if and only if $y \multimap x \in r$.*

Proof. We prove two implications.

The right-to-left implication. Suppose $y \multimap x \in r$ and suppose $p \in r \circ y^\bullet$. We need to show that $p \in x^\bullet$.

By Lemma 9.4.5 $p \in r \circ y^\bullet$ means that for some q with $y \in q$ it is the case that $r \bullet q \subseteq p$. So given some such q , since $y \multimap x \in r$ we have that $(y \multimap x) \bullet y \in p$. By $(\bullet \epsilon)$ $(y \multimap x) \bullet y \leq x$ and since p is up-closed (condition 2 of Definition 6.1.2) we have $x \in p$, thus by Definition 6.3.1 $p \in x^\bullet$ as required.

The left-to-right implication. Suppose $r \circ y^\bullet \subseteq x^\bullet$; so if p is a prime filter and $p \in r \circ y^\bullet$, then $x \in p$. We need to show that $y \multimap x \in r$.

Consider $r \circ y$ from Definition 9.3.1. There are two cases:

- Suppose $\perp \in r \circ y$. Then by Lemma 9.3.2 $y \multimap \perp \in r$ and by Lemma 9.1.7 $y \multimap x \in r$ as required.
- Suppose $\perp \notin r \circ y$. Then by Proposition 9.3.3(3) $r \circ y$ is a prime filter.

By Lemma 6.1.9 $y^\uparrow$ is a filter, and by Lemma 9.3.6(2) there exists a prime filter q with $y^\uparrow \subseteq q$ and $r \bullet q \subseteq r \circ y$. Now $y \in y^\uparrow \subseteq q$ so we have a prime filter q with $y \in q$ and $r \bullet q \subseteq r \circ y$. By Lemma 9.4.5 $p \in r \circ y^\bullet$. Thus by assumption $x \in p$ and by Lemma 9.3.2 $y \multimap x \in r$ as required. $\square$

Theorem 9.4.7 extends Lemma 6.4.2 for the additional structure of $\bullet$ and $\multimap$:

THEOREM 9.4.7. *Suppose $\mathcal{D} \in \text{inDiV}_\bullet$ and $x, y \in |\mathcal{D}|$. Then:*

1. $x^\bullet \bullet y^\bullet = (x \bullet y)^\bullet$.
2. $y^\bullet \multimap x^\bullet = (y \multimap x)^\bullet$.

Proof. We consider each part in turn. Below, p, q , and r range over prime filters in $\mathcal{D}$.

1. By Definition 6.3.1 and Proposition 9.2.6 $r \in x^\bullet \bullet y^\bullet$ if and only if $\exists p, q. (x \in p \wedge y \in q) \wedge r \in p \bullet q$, and by Lemma 9.4.4 this is if and only if $\exists p, q. (x \in p \wedge y \in q) \wedge p \bullet q \subseteq r$.

So suppose there exist prime filters p and q with $x \in p$ and $y \in q$ and $p \bullet q \subseteq r$. Then clearly $x \bullet y \in r$.

Conversely suppose $x \bullet y \in r$ for some prime filter r . By Lemma 9.3.5(2) there exists a prime filter p with $x \in p$ and $p \bullet y \subseteq r$, and by Lemma 9.3.6(2) there exists a prime filter q with $y \in q$ and $p \bullet q \subseteq r$.

2. We reason as follows, using Propositions 9.2.6 and 9.4.6:

$$r \in y^\bullet \multimap x^\bullet \stackrel{\text{Prop 9.2.6}}{\Leftrightarrow} r \circ y^\bullet \subseteq x^\bullet \stackrel{\text{Prop 9.4.6}}{\Leftrightarrow} r \in (y \multimap x)^\bullet \quad \square$$

9.4.4. Properties of the representation

We can now easily extend Definition 6.4.3 (the definition of $\mathcal{D}^\bullet$):

DEFINITION 9.4.8. As in Definition 6.4.3 we take $\mathcal{D}^\bullet$ to have:

- $|\mathcal{D}^\bullet| = \{x^\bullet \mid x \in |\mathcal{D}|\}$
- $(\mathcal{D}^\bullet)^\partial = \mathcal{D}^\partial$ and $\partial_{\mathcal{D}^\bullet} u = (\partial_{\mathcal{D}} u)^\bullet$
- $\pi \cdot (x^\bullet) = (\pi \cdot x)^\bullet$
- $x^\bullet[a \mapsto u] = (x[a \mapsto u])^\bullet$

We give $\mathcal{D}^\bullet$ the actions $X \bullet Y$ and $Y \multimap X$ from Definition 9.2.4.

So Definitions 6.4.3 and 9.4.8 overload the notation $\mathcal{D}^\bullet$ for “the object in inDiV composed of prime filters of $\mathcal{D} \in \text{inDiV}$ ” and “the object of $\text{inDiV}_\bullet$ composed of prime filters of $\mathcal{D} \in \text{inDiV}_\bullet$ ”. The meaning will always be clear.

LEMMA 9.4.9. Suppose $x, y \in |\mathcal{D}|$ and $u \in |\mathcal{D}^\partial|$. Then $(\sigma \bullet)$ and $(\sigma \multimap)$ are valid in $\mathcal{D}^\bullet$:

$$\begin{aligned} (x^\bullet \bullet y^\bullet)[a \mapsto u] &= x^\bullet[a \mapsto u] \bullet (y^\bullet[a \mapsto u]) \\ (\partial b \multimap x^\bullet)[a \mapsto u] &= \partial b \multimap (x^\bullet[a \mapsto u]) \end{aligned}$$

Proof. We reason as follows:

$$\begin{aligned} (x^\bullet \bullet y^\bullet)[a \mapsto u] &= (x \bullet y)^\bullet[a \mapsto u] && \text{Theorem 9.4.7} \\ &= ((x \bullet y)[a \mapsto u])^\bullet && \text{Lemma 6.4.1} \\ &= (x[a \mapsto u] \bullet y[a \mapsto u])^\bullet && (\sigma \bullet) \\ &= (x[a \mapsto u])^\bullet \bullet (y[a \mapsto u])^\bullet && \text{Theorem 9.4.7} \\ &= x^\bullet[a \mapsto u] \bullet y^\bullet[a \mapsto u] && \text{Lemma 6.4.1} \\ (\partial_{\mathcal{D}} b \multimap x^\bullet)[a \mapsto u] &= (\partial_{\mathcal{D}} b \multimap x)^\bullet[a \mapsto u] && \text{Theorem 9.4.7} \\ &= (\partial_{\mathcal{D}} b \multimap x)[a \mapsto u]^\bullet && \text{Lemma 6.4.1} \\ &= (\partial_{\mathcal{D}} b \multimap x[a \mapsto u])^\bullet && (\sigma \multimap) \\ &= \partial_{\mathcal{D}} b \multimap (x[a \mapsto u])^\bullet && \text{Theorem 9.4.7} \quad \square \end{aligned}$$

LEMMA 9.4.10. The adjoint and compatibility axioms from Definition 9.1.1 are valid in $\mathcal{D}^\bullet$. That is:

$$\begin{aligned} (\bullet \epsilon) \quad & (u^\bullet \multimap x^\bullet) \bullet x^\bullet \subseteq x^\bullet \\ (\bullet \eta) \quad & x^\bullet \subseteq u^\bullet \multimap (x^\bullet \bullet u^\bullet) \\ (\bullet \perp) \quad & \emptyset \bullet u^\bullet = \emptyset \quad \quad \quad x^\bullet \bullet \emptyset = \emptyset \\ (\bullet \wedge) \quad & (x^\bullet \cap y^\bullet) \bullet u^\bullet \subseteq (x^\bullet \bullet u^\bullet) \cap (y^\bullet \bullet u^\bullet) \quad x^\bullet \bullet (u^\bullet \cap v^\bullet) \subseteq (x^\bullet \bullet u^\bullet) \cap (x^\bullet \bullet v^\bullet) \\ (\bullet \vee) \quad & (x^\bullet \cup y^\bullet) \bullet u^\bullet = (x^\bullet \bullet u^\bullet) \cup (y^\bullet \bullet u^\bullet) \quad x^\bullet \bullet (u^\bullet \cup v^\bullet) = (x^\bullet \bullet u^\bullet) \cup (x^\bullet \bullet v^\bullet) \\ & \dots \\ (\multimap \forall) \quad & b \# u^\bullet \Rightarrow \bigcap^{\#b} (u^\bullet \multimap x^\bullet) \subseteq u^\bullet \multimap (\bigcap^{\#b} x^\bullet) \end{aligned}$$

Proof. The easiest proof is to combine Theorem 9.4.7 and Lemma 6.4.2 with Lemma 6.3.2 and with the relevant axiom for $\mathcal{D}$.

In the cases of $(\bullet \forall)$ and $(\multimap \forall)$ which have a freshness condition, we use Lemma 2.4.1 to choose a suitably fresh representative of $u^\bullet$ (i.e. a u' such that $(u')^\bullet = u^\bullet$ and $b \# u'$).³² $\square$

³²Lemma 2.4.1 is applied here to the function $u \mapsto u^\bullet$. Equivariance is defined in Definition 2.1.8; Definition 9.4.8 states (amongst other things) that this function is equivariant.

Recall the definition of $\text{inDi}\mathbb{V}_\bullet$ from Definition 9.1.6.

THEOREM 9.4.11 (Second representation theorem). *If $\mathcal{D} \in \text{inDi}\mathbb{V}_\bullet$, then $\mathcal{D}^\bullet$ from Definition 9.4.8 is in $\text{inDi}\mathbb{V}_\bullet$, and the assignment $x \mapsto x^\bullet$ is an isomorphism in $\text{inDi}\mathbb{V}_\bullet$ between $\mathcal{D}$ and $\mathcal{D}^\bullet$.*

Proof. Theorem 6.4.4 handles the purely logical structure ($\cap$, $\emptyset$, $\cup$, and $\bigcap^{\#a}$). Lemmas 9.4.9 and 9.4.10 validate the axioms for $\bullet$ and $\dashv\bullet$. $\square$

9.5. Construction of the topological space $F(\mathcal{D})$, with $\bullet$ and $\dashv\bullet$

Recall the definitions of $\text{inDi}\mathbb{V}_\bullet$ and $\text{inSpect}\mathbb{V}_\bullet$ from Definitions 9.1.6 and 9.2.10. We will now extend Definitions 7.2.1 and 8.1.5:

DEFINITION 9.5.1. Suppose $\mathcal{D} \in \text{inDi}\mathbb{V}_\bullet$. Define $F(\mathcal{D}) \in \text{inSpect}\mathbb{V}_\bullet$ by:

1. $|F(\mathcal{D})| = |\text{Pnt}(\mathcal{D})|$ and $F(\mathcal{D})^\partial = \mathcal{D}^\partial$.
2. $\pi \cdot p = \{\pi \cdot x \mid x \in p\}$.
3. $p[u \leftarrow a] = \{x \mid x[a \mapsto u] \in p\}$.
4. $p \circ q = \bigcap \{(x \bullet y)^\bullet \mid x \in p, y \in q\}$, following Definition 9.4.1.
5. $\text{Opn}(F(\mathcal{D}))$ is the closure of $\{x^\bullet \mid x \in |\mathcal{D}|\}$ under small-supported unions.
6. $\partial_{F(\mathcal{D})}$ maps $u \in |\mathcal{D}^\partial|$ to $(\partial_{\mathcal{D}} u)^\bullet$.³³

In Definition 9.5.1 we claim that $F(\mathcal{D})$ is a nominal spectral space with $\circ$ (Definition 9.2.9). This needs to be proved: Theorem 9.5.3 assembles the various verifications.

Recall from Definition 8.1.5 the map from $f : \mathcal{D} \rightarrow \mathcal{D}'$ to $F(f) : F(\mathcal{D}') \rightarrow F(\mathcal{D})$.

PROPOSITION 9.5.2. *If $f : \mathcal{D} \rightarrow \mathcal{D}'$ is in $\text{inDi}\mathbb{V}_\bullet$, then $F(f)$ commutes with $\bullet$ and $\dashv\bullet$ as specified in Definition 9.5.1. That is, using Theorem 7.4.3:*

1. $F(f)^{-1}(x^\bullet \bullet y^\bullet) = F(f)^{-1}(x^\bullet) \bullet F(f)^{-1}(y^\bullet)$
2. $F(f)^{-1}(\partial b \dashv\bullet x^\bullet) = \partial b \dashv\bullet F(f)^{-1}(x^\bullet)$.

As a corollary, if $f : \mathcal{D} \rightarrow \mathcal{D}'$ then $F(f) : F(\mathcal{D}') \rightarrow F(\mathcal{D})$ is a morphism in the sense of Definition 9.2.10.

Proof. The corollary follows direct from Definition 9.2.10 using Theorem 8.1.10.

For the first part, we reason as follows:

³³It might be helpful to unwind the definitions for this final clause. This is not complicated—it just takes in a lot of definitions!

$u \in |\mathcal{D}^\partial|$ is an element of the termlike σ -algebra over which $F(\mathcal{D})$ has an $\mathfrak{o}$ -action; $\partial_{\mathcal{D}} u \in |\mathcal{D}|$ is an element of $\mathcal{D}$ the (impredicative) nominal distributive lattice with $\mathbb{V}$ and $\bullet$; $(\partial_{\mathcal{D}} u)^\bullet$ is the set of prime filters in $\text{Pnt}(\mathcal{D})$ that contain $\partial_{\mathcal{D}} u$. By Theorem 7.4.3(1) this set of prime filters is compact, that is it is in $\text{Cpct}(\mathcal{T})$, as required in Definition 7.7.1.

$$\begin{aligned}
 r \in F(f)^{-1}(x^\bullet \bullet y^\bullet) &\Leftrightarrow F(f)(r) \in x^\bullet \bullet y^\bullet && \text{Inverse image} \\
 &\Leftrightarrow F(f)(r) \in (x \bullet y)^\bullet && \text{Theorem 9.4.7} \\
 &\Leftrightarrow x \bullet y \in F(f)(r) && \text{Definition 6.3.1} \\
 &\Leftrightarrow f(x \bullet y) \in r && \text{Definition 8.1.5} \\
 &\Leftrightarrow f(x) \bullet f(y) \in r && \text{Definition 9.1.6} \\
 &\Leftrightarrow r \in (f(x) \bullet f(y))^\bullet && \text{Definition 6.3.1} \\
 &\Leftrightarrow r \in f(x)^\bullet \bullet f(y)^\bullet && \text{Theorem 9.4.7} \\
 &\Leftrightarrow r \in F(f)^{-1}(x^\bullet) \bullet F(f)^{-1}(y^\bullet) && \text{Lemma 8.1.7}
 \end{aligned}$$

The reasoning for the second part, for $\dashv$, is similar:

$$\begin{aligned}
 r \in F(f)^{-1}(\partial_{F(\mathcal{D})} b \dashv x^\bullet) &\Leftrightarrow F(f)(r) \in \partial_{F(\mathcal{D})} b \dashv x^\bullet && \text{Inverse image} \\
 &\Leftrightarrow F(f)(r) \in (\partial_{\mathcal{D}} b \dashv x)^\bullet && \text{Theorem 9.4.7} \\
 &\Leftrightarrow \partial_{\mathcal{D}} b \dashv x \in F(f)(r) && \text{Definition 6.3.1} \\
 &\Leftrightarrow f(\partial_{\mathcal{D}} b \dashv x) \in r && \text{Definition 8.1.5} \\
 &\Leftrightarrow \partial_{\mathcal{D}'} b \dashv f(x) \in r && \text{Definition 9.1.6} \\
 &\Leftrightarrow r \in (\partial_{\mathcal{D}'} b \dashv f(x))^\bullet && \text{Definition 6.3.1} \\
 &\Leftrightarrow r \in (\partial_{\mathcal{D}'} b)^\bullet \dashv f(x)^\bullet && \text{Theorem 9.4.7} \\
 &\Leftrightarrow r \in \partial_{F(\mathcal{D}')} b \dashv F(f)^{-1}(x^\bullet) && \text{Lemma 8.1.7} \quad \square
 \end{aligned}$$

Theorem 9.5.3 extends Theorem 8.1.10 from $\text{inDi}\mathcal{V}$ and $\text{inSpect}\mathcal{V}$ to $\text{inDi}\mathcal{V}_\bullet$ and $\text{inSpect}\mathcal{V}_\bullet$:

THEOREM 9.5.3. *F is a functor from $\text{inDi}\mathcal{V}_\bullet$ to $\text{inSpect}\mathcal{V}_\bullet$.*

Proof. This is mostly Theorem 8.1.10 combined with Proposition 9.5.2. We do also need to check the extra conditions on coherence from Definition 9.2.7; these follow easily from Theorem 9.4.7, Lemma 9.4.9, and Theorem 7.4.3. $\square$

Part 1 of Proposition 9.5.2 is stated only for sets of the form $x^\bullet$ and $y^\bullet$. In fact, we note that it can be extended to all $X, Y \in \text{Opn}(F(\mathcal{D}))$:

COROLLARY 9.5.4. *$F(f)^{-1}(X \bullet Y) = F(f)^{-1}(X) \bullet F(f)^{-1}(Y)$ for all $X, Y \in \text{Opn}(F(\mathcal{D}))$ (and not only $X, Y \in \text{Cpct}(F(\mathcal{D}))$).*

Proof. By construction in Definition 7.2.1 every open set in $F(\mathcal{D})$ is a small-supported union of compact opens. By Theorem 7.4.3 compact opens have in $F(\mathcal{D})$ the form $x^\bullet$ and $y^\bullet$ for $x, y \in |\mathcal{D}|$. We use Lemma 9.2.11(1) and Proposition 9.5.2(1). $\square$

Corollary 9.5.4 would not work for $\dashv$, because a result corresponding to Lemma 9.2.11 does not hold for it.

9.6. The duality, in the presence of $\bullet$ and $\dashv$

It is routine to extend Definition 7.8.1—which sends a spectral space $\mathcal{T}$ to the lattice of its compact open sets $G(\mathcal{T})$ —and Definition 8.2.1—which sends a spectral map g to its inverse image function $G(g) = g^{-1}$ —to the case where we also assume $\bullet$ and $\dashv$. We write it out, just to be clear:

DEFINITION 9.6.1. If $\mathcal{T} \in \text{inSpect}\mathcal{V}_\bullet$ (Definition 9.2.9) define $G(\mathcal{T}) \in \text{inDi}\mathcal{V}_\bullet$ (Definition 9.1.6) by:

- $|G(\mathcal{T})| = \text{Cpct}(\mathcal{T})$ and $G(\mathcal{T})^\partial = \mathcal{T}^\partial$.
- $\pi \cdot U = \{\pi \cdot p \mid p \in U\}$ and $U[a \mapsto u] = \{p \mid p[u \leftarrow a] \in U\}$, where $U \in |G(\mathcal{T})|$ and $u \in |G(\mathcal{T})^\partial|$.
- $\top$, $\wedge$, $\perp$, $\vee$, and $\forall$ are interpreted as the whole underlying set, set intersection, the empty set, set union, and $\bigcap^{\#a}$.
- $X \bullet Y$ and $Y \multimap X$ are interpreted as specified in Definition 9.2.4.

Given $g : \mathcal{T} \rightarrow \mathcal{T}' \in \text{inSpect}\mathbf{V}_\bullet$ from Definition 9.2.10, define $G(g) : G(\mathcal{T}') \rightarrow G(\mathcal{T})$ by $G(g)(U) = g^{-1}(U)$.

LEMMA 9.6.2. *Continuing the notation of Definition 9.6.1, suppose $U, V \in |G(\mathcal{T})|$ and suppose $\mathcal{U} \subseteq |G(\mathcal{T})|$. Then*

$$\bigcap_{U \in \mathcal{U}} (V \multimap U) = V \multimap \bigcap_{U \in \mathcal{U}} U.$$

Proof. Using Proposition 9.2.6 $r \in V \multimap \bigcap U$ if and only if $r \circ V \subseteq U$ for every $U \in \mathcal{U}$, and by Proposition 9.2.6 again this is if and only if $r \in V \multimap U$ for every $U \in \mathcal{U}$, which is if and only if $r \in \bigcap_{U \in \mathcal{U}} V \multimap U$. $\square$

LEMMA 9.6.3. *If $\mathcal{T} \in \text{inSpect}\mathbf{V}_\bullet$, then $G(\mathcal{T})$ validates the axioms from Figure 4 (Definition 9.1.1).*

Proof. We consider each axiom in turn. We take $X, Y, X', U \in \text{Cpct}(\mathcal{T})$ (open com-
pacts in $\mathcal{T}$):

- *Axioms $(\sigma \bullet)$ and $(\sigma \multimap)$.* By assumption in Definition 9.2.7.
- *The adjoint axioms $(\bullet \epsilon)$ and $(\bullet \eta)$.* Direct from Lemma 9.2.12.
- *Axioms $(\bullet \perp)$ and $(\bullet \vee)$.* ... are Lemma 9.2.11.
- *Axioms $(\bullet \wedge)$.* If $r \in (X \cap X') \bullet Y$ then $r \in p \circ q$ for some $p \in X \cap X'$ and $q \in Y$. It follows that $r \in p \circ q$ for $p \in X$ and $q \in Y$ and $r \in p \circ q$ for $p \in X'$ and $q \in Y$, and therefore $r \in (X \bullet Y) \cap (X' \bullet Y)$. The second $(\bullet \wedge)$ axiom follows similarly.
- *Axiom $(\bullet \forall)$.* Suppose $b \# U$ and $r \in (\bigcap^{\#a} X) \bullet U$. It follows by Lemma 9.2.11 that $r \in X[a \mapsto w] \bullet U$ for every $w \in |\mathcal{T}^\partial|$. Now by part 1 of Lemma 3.4.7 and condition 6 of Definition 9.2.7, $X[a \mapsto w] \bullet U = (X \bullet U)[a \mapsto w]$. So $r \in (X \bullet U)[a \mapsto w]$ for every $w \in |\mathcal{T}^\partial|$, and by Definition 5.2.1 we have $r \in \bigcap^{\#a} (X \bullet U)$.
- *Axiom $(\multimap \wedge)$.* From Lemma 9.6.2.
- *Axiom $(\multimap \vee)$.* $r \in (U \multimap X) \cup (U \multimap X')$ means $r \circ U \subseteq X$ or $r \circ U \subseteq X'$. In either case, $r \circ U \subseteq X \cup X'$ and this means $r \in U \multimap (X \cup X')$.
- *Axiom $(\multimap \forall)$.* Suppose $b \# U$. We reason as follows:

$$\begin{aligned} \bigcap^{\#b} (U \multimap X) &= \bigcap_{u \in |\mathcal{T}^\partial|} (U \multimap X)[b \mapsto u] && \text{Definition 5.2.1} \\ &\subseteq \bigcap_{u \in |\mathcal{T}^\partial|} U[b \mapsto u] \multimap X[b \mapsto u] && \text{Lemma 9.1.9} \\ &= \bigcap_{u \in |\mathcal{T}^\partial|} U \multimap X[b \mapsto u] && (\sigma \#), b \# U \\ &= U \multimap \bigcap_{u \in |\mathcal{T}^\partial|} X[b \mapsto u] && \text{Lemma 9.6.2} \\ &= U \multimap \bigcap^{\#b} X && \text{Definition 5.2.1} \quad \square \end{aligned}$$

PROPOSITION 9.6.4. G from Definition 9.6.1 is a functor from $\text{inSpect}\mathbb{V}_\bullet^{\text{op}}$ to $\text{inDi}\mathbb{V}_\bullet$.

Proof. The action on objects is handled by Theorem 7.8.3 and Lemma 9.6.3. The action on morphisms is handled by Proposition 8.2.2 and by the two conditions on g^{-1} in Definition 9.2.10. $\square$

We checked in Proposition 9.5.2 that this is true for $g = F(f)^{-1}$, so we can extend Proposition 7.8.4:

PROPOSITION 9.6.5. If $\mathcal{D} \in \text{inDi}\mathbb{V}_\bullet$, then $GF(\mathcal{D})$ is equal to $\mathcal{D}^\bullet$ from Definition 9.4.8, and the map $x \mapsto x^\bullet$ is an isomorphism in $\text{inDi}\mathbb{V}_\bullet$.

Proof. Just as the proof of Proposition 7.8.4; the extra structure of $\bullet$ and $\dashv\bullet$ has no effect. We use Theorem 9.4.11. $\square$

It is routine to check that the $\bullet$ and $\dashv\bullet$ structure is orthogonal to the material of Subsections 8.2 and 8.3, and so we obtain Theorem 9.6.6:

THEOREM 9.6.6 (The duality theorem). $G : \text{inSpect}\mathbb{V}_\bullet^{\text{op}} \rightarrow \text{inDi}\mathbb{V}_\bullet$ defines an equivalence between $\text{inDi}\mathbb{V}_\bullet$ and $\text{inSpect}\mathbb{V}_\bullet^{\text{op}}$.

Theorem 9.6.6 exhibits $\text{inDi}\mathbb{V}_\bullet$ and $\text{inSpect}\mathbb{V}_\bullet$ (Definitions 9.1.6 and 9.2.10) as dual to one another. This is a general result—the abstract nominal algebra structures in $\text{inDi}\mathbb{V}_\bullet$ correspond dually to concrete topological spaces in $\text{inSpect}\mathbb{V}_\bullet$.

It remains to show how $\text{inDi}\mathbb{V}_\bullet$ and $\text{inSpect}\mathbb{V}_\bullet$ relate specifically to the untyped λ -calculus.

Part IV

Application to the λ -calculus

10. The λ -calculus

In this section we sketch the untyped λ -calculus and show how it has been living inside $\text{inDi}\mathbf{V}_\bullet$ all along: this is $\lambda a.x$ in Notation 10.2.1. We make formal that Notation 10.2.1 is ‘a right thing to do’ with Proposition 10.2.4, Definition 10.4.1, and Theorem 10.4.7.

We also briefly unpack what $\lambda a.X$ is when X is an open set in the topological representations in $\text{inSpect}\mathbf{V}_\bullet$. This is Proposition 10.2.6.

Thus, we leverage our topological duality to give both abstract and concrete (i.e. nominal poset flavoured and nominal sets flavoured) semantics for the λ of the untyped λ -calculus.

10.1. Syntax of the λ -calculus

DEFINITION 10.1.1. Define **λ -terms** as usual by

$$s ::= a \mid \lambda a.s \mid s' s$$

where a ranges over atoms (so we use atoms as variable symbols, in nominal style).³⁴

- We treat λ -terms as equal up to α -equivalence.³⁵
- We assume capture-avoiding substitution $s[a:=u]$.
- We write $fa(s)$ for the free atoms (variables) of s .

DEFINITION 10.1.2. Consider λ -terms as a nominal set (Definition 2.1.6) by giving them the natural permutation action:

$$\pi \cdot a = \pi(a) \quad \pi \cdot (\lambda a.s) = \lambda \pi(a). \pi \cdot s \quad \pi \cdot (s' s) = (\pi \cdot s')(\pi \cdot s)$$

Write LmTm for the nominal set of λ -terms with this permutation action.

It is a fact that with the permutation action above, $fa(s)$ the free atoms of s and $\text{supp}(s)$ the atoms in the support of s , coincide.

DEFINITION 10.1.3. Consider λ -terms as a termlike σ -algebra (Definition 3.1.1) by setting

$$s[a \mapsto u] = s[a:=u],$$

so that $[a \mapsto u]$ acting on s is ‘ s with a substituted for u ’.

It is a fact that this does indeed determine a termlike σ -algebra. The nominal algebra axioms of Figure 1 reflect valid properties of capture-avoiding substitution on λ -terms.

³⁴We could allow constants c too, if we wished.

³⁵... using nominal abstract syntax [Gab01, GP01] or by taking equivalence classes or by whatever other method the reader prefers.

10.2. λ , β , and η using adjoints

In objects of $\text{inDi}\mathbb{V}_\bullet$, λ -abstraction arises naturally by combining the ‘logical’ structure $\forall$ and $\leq$ with the ‘combinational’ structure of $\bullet$ and $\multimap$; this is Notation 10.2.1. We shall see that β -reduction and η -expansion arise as natural corollaries of the adjoint properties of $\bullet$ and $\multimap$; this is Proposition 10.2.4.

Proposition 10.2.6 unpacks what this means in $F(\mathcal{D})$, and Definition 10.4.1 and Theorem 10.4.7 show how we can interpret the full untyped λ -calculus.

10.2.1. λ using $\forall$ and $\multimap$

NOTATION 10.2.1. Suppose $\mathcal{D} \in \text{inDi}\mathbb{V}_\bullet$ and $x \in |\mathcal{D}|$. Write $\lambda a.x$ for $\forall a.(\partial a \multimap x)$.

REMARK 10.2.2. We unpack some of Notation 10.2.1. The notation ∂a is explained in detail in Notation 4.4.3. In full,

$$\lambda a.x \text{ and } \forall a.(\partial a \multimap x) \text{ mean } \forall_{\mathcal{D}} a.(\partial_{\mathcal{D}}(a_{\mathcal{D}^\partial}) \multimap_{\mathcal{D}} x).$$

Here $a_{\mathcal{D}^\partial}$ is the copy of a in the termlike σ -algebra $\mathcal{D}^\partial$ and $\partial_{\mathcal{D}}$ maps this to $|\mathcal{D}|$.

$\forall a$ is from Definition 4.1.2. $\multimap$ is a right adjoint to application and is from Definition 9.1.1.

LEMMA 10.2.3. If $b \# u$ then $(\lambda b.x)[a \mapsto u] = \lambda b.(x[a \mapsto u])$.

Proof. We unpack Notation 10.2.1. By assumption in Definition 4.3.3 the σ -action is compatible (Definition 4.2.1) so $(\forall b.\partial b \multimap x)[a \mapsto u] = \forall b.((\partial b \multimap x)[a \mapsto u])$. We use $(\sigma \multimap)$ from Figure 4 (Definition 9.1.1) and $(\sigma \#)$ from Figure 1 (since $b \# a$). $\square$

We now derive β -reduction and η -expansion from the counit and unit axioms $(\bullet\epsilon)$ and $(\bullet\eta)$ respectively:

PROPOSITION 10.2.4. Suppose $\mathcal{D} \in \text{inDi}\mathbb{V}_\bullet$, $x \in |\mathcal{D}|$, $u \in |\mathcal{D}^\partial|$, and a is an atom. Then:

1. $(\lambda a.x) \bullet \partial u \leq x[a \mapsto u]$.
2. If $a \# x$ then $x \leq \lambda a.(x \bullet \partial a)$ (∂u and ∂a from Notation 4.4.3).

Proof. We consider each part in turn.

1. Unfolding Notation 10.2.1 we have $(\lambda a.x) \bullet \partial u = (\forall a.(\partial a \multimap x)) \bullet \partial u$. Renaming using Lemma 4.1.3 if necessary, assume $a \# u$ so that by Theorem 2.3.2 also $a \# \partial u$. By $(\bullet\forall)$ $(\forall a.(\partial a \multimap x)) \bullet \partial u \leq \forall a.((\partial a \multimap x) \bullet \partial u)$. By Lemma 4.2.3 $\forall a.((\partial a \multimap x) \bullet \partial u) \leq ((\partial a \multimap x) \bullet \partial u)[a \mapsto u]$. By Lemma 9.1.9 and $(\sigma \bullet)$ and $(\sigma \#)$, $((\partial a \multimap x) \bullet \partial u)[a \mapsto u] \leq (\partial u \multimap (x[a \mapsto u])) \bullet \partial u$. By $(\bullet\epsilon)$ $(\partial u \multimap (x[a \mapsto u])) \bullet \partial u \leq x[a \mapsto u]$.
2. Suppose $a \# x$; note of Definition 4.1.2 that x is its own a -fresh limit; that is, $\forall a.x = x$. Unfolding Notation 10.2.1 $\lambda a.(x \bullet \partial a) = \forall a.(\partial a \multimap (x \bullet \partial a))$. By $(\bullet\eta)$ $x \leq \partial a \multimap (x \bullet \partial a)$, so by Lemma 4.1.7 $x = \forall a.x \leq \forall a.(\partial a \multimap (x \bullet \partial a))$ as required. $\square$

REMARK 10.2.5. $\mathcal{D} \in \text{inDiV}_\bullet$ gives a model of β -reduction and η -expansion. The reverse inclusions do not follow, but they are not forbidden:

- There exist models such that $x[a \mapsto u] \not\leq (\lambda a.x) \bullet \partial u$ (so that we do not have β -equality) and $a \# x$ for some x and yet $\lambda a.(x \bullet \partial a) \leq x$.
- There also exist models such that $x[a \mapsto u] = (\lambda a.x) \bullet \partial u$ and for all x , $a \# x$ implies $\lambda a.(x \bullet \partial a) = x$. To obtain one, choose any λ -equality theory Π (Definition 10.3.7; $\beta\eta$ -equality would do) and construct Pnt_Π from Definition 11.1.3 and the subsequent constructions.

10.2.2. λ as a sets operation in $F(\mathcal{D})$

We take a moment to perform a sanity check by examining λa (Notation 10.2.1) for the specific case of the sets representation $\mathcal{T} = F(\mathcal{D})$ (Definition 9.5.1) of $\mathcal{D} \in \text{inDiV}_\bullet$.

PROPOSITION 10.2.6. Suppose $\mathcal{D} \in \text{inDiV}_\bullet$, $X \in \text{Opn}(F(\mathcal{D}))$, and $u \in |F(\mathcal{D})^\partial| = |\mathcal{D}^\partial|$, and let p range over elements of $|F(\mathcal{D})|$, which are prime filters in $\mathcal{D}$. Then for every $u \in |\mathcal{D}^\partial|$,

$$p \in \lambda a.X \text{ implies } p \circ \partial u \subseteq X[a \mapsto u].$$

Proof. Recall the unpacking of λa from Remark 10.2.2. We reason as follows:

$$\begin{aligned} p \in \lambda a.X &\Leftrightarrow p \in \forall a.(\partial a \multimap X) && \text{Notation 10.2.1} \\ &\Leftrightarrow \forall u \in |F(\mathcal{D})^\partial|. p \in (\partial a \multimap X)[a \mapsto u] && \text{Proposition 4.2.5(1)} \\ &\Leftrightarrow \forall u \in |\mathcal{D}^\partial|. p \in (\partial a \multimap X)[a \mapsto u] && F(\mathcal{D})^\partial = \mathcal{D}^\partial \text{ by Thm 7.7.3} \\ &\Rightarrow \forall u \in |\mathcal{D}^\partial|. p \in (\partial a)[a \mapsto u] \multimap X[a \mapsto u] && \text{Lemma 9.1.9} \\ &\Leftrightarrow \forall u \in |\mathcal{D}^\partial|. p \in \partial u \multimap X[a \mapsto u] && (\sigma a), \partial \text{ morphism (Def 4.3.5)} \\ &\Leftrightarrow \forall u \in |\mathcal{D}^\partial|. p \circ \partial u \subseteq X[a \mapsto u] && \text{Proposition 9.2.6} \quad \square \end{aligned}$$

REMARK 10.2.7. Continuing the notation of Proposition 10.2.6, one might expect $p \in \lambda a.X$ to also be *equivalent* to $\forall u \in |\mathcal{D}^\partial|. p \circ \partial u \subseteq X[a \mapsto u]$. This seems to not be the case, because Lemma 9.1.9 used in the proof above is an inequality and not an equality.

10.3. Idioms

It is convenient to generalise λ -syntax a little. Recall from Definition 3.1.1 that a *termlike σ -algebra* expresses in nominal algebra the property of ‘having a substitution action over itself’.

DEFINITION 10.3.1. A $(\lambda\text{-})$ **idiom** is a termlike σ -algebra $\mathcal{U}$ equipped with equivariant functions

$$\begin{aligned} \bullet_{\mathcal{U}} : \mathcal{U} \times \mathcal{U} &\rightarrow \mathcal{U} \quad \text{and} \\ \lambda_{\mathcal{U}} : \mathbb{A} \times \mathcal{U} &\rightarrow \mathcal{U} \end{aligned}$$

such that for all $a \in \mathbb{A}$ and $x, y \in |\mathcal{U}|$ and $u \in |\mathcal{U}|$:

1. $a \# \lambda_{\mathcal{U}} a.x$ (this justifies quantifier notation: λ abstracts the atoms argument a).
2. If $b \# u$ then $(\lambda_{\mathcal{U}} b.x)[a := u]_{\mathcal{U}} = \lambda_{\mathcal{U}} b.(x[a := u]_{\mathcal{U}})$.
3. $(x \bullet_{\mathcal{U}} y)[a := u]_{\mathcal{U}} = (x[a := u]_{\mathcal{U}}) \bullet_{\mathcal{U}} (y[a := u]_{\mathcal{U}})$.
4. If $x, y \in |\mathcal{U}|$ then $x \bullet_{\mathcal{U}} y \in |\mathcal{U}|$, and $\lambda_{\mathcal{U}} a.x \in |\mathcal{U}|$.

Above, we use the fact that because $\mathcal{U}$ is a termlike σ -algebra, it interprets atoms as $\text{atm}_{\mathcal{U}}(a)$ and as a σ -action $x[a:=u]_{\mathcal{U}}$.

EXAMPLE 10.3.2. The canonical example of a λ -idiom is the **syntactic idiom** LmTm ; λ -terms up to α -equivalence, with their natural substitution, application, and λ -actions.

NOTATION 10.3.3. • We write $a_{\mathcal{U}}$ for $\text{atm}_{\mathcal{U}}(a)$, or just a .

- We write $x[a:=u]$ for $x[a:=u]_{\mathcal{U}}$.
- We write xy for $x \bullet_{\mathcal{U}} y$.
- We write $\lambda a.x$ for $\lambda_{\mathcal{U}} a.x$.

The conditions of Definition 10.3.1 can now be written using Corollary 2.3.5, thus:

$$\begin{aligned} b \# x &\Rightarrow \lambda a.x = \lambda b.(b \ a) \cdot x \\ b \# u &\Rightarrow (\lambda b.x)[a:=u] = \lambda a.(x[a:=u]) \\ &\quad (xy)[a:=u] = x[a:=u] y[a:=u] \end{aligned}$$

Note that it follows already from axiom (σa) in Figure 1 that $a_{\mathcal{U}}[a:=u] = u$.

NOTATION 10.3.4. In what follows, what the variables, substitution, application, and λ of an idiom $\mathcal{U}$ have to be, will always be clear. For the rest of this section fix some λ -idiom $\mathcal{U}$.

NOTATION 10.3.5. We call elements of $|\mathcal{U}|$ **phrases**. We let s and t range over phrases in $|\mathcal{U}|$, and also u and v range over phrases in $|\mathcal{U}|$.

REMARK 10.3.6. Phrases of a λ -idiom ‘look like’ terms of λ -syntax up to α -equivalence, inasmuch as they must support variables, substitution, a binary operator which we suggestively call application, and a variable-abstractor which we suggestively call λ . We noted in Example 10.3.2 that $\mathcal{U}$ might be the syntactic idiom, which is precisely λ -syntax.

However, we do not insist that phrases be λ -terms; they need not even be syntax. They just have to support nominal algebraic models of variables, substitution, application and a λ -abstraction. Nothing about the constructions that follow immediately below depends on $\mathcal{U}$ being syntactic.

DEFINITION 10.3.7. Suppose $\mathcal{U}$ is a λ -idiom. Call a preorder $\mathcal{R}$ on phrases **compatible** when for all $s, s', t, t' \in |\mathcal{U}|$ and $u \in |\mathcal{U}|$.³⁶

1. If $s \mathcal{R} s'$ and $t \mathcal{R} t'$ then $st \mathcal{R} s't'$.
2. If $s \mathcal{R} s'$ then $\lambda a.s \mathcal{R} \lambda a.s'$.
3. If $s \mathcal{R} s'$ then $s[a:=u] \mathcal{R} s'[a:=u]$.
4. $(\lambda a.s)u \mathcal{R} s[a:=u]$. This is **β -reduction**.
5. If a is not free in s then $s \mathcal{R} \lambda a.(sa)$. This is **η -expansion**.
6. If $s \mathcal{R} s'$ then $\pi \cdot s \mathcal{R} \pi \cdot s'$, or following Definition 2.1.8: $\mathcal{R}$ is *equivariant*.³⁷

Then we write:

³⁶ $\mathcal{R}$ being a preorder means precisely that it is transitive and reflexive. (A partial order is an antisymmetric preorder.)

³⁷See the discussion in Subsection 2.2.2. Another description of this condition is that $\mathcal{R}$ is an element of $\text{Nom}(\text{LmTm} \times \text{LmTm})$ (Subsection 2.5.1) with $\text{supp}(\mathcal{R}) = \emptyset$.

- A **λ -reduction theory** is a compatible preorder on an idiom $\mathcal{U}$.
- A **λ -equality theory** is a compatible equivalence relation on $\mathcal{U}$.³⁸

Π will range over λ -reduction theories.

DEFINITION 10.3.8. Call a pair $s \rightarrow t$ a **(λ) -reduction axiom**. We let $\mathcal{T}$ range over sets of reduction axioms.

- Write $\text{rew}(\mathcal{T})$ for the least λ -reduction theory Π such that $\mathcal{T} \subseteq \Pi$.
- Write $\text{equ}(\mathcal{T})$ for the least λ -equality theory Π such that $\mathcal{T} \subseteq \Pi$.

NOTATION 10.3.9. Suppose $\mathcal{T}$ is a set of reduction axioms. Then:

- We may write $s \rightarrow_{\mathcal{T}} t$ or $\mathcal{T} \vdash s \rightarrow t$ for $(s \rightarrow t) \in \text{rew}(\mathcal{T})$.
- We may write $s =_{\mathcal{T}} t$ or $\mathcal{T} \vdash s = t$ for $(s \rightarrow t) \in \text{equ}(\mathcal{T})$.

We conclude with an easy technical lemma:

LEMMA 10.3.10. Suppose $\mathcal{T}$ is a set of reduction axioms and write $\Pi = \text{rew}(\mathcal{T})$. Then:

- $\text{rew}(\mathcal{T}) = \text{rew}(\Pi)$.
- $\mathcal{T} \vdash s \rightarrow t$ if and only if $\Pi \vdash s \rightarrow t$.

Proof. Direct from Definition 10.3.8 and Notation 10.3.9. □

10.4. A sound denotation for the λ -calculus

Any $\mathcal{D} \in \text{inDi}\mathbb{V}_{\bullet}$ has the structure of $\mathbb{V}$, $\bullet$, and $\rightarrow$, so we can immediately interpret the λ -calculus in $\mathcal{D}$. Lo and behold, the interpretation is sound. This is Definition 10.4.1 and Theorem 10.4.7.

The denotation we obtain is *absolute*, meaning that a variable/atom a is interpreted ‘as itself’—there is no valuation. Slightly more formally, a denotation is absolute when variable symbols in the syntax map to fixed entities in the denotation. In the case of this paper, a (more precisely: $a_{\mathcal{U}}$) is interpreted as ∂a (more precisely: $\partial_{\mathcal{D}} a_{\mathcal{D}\partial}$, see Notation 4.4.3).

The role of a valuation is played by the σ -action. If we have some $x \in |\mathcal{D}|$ and want to ‘evaluate’ any a in it to become u , then we just apply $[a \mapsto u]$. This nominal approach to valuations using σ -algebras is *more general* than the usual Tarski denotation based on valuations; to see why, see the discussion in [Gab16, Remark 8.18].

DEFINITION 10.4.1. Suppose $\mathcal{D} \in \text{inDi}\mathbb{V}_{\bullet}$. Define a **denotation** of λ -terms by the rules in Figure 5 (λ is from Notation 10.2.1; $\bullet$ is from Definition 9.2.4),³⁹ and:

³⁸An equivalence relation is a symmetric preorder, so a λ -equality theory is, as we expect, a symmetric λ -reduction theory.

³⁹In the case for $\llbracket a \rrbracket^{\mathcal{D}}$, $a_{\mathcal{D}\partial}$ is the copy of a in the termlike σ -algebra $\mathcal{D}^{\partial}$ (Definition 3.1.1) and $\partial_{\mathcal{D}}$ is the function mapping $\mathcal{D}^{\partial}$ to $\mathcal{D}$ (see Definition 4.4.1 and Notation 4.4.3). We have written $\partial_{\mathcal{D}} a_{\mathcal{D}\partial}$ as just ∂a , but here we prefer the more careful notation.

Of course, if we wanted to be *really* careful we would also mention that $a_{\mathcal{D}\partial}$ is itself shorthand for $\text{atm}_{\mathcal{D}\partial}(a)$ from Definition 3.1.1. But the reader probably is not interested in that high level of pedantry, and may even be confused by it, so we will not labour the point further.

$$\begin{aligned}\llbracket a \rrbracket^{\mathcal{D}} &= \partial_{\mathcal{D}} a_{\mathcal{D}^{\partial}} \\ \llbracket \lambda a.s \rrbracket^{\mathcal{D}} &= \lambda a. \llbracket s \rrbracket^{\mathcal{D}} \\ \llbracket s' s \rrbracket^{\mathcal{D}} &= \llbracket s' \rrbracket^{\mathcal{D}} \bullet \llbracket s \rrbracket^{\mathcal{D}}\end{aligned}$$

Figure 5: Denotation of λ -terms

- Write $\mathcal{D} \models s \leq t$ when $\llbracket s \rrbracket^{\mathcal{D}} \leq \llbracket t \rrbracket^{\mathcal{D}}$.
- Write $\mathcal{D} \models \mathcal{T}$ when $\mathcal{D} \models s \leq t$ for every $(s \rightarrow t) \in \mathcal{T}$.
- Write $\mathcal{T} \models s \leq t$ when $\forall \mathcal{D} \in \text{inDiV}_{\bullet}. (\mathcal{D} \models \mathcal{T} \Rightarrow \mathcal{D} \models s \leq t)$.

REMARK 10.4.2. Suppose $\mathcal{D} \in \text{inDiV}_{\bullet}$. Recall from Definition 4.3.3 that $\mathcal{D}^{\partial}$ is the termlike σ -algebra over which substitution in $\mathcal{D}$ is defined, and recall that (since $\mathcal{D} \in \text{inDiV}_{\bullet}$ is *impredicative*; see Definition 4.4.1) we assume a σ -algebra morphism $\partial_{\mathcal{D}}$ from $\mathcal{D}^{\partial}$ to $\mathcal{D}$.

Recall from Notation 4.4.3 that we write $\partial\mathcal{D}$ for the sets image of $\partial_{\mathcal{D}}$, i.e. $\partial\mathcal{D} = \{\partial_{\mathcal{D}} u \mid u \in \mathcal{D}^{\partial}\} \subseteq |\mathcal{D}|$, and recall that we call this image the **programs** of $\mathcal{D}$.

DEFINITION 10.4.3. Call $\mathcal{D} \in \text{inDiV}_{\bullet}$ **replete** if $\partial\mathcal{D}$ is closed under application and λ . That is:

- If $x, y \in \partial\mathcal{D}$ then $x \bullet y \in \partial\mathcal{D}$.
- If $x \in \partial\mathcal{D}$ then $\lambda a.x \in \partial\mathcal{D}$.

REMARK 10.4.4. Note that $\partial a \in \partial\mathcal{D}$ is a fact, where ∂a is shorthand for $\partial_{\mathcal{D}} a_{\mathcal{D}^{\partial}}$. If $\mathcal{D}$ is replete then programs are closed under taking variables, application, or λ -abstraction, and intuitively this tells us the following:

If $\mathcal{D}$ is replete then its programs include denotations for all λ -terms.

This intuition is exactly the notion of repleteness used in [GG10] (we called it *faithful* there, but that terminology clashes with faithfulness of functors in category theory). In this paper we are using nominal techniques, so we can give a name-based semantic treatment of λ , so that Definition 10.4.3 can be more abstract than it needed to be in [GG10], and it needs make no explicit mention of λ -term syntax.

REMARK 10.4.5. Definition 10.4.3 is needed for Lemma 10.4.6. In any case, we are most interested in $\mathcal{D}$ that are replete, since we are interested in models of the λ -calculus and we would expect λ -terms to denote programs.

So if $\mathcal{D}$ is replete then Definition 10.4.1 generates programs, which can be substituted for in $\mathcal{D}$, and we can express Lemma 10.4.6:

LEMMA 10.4.6. Suppose $\mathcal{D} \in \text{inDiV}_{\bullet}$ is replete. Then $\llbracket s \rrbracket^{\mathcal{D}}[a \mapsto \llbracket u \rrbracket^{\mathcal{D}}] = \llbracket s[a := u] \rrbracket^{\mathcal{D}}$.
($\llbracket u \rrbracket^{\mathcal{D}}$ always exists, but repleteness ensures that $\llbracket u \rrbracket^{\mathcal{D}} \in \partial\mathcal{D}$ so that the substitution $[a \mapsto \llbracket u \rrbracket^{\mathcal{D}}]$ also exists.)

Proof. By induction on s .

- *The case of a .* By Definition 10.4.1 $\llbracket a \rrbracket^{\mathcal{D}} = \partial_{\mathcal{D}} a_{\mathcal{D}^{\partial}}$. We use Lemma 4.4.5.

- *The case of $\lambda b.s$.* Renaming if necessary assume $b\#u$. We reason as follows:

$$\begin{aligned}
 \llbracket (\lambda b.s)[a:=u] \rrbracket^{\mathcal{D}} &= \llbracket \lambda b.(s[a:=u]) \rrbracket^{\mathcal{D}} && \text{Fact of } \lambda\text{-terms, } b\#u \\
 &= \lambda b.\llbracket s[a:=u] \rrbracket^{\mathcal{D}} && \text{Definition 10.4.1} \\
 &= \lambda b.(\llbracket s \rrbracket^{\mathcal{D}}[a \mapsto \llbracket u \rrbracket^{\mathcal{D}}]) && \text{ind. hyp.} \\
 &= (\lambda b.\llbracket s \rrbracket^{\mathcal{D}})[a \mapsto \llbracket u \rrbracket^{\mathcal{D}}] && \text{Lemma 10.2.3} \\
 &= \llbracket \lambda b.s \rrbracket^{\mathcal{D}}[a \mapsto \llbracket u \rrbracket^{\mathcal{D}}] && \text{Definition 10.4.1}
 \end{aligned}$$

In the use of Lemma 10.2.3 above we know $b\#\llbracket u \rrbracket^{\mathcal{D}}$ by Theorem 2.3.2 since $b\#u$.

- *The case of $s's$.* Routine using the inductive hypothesis and $(\sigma\bullet)$ from Figure 4 (Definition 9.1.1). $\square$

Recall the notation $\mathcal{T} \vdash s \rightarrow t$ from Notation 10.3.9, applied here to the idiom LmTm (λ -terms). Recall the notation $\mathcal{T} \models s \leq t$ from Definition 10.4.1.

THEOREM 10.4.7 (Soundness). *Suppose $\mathcal{T}$ is a set of reduction axioms. Then*

$$\mathcal{T} \vdash s \rightarrow t \text{ implies } \mathcal{T} \models s \leq t.$$

(The reverse implication also holds; see Theorem 11.9.5.)

Proof. Suppose $\mathcal{D} \in \text{inDiV}_{\bullet}$ is replete and suppose $\mathcal{D} \models \mathcal{T}$ (Definition 10.4.1). We consider the rules defining a compatible relation on λ -terms (Definition 10.3.7):

1. If $s \mathcal{R} s'$ and $u \mathcal{R} u'$ then $su \mathcal{R} s'u'$. We use Lemma 9.1.7.
2. If $s \mathcal{R} s'$ then $\lambda a.s \mathcal{R} \lambda a.s'$. We use Lemmas 4.1.7 and 9.1.7.
3. If $s \mathcal{R} s'$ then $s[a:=u] \mathcal{R} s'[a:=u]$. We use Lemmas 10.4.6 and 4.2.2.
4. $(\lambda a.s)t \mathcal{R} s[a:=t]$. We use Lemma 10.4.6 and part 1 of Proposition 10.2.4.
5. If a is not free in s then $s \mathcal{R} \lambda a.(sa)$. If $a\#s$ then by Theorem 2.3.2 also $a\#\llbracket s \rrbracket^{\mathcal{D}}$. We use part 2 of Proposition 10.2.4.
6. If $s \mathcal{R} s'$ then $\pi \cdot s \mathcal{R} \pi \cdot s'$. From Theorem 2.3.2. $\square$

10.5. Interlude: axiomatising the λ -calculus in nominal algebra

Some words on where we are and where we are going.

Nominal algebra considers equality over nominal sets.⁴⁰ It was introduced in two papers [GM06a, GM06b] where it was applied to axiomatise to substitution and first-order logic respectively.⁴¹ Both applications feature α -equivalence and freshness side-conditions, which are of course just what nominal sets were developed to model, so this was natural.

See [GP01] or see [Gab11b, Gab13] for surveys.

In [GM08b, GM10] nominal algebra was applied to the λ -calculus, extending an incomplete axiomatisation from [FG07]—Henkin style models of such axioms were considered in [GM11] and found to have some interesting properties. In particular the axiomatisation is sound and complete—so the axioms below *really do* axiomatise

⁴⁰It is descended from nominal rewriting, which considers rewriting over nominal terms [FGM04, FG07].

⁴¹The papers wrote axioms and proved them sound and complete. So we really did check that the axioms do what one would expect them to do; no more and no less.

the λ -calculus; and this proof, in greatly strengthened form, has become the duality, soundness, and completeness results of the current paper.

So an axiomatisation of the λ -calculus is implicit in this paper. The reader could extract it by tracing through Notation 10.2.1 and the axioms of $\text{inDi}\mathbb{V}_\bullet$. We do not have to write out this theory to prove soundness in this paper, because the notion of λ -calculus we use in this paper is the standard one based on λ -term syntax and reduction.

Yet the axiomatisation is there in the background, and for the reader's convenience it might be illuminating to write it out.

Consider a nominal set $\mathcal{X}$.

We assume equivariant functions $\text{atm} : \mathbb{A} \rightarrow \mathcal{X}$ and $\text{sub} : \mathcal{X} \times \mathbb{A} \times \mathcal{X} \rightarrow \mathcal{X}$ and impose the axioms of a termlike σ -algebra from Figure 1:

$$\begin{array}{ll} (\sigma\mathbf{a}) & a[a \mapsto x] = x \\ (\sigma\mathbf{id}) & x[a \mapsto a] = x \\ (\sigma\#) & a\#x \Rightarrow x[a \mapsto u] = x \\ (\sigma\alpha) & b\#x \Rightarrow x[a \mapsto u] = ((b\ a) \cdot x)[b \mapsto u] \\ (\sigma\sigma) & a\#v \Rightarrow x[a \mapsto u][b \mapsto v] = x[b \mapsto v][a \mapsto u][b \mapsto v] \end{array}$$

Here we sugar $\text{atm}(a)$ to just a and $\text{sub}(x, a, u)$ to just $x[a \mapsto u]$.

Next we assume equivariant functions $\text{app} : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ and $\text{lam} : \mathbb{A} \times \mathcal{X} \rightarrow \mathcal{X}$, and impose the axioms of β - and η -equality:

$$\begin{array}{ll} (\lambda\alpha) & b\#x \Rightarrow \lambda a.x = \lambda b.(b\ a) \cdot x \\ (\beta=) & (\lambda a.x)y = x[a \mapsto y] \\ (\eta=) & a\#x \Rightarrow \lambda a.(xa) = x \end{array}$$

Here we sugar $\text{app}(x, y)$ to xy and $\text{lam}(a, x)$ to $\lambda a.x$.

A few notes on this axiomatisation:

- The axiomatisation of [GM08b, GM10] identified substitution with a β -reduct. The axiomatisation above distinguishes substitution and β -reducts. This turns out to be important for making the results in this paper work; for more discussion see the Conclusions.
- The body of this paper is based on lattices, so we do not *assume* β - or η -equality; we only assume β -reduction and η -expansion. (The equalities might happen to be valid anyway, see for instance λ -equality theories in Definition 10.3.7.) This is also important for this paper.

In summary the axiomatisation above is a special case of a generalisation of [GM08b, GM10], which is itself a complete extension of a rewrite theory from [FG07, GM06a].

The axiomatisation above is also what we are aiming for, and models of λ -equality theories constructed in Section 11 are models of the axioms above, though the demands of our main results are such that we do not phrase matters in that specific form.

11. Representation of the λ -calculus in $\text{inSpect}\mathbb{V}_\bullet$

In Section 10 we showed how any $\mathcal{D} \in \text{inDi}\mathbb{V}_\bullet$ / (dually) any $\mathcal{T} \in \text{inSpect}\mathbb{V}_\bullet$, gives a sound abstract / (dually) concrete interpretation of the untyped λ -calculus.

The next step is to prove completeness. This is Theorem 11.9.5. The method is to construct a nominal spectral space Pnt_{Π} out of a λ -reduction theory Π , in which only those subset inclusions are valid that are insisted on by Π .

Pnt_{Π} is a rich structure. Notable technical definitions and results are Definition 11.1.1 and Proposition 11.1.6 ($a \#_{\sigma} p$ and its equivalence with $a \# p$), completeness under small-supported sets unions and intersections (Proposition 11.1.7), the σ -action on points (Definition 11.3.1) and its two characterisation in Subsection 11.3.2—one in terms of the now-ubiquitous $\mathcal{U}$.

For this section, fix the following data:

- Fix a λ -idiom $\mathcal{U}$ (Definition 10.3.1).
- Fix a λ -reduction theory Π on $\mathcal{U}$ (Definition 10.3.7).

s, s', s'', t, u , and v will range over elements of $|\mathcal{U}|$.

11.1. Π -points and σ -freshness

Given a subset $p \subseteq |\mathcal{U}|$, we can suggest two notions of ‘ a is fresh for p ’:

- One inherited from nominal techniques: $\mathcal{U}b.(b \ a) \cdot p = p$. We write this $a \# p$.
- One inherited from our syntactic intuitions: if $s \in p$ then $\forall u \in |\mathcal{U}|. s[a:=u] \in p$. We will make this formal in Definition 11.1.1 and write it $a \#_{\sigma} p$.

A *point* is then defined to be a set of phrases for which these two notions of freshness coincide. This is Definition 11.1.3. The interesting part is condition 2, which is not obviously just “ $a \# p \Leftrightarrow a \#_{\sigma} p$ ”—for this, see Proposition 11.1.6 and Lemma 11.1.5, which work from surprisingly little in the way of assumptions.

We conclude with Proposition 11.1.7, an important result asserting that Pnt_{Π} is complete for small-supported diagrams (i.e. small-supported sets of points have an intersection that is also a point). This has some useful consequences: for instance it makes possible the use of $\bigcap$ in Definitions 11.5.1, and 11.3.1, and also in Corollary 11.3.7.

DEFINITION 11.1.1. Suppose $p \subseteq |\mathcal{U}|$.⁴² Define $a \#_{\sigma} p$ by:

$$a \#_{\sigma} p \quad \text{when} \quad \forall s \in |\mathcal{U}|. \forall u \in |\mathcal{U}|. (s \in p \Rightarrow s[a:=u] \in p)$$

If $a \#_{\sigma} p$ we say that a is σ -**fresh** for p .

REMARK 11.1.2. We rewrite Definition 11.1.1 twice:

1. $a \#_{\sigma} p$ when $\forall s \in |\mathcal{U}|. (s \in p \Rightarrow \forall u \in |\mathcal{U}|. s[a:=u] \in p)$.
2. $a \#_{\sigma} p$ when $\forall u \in |\mathcal{U}|. p \subseteq p[u \leftarrow a]$.

$p[u \leftarrow a]$ is from Definition 3.3.1; but see also Definition 11.2.3 and Lemma 11.2.6 below.

⁴² p need not be small-supported, but it will turn out that we are most interested in the case where it is.

Recall the notion of a λ -reduction theory from Definition 10.3.7.

DEFINITION 11.1.3. Suppose Π is a λ -reduction theory. Call a subset $p \subseteq |\mathcal{U}|$ a **Π -point** when:

1. $\forall s, t \in |\mathcal{U}|. (s \in p \wedge s \rightarrow_{\Pi} t) \Rightarrow t \in p$. We call p **closed under Π** .
2. $\forall a. a \#_{\sigma} p$. We call p **σ -supported**.

Write $|Pnt_{\Pi}|$ for the set of Π -points.

REMARK 11.1.4. We will give Pnt_{Π} an $\mathfrak{o}$ -algebra structure in Proposition 11.2.4 and Corollary 11.2.5. We consider examples of points, and sets that are not points:

1. $\emptyset$ (the empty set) is a point.

This will be useful in Proposition 11.8.3 to prove that the set of all points is compact and covered by $\{\emptyset^{\bullet}\}$ (the $\cdot^{\bullet}$ notation will be defined in Definition 11.3.1).

2. $\mathcal{U}$ is a point.

3. Take $\mathcal{U}$ to be the syntactic idiom $LmTm$ from Example 10.3.2 and take Π to be $rew(\emptyset)$ from Definition 10.3.8—that is, the minimal λ -reduction theory (Definition 10.3.7), containing just β -reduction and η -expansion.

Take p to be $\{a, \lambda b.ab, \lambda c.\lambda b.abc, \dots, b, \lambda b'.bb', \dots\}$ the set of all η -expansions of variable symbols.

This is not a point, because by Theorem 2.3.2 $a \# p$ and yet $a[a := \lambda a.a] = \lambda a.a \notin p$. In the terminology of condition 2 of Definition 11.1.3, this p is not σ -supported.

LEMMA 11.1.5. Suppose $p \subseteq |\mathcal{U}|$ and suppose p satisfies condition 2 of Definition 11.1.3. Then:

1. $a \#_{\sigma} p$ implies $\forall b. (b a) \cdot p = p$.
2. As a corollary, if $p \in |Pnt_{\Pi}|$ then p has small support.

Proof. Suppose $a \#_{\sigma} p$ and consider any $b \#_{\sigma} p$; by condition 2 of Definition 11.1.3 there are cosmall many such b . We will prove $(b a) \cdot p = p$. The permutation action is pointwise (Definition 2.2.3) so it suffices to show that for any $s \in |\mathcal{U}|$, $s \in p$ implies $(b a) \cdot s \in p$.

By Lemma 3.2.7 $(b a) \cdot s = s[a := c][b := a][c := b]$ for fresh c (so $c \# s$ and c is distinct from a and b , and by condition 2 of Definition 11.1.3 $c \#_{\sigma} p$). Now $s \in p$ and $a \#_{\sigma} p$ so $s[a := c] \in p$. Also $b \#_{\sigma} p$ so $s[a := c][b := a] \in p$. Also $c \#_{\sigma} p$ so $s[a := c][b := a][c := b] \in p$, and we are done.

For the corollary, from part 1 of this result $\forall a. \forall b. (b a) \cdot p = p$. We use Lemma 2.3.10. $\square$

We discussed at the introduction to this subsection why Proposition 11.1.6 is interesting:

PROPOSITION 11.1.6. Suppose $p \in |Pnt_{\Pi}|$. Then

$$a \# p \quad \text{if and only if} \quad a \#_{\sigma} p.$$

Proof. First, suppose $a \# p$. By condition 2 of Definition 11.1.3 $\mathcal{W}b.b \#_{\sigma} p$ and so by Theorem 2.3.9 (since by Lemma 11.1.5(2) p has small support) we have $a \#_{\sigma} p$.

Conversely, if $a \#_{\sigma} p$ then by Lemma 11.1.5(1) $\mathcal{W}b.(b a) \cdot p = p$ and so using Corollary 2.3.5(3) (since by Lemma 11.1.5(2) p has small support) $a \# p$. $\square$

We conclude the subsection with Proposition 11.1.7, a useful result with an attractive proof:

PROPOSITION 11.1.7. *Suppose $\mathcal{P} \subseteq |\text{Pnt}_{\Pi}|$ is small-supported. Then*

$$\bigcap \mathcal{P} \in \text{Pnt}_{\Pi} \quad \text{and} \quad \bigcup \mathcal{P} \in \text{Pnt}_{\Pi}.$$

In words: small-supported intersections and unions of points, are points.

Proof. We check the conditions of Definition 11.1.3 for $\bigcap \mathcal{P} \in \text{Pnt}_{\Pi}$.

1. Condition 1 is by a routine calculation.
2. For condition 2, suppose a is fresh (so $a \# \mathcal{P}$, because we assumed $\mathcal{P}$ has small support). To prove $a \#_{\sigma} \bigcap \mathcal{P}$ we must show $s \in \bigcap \mathcal{P}$ implies $\forall u \in |\mathcal{U}|. s[a:=u] \in \bigcap \mathcal{P}$. Consider $s \in \bigcap \mathcal{P}$ and any $p \in \mathcal{P}$, so $s \in p$. We want to prove $\forall u \in |\mathcal{U}|. s[a:=u] \in p$. We cannot do this directly since we do not necessarily know that $a \# p$.⁴³

So choose fresh c (so $c \# \mathcal{P}, p, s$). Since $p \in \mathcal{P}$ also $(c a) \cdot p \in (c a) \cdot \mathcal{P} \stackrel{\text{Cor 2.3.5}}{=} \mathcal{P}$. So $s \in (c a) \cdot p$ and therefore $(c a) \cdot s \in p$. By assumption $c \# p$ so by Proposition 11.1.6 $c \#_{\sigma} p$ so $\forall u. ((c a) \cdot s)[c:=u] \in p$. We α -convert, and conclude that $\forall u \in |\mathcal{U}|. s[a:=u] \in p$ as required.

The reasoning for $\bigcup \mathcal{P} \in \text{Pnt}_{\Pi}$ is almost identical. $\square$

11.2. Constructing Π -points, and their amgis-algebra structure

Recall the notion of idiom $\mathcal{U}$ from Definition 10.3.1, the notion of λ -reduction theory Π from Definition 10.3.7, and the notion of point p from Definition 11.1.3.

DEFINITION 11.2.1. Suppose $s \in |\mathcal{U}|$. Define the **(Π -)principal filter** $s \uparrow_{\Pi}$ by

$$s \uparrow_{\Pi} = \{s' \in |\mathcal{U}| \mid s \rightarrow_{\Pi} s'\}.$$

LEMMA 11.2.2. *If $s \in |\mathcal{U}|$ then $s \uparrow_{\Pi}$ is a point.*

Proof. We check the conditions of Definition 11.1.3.

1. By transitivity if $s'' \in s \uparrow_{\Pi}$, meaning $s \rightarrow_{\Pi} s''$, and $s'' \rightarrow_{\Pi} s'$ then $s \rightarrow_{\Pi} s'$.
2. Suppose $a \# s$ and suppose $s' \in s \uparrow_{\Pi}$. By condition 3 of Definition 10.3.7 also $s[a:=u] \rightarrow_{\Pi} s'[a:=u]$ and by $(\sigma \#)$ $s[a:=u] = s$. It follows that $s'[a:=u] \in s \uparrow_{\Pi}$ for every u . Thus $s \uparrow_{\Pi}$ is σ -supported. $\square$

⁴³We assumed $\mathcal{P} \subseteq |\text{Pnt}_{\Pi}|$ is small-supported, not *strictly* small-supported. See Lemma 2.5.1.

DEFINITION 11.2.3. Give $p \subseteq |\mathcal{U}|$ a permutation action and an $\mathfrak{o}$ -action following Definitions 3.3.1 and 7.2.1:

$$\pi \cdot p = \{\pi \cdot r \mid r \in p\} \quad p[u \leftarrow a] = \{s \mid s[a := u] \in p\} \quad (u \in |\mathcal{U}|)$$

Write Pnt_{Π} for (what will prove will be) the $\mathfrak{o}$ -algebra with underlying set $|Pnt_{\Pi}|$ and the permutation and $\mathfrak{o}$ -actions defined above.

Our notation suggests that $[u \leftarrow a]$ is an $\mathfrak{o}$ -action. This is true, but we must prove it: this is Proposition 11.2.4 and Corollary 11.2.5.

PROPOSITION 11.2.4. *Suppose $u \in |\mathcal{U}|$. Then $p \in |Pnt_{\Pi}|$ implies $p[u \leftarrow a] \in |Pnt_{\Pi}|$.*

In words: if p is a Π -point then so is $p[u \leftarrow a]$.

Proof. We check the conditions of Definition 11.1.3, freely using Proposition 3.3.2:

1. *Suppose $s[a := u] \in p$ and $s \rightarrow_{\Pi} s'$.*
By condition 3 of Definition 10.3.7 $s[a := u] \rightarrow_{\Pi} s'[a := u]$. By assumption p is closed under Π , and so $s'[a := u] \in p$.
2. *Suppose b is fresh (so $b \# p, a, u$) and consider $s[a := u] \in p$.*
By assumption p is σ -supported and $b \# p$ so by Proposition 11.1.6 $b \#_{\sigma} p$. Therefore

$$\forall v. s[a := u][b := v] \in p$$

Now $b \# u$, so by $(\sigma\sigma)$ for any $v' \in |\mathcal{U}|$, $s[b := v'][a := u] = s[a := u][b := v'[a := u]]$. It follows (taking all v of the form $v'[a := u]$ above) that $\forall v. s[b := v][a := u] \in p$, and so $\forall v. s[b := v] \in p[u \leftarrow a]$ as required. $\square$

Recall Pnt_{Π} from Definition 11.1.3:

COROLLARY 11.2.5. *Pnt_{Π} is indeed an $\mathfrak{o}$ -algebra.*

Proof. Proposition 11.2.4 proves that $[u \leftarrow a]$ maps points to points. We use Proposition 3.3.5. $\square$

We conclude with a technical result which will be useful for Lemma 11.4.1:

LEMMA 11.2.6. *If $p \in |Pnt_{\Pi}|$ and $a \# p$ and $u \in |\mathcal{U}|$ then $p \subseteq p[u \leftarrow a]$.*

Proof. Suppose $a \# p$. By Proposition 11.1.6 $a \#_{\sigma} p$, so by Definition 11.1.1 $s \in p$ implies $s[a := u] \in p$. The result follows by Proposition 3.3.2. $\square$

11.3. *The left adjoint $p[a \mapsto u]$ to the amgis-action $p[u \leftarrow a]$*

Recall from the start of the Section that we fixed some λ -reduction theory Π .

By Proposition 11.1.7 a small-supported intersection of points is a point. This suggests that we could build a left adjoint to $[u \leftarrow a]$ on points by taking a suitable intersection. We do this in Definition 11.3.1.

This left adjoint turns out to be very well-behaved. It has interesting characterisations (Subsection 11.3.2) which give us strong proof-methods for reasoning on it. Furthermore

it is a σ -action; this is Proposition 11.4.6. So Pnt_{Π} is both an $\mathfrak{o}$ -algebra and a σ -algebra.⁴⁴ We shall see in Definition 11.5.1 how this σ -action enables us to interpret $\forall$ on points.

Even better, the σ -action commutes with $\cdot$ from Definition 11.7.1, which is key to how points are used to generate compact sets; this is part 1 of Theorem 11.7.9. Thus we can study the behaviour of substitution on open sets by understanding the behaviour of the left adjoint to the $\mathfrak{o}$ -algebra action of points.

In short, most of the rest of this section depends on Definition 11.3.1 and the results that follow it in this subsection.

11.3.1. Basic definition

Recall from Definition 11.2.3 the $\mathfrak{o}$ -action on points $p[u \leftarrow a]$. We can build a left adjoint for it:

DEFINITION 11.3.1. Given $p \subseteq |\mathcal{U}|$ with small support and $u \in |\mathcal{U}|$, define $p[a \mapsto u]$ by:

$$p[a \mapsto u] = \bigcap \{q \in |Pnt_{\Pi}| \mid \forall c. ((c a) \cdot p \subseteq q[u \leftarrow c])\}$$

By Proposition 11.1.7, Definition 11.3.1 does indeed define a point.

REMARK 11.3.2. Definition 11.3.1 looks like a repeat of Definition 3.4.1, but they are not quite the same because p and q above have the same type (subsets of $|\mathcal{U}|$) whereas in Definition 3.4.1 p and X have different types (a point and a set of points, respectively).

Points here have small support, so the proof of Proposition 11.3.3 is (almost) a replay of the proof of part 2 of Proposition 3.4.3 (only for subset inclusion instead of sets membership).

PROPOSITION 11.3.3. If $a \# u, q$ then $p \subseteq q[u \leftarrow a]$ if and only if $p[a \mapsto u] \subseteq q$.

Proof. Suppose $a \# u, q$. From Definition 11.3.1, $p[a \mapsto u] \subseteq q$ if and only if $\forall c. (c a) \cdot p \subseteq q[u \leftarrow c]$. By Corollary 2.3.5 $(c a) \cdot u = u$ and $(c a) \cdot q = q$, so (applying $(c a)$ to both sides of the subset inclusion) this is if and only if $\forall c. p \subseteq q[u \leftarrow a]$, that is: $p \subseteq q[u \leftarrow a]$. $\square$

The interested reader can also find two corollaries of Proposition 11.3.3 in Subsection 11.3.3.

LEMMA 11.3.4. 1. If $b \# p$ then $p[a \mapsto u] = ((b a) \cdot p)[b \mapsto u]$.

2. As a corollary, if $a \# u$ then $a \# p[a \mapsto u]$.

Proof. Suppose c is fresh (so $c \# p$). By Theorem 2.3.9 it suffices to show that $(c a) \cdot p \subseteq q[u \leftarrow c]$ if and only if $(c b) \cdot ((b a) \cdot p) \subseteq q[u \leftarrow c]$. It would suffice to prove that $(c a) \cdot p = (c b) \cdot ((b a) \cdot p)$. This follows from Corollary 2.3.5 and our assumption that $b \# p$.

The corollary follows using Corollary 2.3.5. $\square$

⁴⁴This may well be a special case of a general result deserving its own paper. Here, we are simply grateful and press on.

11.3.2. Two characterisations of $p[a \mapsto u]$

DEFINITION 11.3.5. If $p \subseteq |\mathcal{U}|$ is small-supported and $u \in |\mathcal{U}|$ define

$$p[a := u] = \bigcup \{s[a := u] \uparrow_{\Pi} \mid s \in p\}.$$

By Proposition 11.1.7 (since p is small-supported) and Lemma 11.2.2, $p[a := u]$ is a point.

For the rest of this subsection, we assume $p, q \in |Pnt_{\Pi}|$.

LEMMA 11.3.6. If $a \# u$, q then $p[a \mapsto u] \subseteq q$ if and only if $p[a := u] \subseteq q$.

As a corollary, if $a \# u$ then $p[a := u] \subseteq p[a \mapsto u]$.

Proof. By Proposition 11.3.3 (since $a \# u$, q) $p[a \mapsto u] \in q$ if and only if $p \subseteq q[u \leftarrow a]$. From Proposition 3.3.2 and condition 1 of Definition 11.1.3 this happens if and only if $p[a := u] \subseteq q$.

The corollary follows since $p[a \mapsto u] \subseteq p[a \mapsto u]$ and by Lemma 11.3.4(2) $a \# p[a \mapsto u]$. $\square$

COROLLARY 11.3.7. If $a \# u$ then

$$p[a \mapsto u] = \bigcap \{q \mid a \# q \wedge p[a := u] \subseteq q\}. \quad (\text{Characterisation 1})$$

Proof. If $a \# q$ and $p[a := u] \subseteq q$ then by Lemma 11.3.6 also $p[a \mapsto u] \subseteq q$. Therefore

$$p[a \mapsto u] \subseteq \bigcap \{q \mid a \# q \wedge p[a := u] \subseteq q\}.$$

Furthermore by Lemma 11.3.4(2) $a \# p[a \mapsto u]$ and by Lemma 11.3.6 $p[a := u] \subseteq p[a \mapsto u]$. Therefore

$$\bigcap \{q \mid a \# q \wedge p[a := u] \subseteq q\} \subseteq p[a \mapsto u]. \quad \square$$

Recall from Definition 2.6.1 the notion of $\mathcal{N}a.p$ (the $\mathcal{N}$ -quantifier, for sets).

LEMMA 11.3.8. If $p \in |Pnt_{\Pi}|$ then also $\mathcal{N}a.p \in |Pnt_{\Pi}|$.

Proof. We check the conditions of Definition 11.1.3:

1. Suppose $s \in \mathcal{N}a.p$ and $s \rightarrow_{\Pi} t$. Then $\mathcal{N}b.(b a) \cdot s \in p$. By condition 6 of Definition 10.3.7 (equivariance) $(b a) \cdot s \rightarrow_{\Pi} (b a) \cdot t$ (for any b). It follows by condition 1 of Definition 11.1.3 that $\mathcal{N}b.(b a) \cdot t \in p$, so that $b \in \mathcal{N}a.p$.
2. Consider some fresh c (so $c \# p$) and consider $s \in \mathcal{N}a.p$ and some fresh b (so $b \# p, s$), so that $(b a) \cdot s \in p$. By condition 2 of Definition 11.1.3, $c \#_{\sigma} p$, so that $\forall u.((b a) \cdot s)[c := u] \in p$. It follows that $\forall u.(b a) \cdot (s[c := u]) \in p$, and therefore $\forall u.s[c := u] \in \mathcal{N}a.p$. $\square$

Lemma 11.3.9 gives a striking characterisation connecting the left adjoint to the τ -action, the pointwise substitution action, and the $\mathcal{N}$ -quantifier for sets (Definitions 11.3.1, 11.3.5, and 2.6.1):

LEMMA 11.3.9. If $a \# u$ then $p[a := u] \subseteq \mathcal{N}a.(p[a := u])$. As a corollary, if $a \# u$ then

$$p[a \mapsto u] = \mathcal{N}a.(p[a := u]). \quad (\text{Characterisation 2})$$

Proof. Suppose $s \in p[a:=u]$. This means there is $s' \in p$ such that $s'[a:=u] \rightarrow_{\Pi} s$. By Lemma 3.2.6 $a\#s'[a:=u]$ and it follows for any fresh c that $s'[a:=u] \rightarrow_{\Pi} (c\ a) \cdot s$, so that $(c\ a) \cdot s \in p[a:=u]$. Thus, $s \in \text{va.}(p[a:=u])$.

The corollary follows from Lemma 2.6.3 and Corollary 11.3.7. $\square$

11.3.3. Additional lemmas about the σ -action as an adjoint

Lemmas 11.3.10 and 11.3.11 describe a unit and counit style interaction between $[a \mapsto u]$ and $[u \leftarrow a]$ acting on points. We will not use these lemmas later—we will use the result they come from, Proposition 11.3.3, directly instead.

The lemmas are still worth looking at, because they are subject to freshness side-conditions and so are not quite exactly what one might assume. They and the related results in this Section suggest a theory of ‘nominal adjoints’, in the spirit of the theories of nominal abstract syntax, unification, rewriting, and algebra which we have already seen [GP01, UPG04, FG07, GM09a].

LEMMA 11.3.10. *If $a\#u$ then $p \subseteq p[a \mapsto u][u \leftarrow a]$.*

Proof. By Lemma 11.3.4(2) (since $a\#u$) $a\#p[a \mapsto u]$. It is a fact that $p[a \mapsto u] \subseteq p[a \mapsto u]$, therefore by Proposition 11.3.3 (since $a\#u, p[a \mapsto u]$) $p \subseteq p[a \mapsto u][u \leftarrow a]$. $\square$

LEMMA 11.3.11. 1. *If $a\#u, p$ then $p[u \leftarrow a][a \mapsto u] \subseteq p$.*

2. *If $a\#u, p$ then $p \subseteq p[u \leftarrow a][a \mapsto u]$.*

3. *As a corollary, if $a\#u, p$ then $p[u \leftarrow a][a \mapsto u] = p$.*

Proof. 1. It is a fact that $p[u \leftarrow a] \subseteq p[u \leftarrow a]$. We use Proposition 11.3.3 (since $a\#u, p$).

2. Suppose $s \in p$. Using Lemma 11.3.4(1) to rename if necessary, we may assume $a\#s$. So by $(\sigma\#)$ from Figure 1 $s[a:=u] = s$, so that $s \in p[u \leftarrow a]$ and therefore $s \in p[u \leftarrow a][a:=u] \stackrel{L\ 11.3.6}{\subseteq} p[u \leftarrow a][a \mapsto u]$.

3. From parts 1 and 2 of this result. $\square$

11.4. The left adjoint $p[a \mapsto u]$ as a σ -action on points

11.4.1. It is indeed a σ -action

We prove Proposition 11.4.6, that $p[a \mapsto u]$ is indeed a σ -action on points. Fix $p \in |Pnt_{\Pi}|$ and $u \in |\mathcal{U}|$.

LEMMA 11.4.1. *If $a\#p$ then $p[a \mapsto u] = p$.*

Proof. Using Lemma 11.3.4(1) and Corollary 2.3.5 assume without loss of generality that $a\#u$ as well as $a\#p$.

By Lemma 11.2.6 (since $a\#p$) $p \subseteq p[u \leftarrow a]$ so by Proposition 11.3.3 (since $a\#u, p$) $p[a \mapsto u] \subseteq p$.

Now suppose $s \notin p[a \mapsto u]$. We will show that $s \notin p$.

Unpacking Definition 11.3.1, $s \notin p[a \mapsto u]$ implies that there exists $q \in |Pnt_{\Pi}|$ such that $s \notin q$ and for fresh c (so $c\#p, q, u, s$) $(c\ a) \cdot p \subseteq q[u \leftarrow c]$. Now by $(\sigma\#)$ $s[c:=u] = s$. Thus if $s \notin q$ then by Proposition 3.4.3 also $s \notin q[u \leftarrow c]$. Therefore $s \notin (c\ a) \cdot p$. By Corollary 2.3.5 since $a\#p$ and $c\#p$ also $(c\ a) \cdot p = p$, so $s \notin p$ as required. $\square$

LEMMA 11.4.2. *If $b \# p$ then $p[a \mapsto b] = (b a) \cdot p$.*

Proof. By Lemma 11.3.9 $s \in p[a \mapsto b]$ if and only if $\forall c. (c a) \cdot s \in p[a := b]$, and by Definitions 11.2.1 and 11.3.5 this is if and only if $\forall c. \exists s' \in p. s'[a := b] \rightarrow_{\Pi} (c a) \cdot s$.

Now by assumption $b \# p$ so if $b \in \text{supp}(s')$ then by condition 2 of Definition 11.1.3 also $s'[b := a] \in p$. So we may assume without loss of generality of the ' $\exists s' \in p$ ' above that the s' chosen satisfies $b \# s'$, so that $s'[a := b] = (b a) \cdot s'$.

Thus this is if and only if $\forall c. \exists s' \in p. (b a) \cdot s' \rightarrow_{\Pi} (c a) \cdot s$. Rearranging the permutations, this is if and only if $\forall c. \exists s' \in p. (b a) \cdot ((c b) \cdot s') \rightarrow_{\Pi} s$.

Again, since $c, b \# p$, by Corollary 2.3.5 $(c b) \cdot p = p$ so that $s' \in p$ if and only if $(c b) \cdot s' \in p$.

Thus this is if and only if $\forall c. \exists s' \in p. (b a) \cdot s' \rightarrow_{\Pi} s$, and by condition 1 of Definition 11.1.3 this is if and only if $s \in (b a) \cdot p$. $\square$

REMARK 11.4.3. Lemma 11.4.2 is remarkable. There is no reason to expect that $p[a \mapsto b] = (b a) \cdot p$ should hold—for contrast, we needed to impose this as condition 2 when we constructed $\text{Pow}_{\sigma}(\mathcal{P})$ in Definition 3.4.6. Here, it works without requiring conditions.

Corollary 11.4.4 is a repeat of Corollary 3.4.9, but for points. We will use it in Proposition 11.4.6:

COROLLARY 11.4.4. $p[a \mapsto a] = p$.

Proof. Choose any $b \# p$. By Lemma 11.3.4(1) $p[a \mapsto a] = ((b a) \cdot p)[b \mapsto a]$. By Proposition 2.3.4 $a \# (b a) \cdot p$, and by Lemma 11.4.2 $((b a) \cdot p)[b \mapsto a] = (b a) \cdot ((b a) \cdot p) = p$. $\square$

LEMMA 11.4.5. *If $a \# v$ then $p[a \mapsto u][b \mapsto v] = p[b \mapsto v][a \mapsto u[a \mapsto v]]$.*

Proof. Using Lemma 11.3.4(1) assume without loss of generality that $a \# u$ and $b \# v$. Then the result follows using Proposition 11.3.3 and Corollary 11.2.5, from $(\tau\sigma)$. $\square$

Proposition 11.4.6 does not hold in the general case of $F(\mathcal{D})$ from Definition 7.2.1, but it holds specifically for Pnt_{Π} . The underlying reason this happens is Proposition 11.1.7, which allows us to build (small-supported) intersections of points and so construct $p[a \mapsto u]$ in Definition 11.3.1:

PROPOSITION 11.4.6. *Pnt_{Π} with the action $[a \mapsto u]$ from Definition 11.3.1 is indeed a σ -algebra.*

Proof. The interesting part is to check the axioms of Figure 1:

- (σid) is Corollary 11.4.4.
- $(\sigma \#)$ is Lemma 11.4.1.
- $(\sigma \alpha)$ is Lemma 11.3.4(1).
- $(\sigma \sigma)$ is Lemma 11.4.5.

$\square$

11.4.2. The σ -action distributes over union and subset

LEMMA 11.4.7. Suppose $\mathcal{P} \subseteq |\text{Pnt}_{\Pi}|$ is strictly small-supported (Subsection 2.5.2). Then

$$(\bigcup \mathcal{P})[a \mapsto u] = \bigcup \{p[a \mapsto u] \mid p \in \mathcal{P}\}.$$

By Lemma 2.5.5 this holds in particular if $\mathcal{P}$ is finite.

Proof. We use Lemma 11.3.4(1) to assume without loss of generality that $a \# u$. Take any $r \in |\text{Pnt}_{\Pi}|$ such that $a \# r$. We reason as follows:

$$\begin{aligned} (\bigcup \mathcal{P})[a \mapsto u] \subseteq r &\Leftrightarrow \bigcup \mathcal{P} \subseteq r[u \leftarrow a] && \text{Proposition 11.3.3 } a \# u, r \\ &\Leftrightarrow \forall p \in \mathcal{P}. p \subseteq r[u \leftarrow a] && \text{Fact of sets} \\ &\Leftrightarrow \forall p \in \mathcal{P}. p[a \mapsto u] \subseteq r && \text{Proposition 11.3.3 } a \# u, r \\ &\Leftrightarrow \bigcup \{p[a \mapsto u] \mid p \in \mathcal{P}\} \subseteq r && \text{Fact of sets} \end{aligned}$$

By Lemma 11.3.4(2) $a \# (\bigcup \mathcal{P})[a \mapsto u]$. Also, $a \# p[a \mapsto u]$ for every $p \in \mathcal{P}$ so that by Lemma 2.5.3(2) $a \# \bigcup \{p[a \mapsto u] \mid p \in \mathcal{P}\}$. Taking first $r = (\bigcup \mathcal{P})[a \mapsto u]$ and then $r = \bigcup \{p[a \mapsto u] \mid p \in \mathcal{P}\}$ we obtain two subset inclusions and thus an equality. $\square$

REMARK 11.4.8. Lemma 11.4.7 does not contain a second part proving distributivity of substitution over sets intersection of points. This is why, as we will note in Remark 11.5.2, we do not consider an operation $p \vee q = p \cap q$ in Definition 11.5.1; it would *not* satisfy $(p \cap q)[a \mapsto u] = p[a \mapsto u] \cap q[a \mapsto u]$ or equivalently $(p \vee q)[a \mapsto u] = p[a \mapsto u] \vee q[a \mapsto u]$. To get this kind of property we need the topologies, developed below. See in particular Corollary 11.7.10.

LEMMA 11.4.9. If $p \subseteq q$ then $p[a \mapsto u] \subseteq q[a \mapsto u]$.

Proof. Using Lemma 11.4.7, since $p \subseteq q$ if and only if $p \cup q = q$. $\square$

11.5. Some further operations on points

11.5.1. The operations: $\wedge$, $\forall$, $\bullet$, and $\multimap$ on points

Recall from the start of the Section that we fixed some λ -reduction theory Π . Recall also $p[a \mapsto u]$ from Definition 11.3.1:

DEFINITION 11.5.1. Suppose $p, q \subseteq |\mathcal{U}|$. Define the following operations:

$$\begin{aligned} p \wedge q &= p \cup q \\ \forall a. p &= \bigcup_{u \in |\mathcal{U}|} p[a \mapsto u] \\ p \bullet q &= \bigcup \{(st) \uparrow_{\Pi} \mid s \in p, t \in q\} \\ q \multimap p &= \bigcap \{r \in |\text{Pnt}_{\Pi}| \mid p \subseteq r \bullet q\} \end{aligned}$$

REMARK 11.5.2. Two things about Definition 11.5.1 might seem odd:

- $p \wedge q$ and $\forall a. p$ are sets *unions*— $p \cup q$ and $\bigcup_u p[a \mapsto u]$ respectively—and not sets intersections. This is a contravariance typical in duality results. See Proposition 11.7.12, and see Lemma 11.7.4 for a clearer view of the contravariance in this case.

- There is no $p \vee q$, even though in Proposition 11.1.7 we proved that a finite sets intersection of points is a point. This is because the operation of taking a sets intersection of points does not interact correctly with the σ -action, see Remark 11.4.8. For that, we need to consider sets of points; see Corollary 11.7.10.

LEMMA 11.5.3. *If p and q are Π -points then so are $p \wedge q$, $\forall a.p$, $p \bullet q$, and $q \rightarrow p$.*

Proof. All from Proposition 11.1.7, and for $p \bullet q$ also Lemma 11.2.2. $\square$

LEMMA 11.5.4. *Suppose $p, q \subseteq |\mathcal{U}|$ and suppose $r \in |\text{Pnt}_\Pi|$. Then:*

1. $q \rightarrow p \subseteq r$ if and only if $p \subseteq r \bullet q$.
2. If $p, q \in |\text{Pnt}_\Pi|$ then $p \subseteq (q \rightarrow p) \bullet q$ and $q \rightarrow (p \bullet q) \subseteq p$.
3. If $p \subseteq p'$ then $q \rightarrow p \subseteq q \rightarrow p'$.

Proof. Part 1 is from Definition 11.5.1. Part 2 follows using Lemma 11.5.3 since $q \rightarrow p \subseteq q \rightarrow p$ and $p \bullet q \subseteq p \bullet q$. For part 3 we note by part 2 that $p' \subseteq (q \rightarrow p') \bullet q$, deduce that $p \subseteq (q \rightarrow p') \bullet q$, and use part 1 to conclude that $q \rightarrow p \subseteq q \rightarrow p'$. $\square$

LEMMA 11.5.5. *Suppose $p \subseteq |\mathcal{U}|$ is small-supported. Then $a \# \forall a.p$.*

As a corollary, $\text{supp}(\forall a_1, \dots, a_n.p) \subseteq \text{supp}(p) \setminus \{a_1, \dots, a_n\}$.

Proof. The first part is from Definition 11.5.1 using Lemma 11.3.4, along with Theorem 2.3.2 and Corollary 2.3.5(3). The corollary follows using the first part and Theorem 2.3.2. $\square$

LEMMA 11.5.6. *Suppose $p, q \in \text{Pnt}_\Pi$ and $s \in |\mathcal{U}|$. Then:*

1. $p \subseteq \forall a.p$.
2. If $a \# p$ then $p = \forall a.p$.
3. If $p \subseteq q$ then $\forall a.p \subseteq \forall a.q$.
4. If $a \# q$ then $(\forall a.p) \subseteq q$ if and only if $p \subseteq q$.

Proof. We consider each part in turn:

1. We take $u=a$ and use Corollary 11.4.4.
2. Using Lemma 11.4.1.
3. From Lemma 11.4.9.
4. If $\forall a.p \subseteq q$ then $p \subseteq q$ using part 1 of this result, and if $p \subseteq q$ then $\forall a.p \overset{\text{pt 3}}{\subseteq} \forall a.q \overset{\text{pt 2}}{=} q$. $\square$

11.5.2. $\rightarrow$ and $\forall$ make λ

LEMMA 11.5.7. $q \rightarrow p = \{s' \in |\mathcal{U}| \mid \forall r. (p \subseteq r \bullet q \Rightarrow s' \in r)\}$.

As a corollary, $s' \in t \uparrow_\Pi \rightarrow s \uparrow_\Pi$ if and only if $s't \rightarrow_\Pi s$.

Proof. The first part just unpacks Definition 11.5.1.

Now suppose $s' \in t \uparrow_\Pi \rightarrow s \uparrow_\Pi$. By the first part, there exists some t' with $t \rightarrow_\Pi t'$ and $s't' \rightarrow_\Pi s$. By condition 1 of Definition 10.3.7 $s't \rightarrow_\Pi s't'$. It follows that $s't \rightarrow_\Pi s$.

Conversely suppose $s't \rightarrow_\Pi s$ and choose any s'' with $s \rightarrow_\Pi s''$. It follows that $s't \rightarrow_\Pi s''$ and since $t \in t \uparrow_\Pi$, we are done. $\square$

REMARK 11.5.8. For the reader's convenience we apply Lemma 11.5.7 to some concrete cases. Suppose $\mathcal{U} = \text{LmTm}$.

- Take $q = a \uparrow_{\Pi} = p$. Then $s' \in a \uparrow_{\Pi} \multimap a \uparrow_{\Pi}$ if and only if $s' a \rightarrow_{\Pi} a$. We can calculate that $\lambda a.a \in q \multimap p$ and also $\lambda b.a \in q \multimap p$.
- Assume some implementation of ordered pairs (s, t) and π_1 and π_2 for first and second projection, and take $q = (a, b) \uparrow_{\Pi}$ and $p = a \uparrow_{\Pi}$. Then $s' \in (a, b) \uparrow_{\Pi} \multimap a \uparrow_{\Pi}$ if and only if $s'(a, b) \rightarrow_{\Pi} a$. We can calculate that $\lambda b.a \in q \multimap p$ and $\pi_1 \in q \multimap p$.

So we can think of $\multimap$ as a kind of pattern-matching. We refine this to model λ in Proposition 11.5.10.

Recall $s \uparrow_{\Pi}$ from Definition 11.2.1, which is a point by Lemma 11.2.2.

LEMMA 11.5.9. $s \uparrow_{\Pi} \bullet t \uparrow_{\Pi} = (st) \uparrow_{\Pi}$.

Proof. Unpacking Definitions 11.2.1 and 11.5.1, $u \in s \uparrow_{\Pi} \bullet t \uparrow_{\Pi}$ when $s \rightarrow_{\Pi} s'$ and $t \rightarrow_{\Pi} t'$ and $s' t' \rightarrow_{\Pi} u$. Also, $u \in (st) \uparrow_{\Pi}$ when $st \rightarrow_{\Pi} u$. It is a fact that these two conditions are equivalent. $\square$

Proposition 11.5.10 connects $\forall a$ and $\multimap$ on points, with λa on $\mathcal{U}$. It will also be useful later in Corollary 11.7.14. We suggested in Remark 11.5.8 that $\multimap$ is a kind of pattern-matching; by that view, what we do now is pattern-matching on a universally quantified atom:

PROPOSITION 11.5.10. $\forall a.(a \uparrow_{\Pi} \multimap s \uparrow_{\Pi}) = (\lambda a.s) \uparrow_{\Pi}$.

Proof. We prove two subset inclusions:

- *Proof that $\forall a.(a \uparrow_{\Pi} \multimap s \uparrow_{\Pi}) \subseteq (\lambda a.s) \uparrow_{\Pi}$.* By condition 4 of Definition 10.3.7 $(\lambda a.s) a \rightarrow_{\Pi} s$, so from Definition 11.2.1⁴⁵ $s \uparrow_{\Pi} \subseteq ((\lambda a.s) a) \uparrow_{\Pi} \stackrel{\text{L11.5.9}}{=} (\lambda a.s) \uparrow_{\Pi} \bullet a \uparrow_{\Pi}$. It follows by Lemma 11.5.4 that $a \uparrow_{\Pi} \multimap s \uparrow_{\Pi} \subseteq (\lambda a.s) \uparrow_{\Pi}$. By condition 1 of Definition 10.3.1 $a \# \lambda a.s$ so by Theorem 2.3.2 $a \# (\lambda a.s) \uparrow_{\Pi}$, and therefore by part 4 of Lemma 11.5.6 $\forall a.(a \uparrow_{\Pi} \multimap s \uparrow_{\Pi}) \subseteq (\lambda a.s) \uparrow_{\Pi}$.
- *Proof that $(\lambda a.s) \uparrow_{\Pi} \subseteq \forall a.(a \uparrow_{\Pi} \multimap s \uparrow_{\Pi})$.* By condition 4 of Definition 10.3.7 $(\lambda a.s) a \rightarrow_{\Pi} s$, so by Lemma 11.5.7 $\lambda a.s \in a \uparrow_{\Pi} \multimap s \uparrow_{\Pi}$. By part 1 of Lemma 11.5.6 we have that $\lambda a.s \in \forall a.(a \uparrow_{\Pi} \multimap s \uparrow_{\Pi})$. By condition 1 of Definition 11.1.3 we conclude $(\lambda a.s) \uparrow_{\Pi} \subseteq \forall a.(a \uparrow_{\Pi} \multimap s \uparrow_{\Pi})$ as required. $\square$

11.6. How the σ -action on points commutes

The set of points Pnt_{Π} has plenty of structure. It is a nominal set, it has an $\circ$ -action $p[u \leftarrow a]$ (Corollary 11.2.5), a σ -action $p[a \rightarrow u]$ (Proposition 11.4.6) and a subsidiary pointwise version $p[a := u]$ (Definition 11.3.5). It is a fresh semi-lattice (a top element, $\wedge$, and $\vee$; see Remark 11.1.4 and Definition 11.5.1) and has $\bullet$ and $\multimap$ (Definition 11.5.1) and even $\mathcal{U}a.p$ a sets version of the $\mathcal{U}$ -quantifier (Lemma 11.3.8). There is also a map from $|\mathcal{U}|$ to points given by $s \in |\mathcal{U}|$ maps to $s \uparrow_{\Pi}$ (Definition 11.2.1).

⁴⁵See also the later and more comprehensive Lemma 11.7.7.

In this subsection we consider useful ways in which the σ -action commutes with some of this structure. These commutation results will later be useful in proving that sets of points have the structure of an impredicative lattice with $\forall$ and $\bullet$.

LEMMA 11.6.1. $s \uparrow_{\Pi}[a \mapsto u] = s[a := u] \uparrow_{\Pi}$.

Proof. Using $(\sigma\alpha)$ and Lemma 11.3.4(1) assume without loss of generality that $a \# u$. We prove two subset inclusions.

- *Proof that $s \uparrow_{\Pi}[a \mapsto u] \subseteq s[a := u] \uparrow_{\Pi}$.* By Lemma 3.2.6 $a \# s[a := u]$ and by Theorem 2.3.2 also $a \# s[a := u] \uparrow_{\Pi}$. Thus by Lemma 11.3.6 to it suffices to prove $s \uparrow_{\Pi}[a := u] \subseteq s[a := u] \uparrow_{\Pi}$. Suppose $s \rightarrow_{\Pi} t$. By condition 3 of Definition 10.3.7 $s[a := u] \rightarrow_{\Pi} t[a := u]$. The result follows.
- *Proof that $s[a := u] \uparrow_{\Pi} \subseteq s \uparrow_{\Pi}[a \mapsto u]$.*

By condition 1 of Definition 11.1.3 it suffices to note that $s[a := u] \in s \uparrow_{\Pi}[a \mapsto u]$. $\square$

LEMMA 11.6.2. • $(p \bullet q)[a := u] = p[a := u] \bullet q[a := u]$.
 • $(\forall a. p) \bullet (\forall a. q) = \forall a. (p \bullet q)$ (Definition 2.6.1; Lemma 11.3.8).

Proof. From Definitions 11.5.1 and 11.3.5, we have that $r \in (p \bullet q)[a := u]$ when there exist $s \in p$ and $t \in q$ such that $(st)[a := u] \rightarrow_{\Pi} r$, and that $r \in p[a := u] \bullet q[a := u]$ when there exist $s \in p$ and $t \in q$ such that $s[a := u] t[a := u] \rightarrow_{\Pi} r$. By Definition 10.3.1(3) $(st)[a := u] = s[a := u] t[a := u]$.

For the second part, $r \in (\forall a. p) \bullet (\forall a. q)$ when there exist s and t such that $\mathcal{W}b. (b a) \cdot s \in p$ and $\mathcal{W}b. (b a) \cdot t \in q$ and $r = st$. It is a fact that the $\mathcal{W}$ -quantifier distributes over conjunction [Gab11b, Theorem 6.6] (provided p and q are small-supported; by Lemma 11.1.5(2) they are) so also $\mathcal{W}b. (b a) \cdot (st) \in p \bullet q$. The result follows. $\square$

COROLLARY 11.6.3. Suppose $p, q \in |Pnt_{\Pi}|$. Then:

- $(p \wedge q)[a \mapsto u] = p[a \mapsto u] \wedge q[a \mapsto u]$.
- $(p \bullet q)[a \mapsto u] = p[a \mapsto u] \bullet q[a \mapsto u]$.

Proof. For the first part, by Definition 11.5.1 $p \wedge q = p \cup q$. We use Lemma 11.4.7.

For the second part, we use Lemma 11.3.4(1) to assume without loss of generality that $a \# u$. By Lemma 11.3.9 $(p \bullet q)[a \mapsto u] = \forall a. ((p \bullet q)[a := u])$ and $p[a \mapsto u] \bullet q[a \mapsto u] = (\forall a. (p[a := u])) \bullet (\forall a. (q[a := u]))$. We use Lemma 11.6.2. $\square$

Proposition 11.6.4 is the key technical result to proving $(\sigma \dashv \bullet)$ valid in Proposition 11.8.18:

PROPOSITION 11.6.4. Suppose $u \in |\mathcal{U}|$ and $p \in |Pnt_{\Pi}|$. Then

$$(b \uparrow_{\Pi} \dashv \bullet p)[a \mapsto u] = b \uparrow_{\Pi} \dashv \bullet (p[a \mapsto u]).$$

Proof. For the right-to-left inclusion $b \uparrow_{\Pi} \dashv \bullet (p[a \mapsto u]) \subseteq (b \uparrow_{\Pi} \dashv \bullet p)[a \mapsto u]$ we reason as follows:

$$\begin{aligned} b \uparrow_{\Pi} \dashv \bullet (p[a \mapsto u]) &\subseteq (b \uparrow_{\Pi} \dashv \bullet p)[a \mapsto u] \\ &\Leftrightarrow p[a \mapsto u] \subseteq (b \uparrow_{\Pi} \dashv \bullet p)[a \mapsto u] \bullet (b \uparrow_{\Pi}) && \text{Lemma 11.5.4} \\ &\Leftrightarrow p[a \mapsto u] \subseteq (b \uparrow_{\Pi} \dashv \bullet p)[a \mapsto u] \bullet (b \uparrow_{\Pi}[a \mapsto u]) && \text{Lemma 11.4.1} \\ &\Leftrightarrow p[a \mapsto u] \subseteq ((b \uparrow_{\Pi} \dashv \bullet p) \bullet b \uparrow_{\Pi})[a \mapsto u] && \text{Corollary 11.6.3} \\ &\Leftarrow p[a \mapsto u] \subseteq p[a \mapsto u] && \text{Ls 11.4.9 \& 11.5.4} \end{aligned}$$

For the left-to-right inclusion $(b \uparrow_{\Pi} \multimap p)[a \mapsto u] \subseteq b \uparrow_{\Pi} \multimap (p[a \mapsto u])$, using Lemma 11.3.4(1) to rename if necessary assume $a \# u$. Using Theorem 2.3.2 also note that $a \# b \uparrow_{\Pi} \multimap (p[a \mapsto u])$. Thus by Lemma 11.3.6 and part 3 of Lemma 11.5.4 it suffices to prove that $(b \uparrow_{\Pi} \multimap p)[a := u] \subseteq b \uparrow_{\Pi} \multimap (p[a := u])$.

Unpacking Definitions 11.5.1 (for $\multimap$) 11.2.1 (for $\uparrow_{\Pi}$) and 11.3.5 (for $[a := u]$ on points), this simplifies to showing that for all s ,

$$\exists s'. (s' b \in p \wedge s = s'[a := u]) \Rightarrow \exists t'. (t' \in p \wedge s b = t'[a := u]).$$

So suppose s' is such that $s' b \in p$ and $s = s'[a := u]$. Take $t' = s' b$. So $t'[a := u] = (s' b)[a := u] = s b$ as required. $\square$

11.7. Operations on sets of points

11.7.1. Basic definitions

DEFINITION 11.7.1. Suppose $p, q \in |Pnt_{\Pi}|$. Define the following operations:

$$\begin{aligned} p^{\bullet} &= \{q \in |Pnt_{\Pi}| \mid p \subseteq q\} \\ p \circ q &= (p \bullet q)^{\bullet} \end{aligned}$$

REMARK 11.7.2. $p^{\bullet}$ and $\circ$ generate sets of points. We use these to build a topology and a nominal spectral space with $\circ$ over Pnt_{Π} in Definition 11.8.1 and so to prove Theorem 11.9.5 (completeness).

$p \circ q$ is a *combination operator* in the sense of Definition 9.2.1, so that we get operations $\bullet$ and $\multimap$ on sets of points from Definition 9.2.4. Unpacking Definition 11.5.1 (for $p \bullet q$) and 11.7.1 (for $(p \bullet q)^{\bullet}$) it is not hard to see that $p \circ q$ is the set of Π -points r such that $\{st \mid s \in p, t \in q\} \subseteq r$.

Thus we have two operations named $\bullet$; one on points from Definition 11.5.1 and one on sets of points from Definition 11.7.1. Similarly, we have two operations named $\multimap$. It will always be clear from context which is meant, and they are related by Proposition 11.7.12.

REMARK 11.7.3. It might be worth mentioning that $p^{\bullet}$ from Definition 11.7.1 is not a repeat of $x^{\bullet}$ from Definition 6.3.1:

- In Definition 6.3.1 we assumed an underlying nominal distributive lattice with $\forall$.
- Definition 11.7.1 is constructed using Pnt_{Π} . Now we consider Definition 11.5.1 and see that if we ignore $\bullet$ and $\multimap$ then Pnt_{Π} is almost a nominal distributive lattice with $\forall$ —but it lacks a disjunction (this is discussed in Remark 11.5.2). We could reasonably call it a semilattice with $\forall$.

So Pnt_{Π} is a specific structure with its own properties.

Lemma 11.7.4 is in the spirit of Lemma 6.3.2:

LEMMA 11.7.4. Suppose $p, q \in |Pnt_{\Pi}|$. Then the following conditions are equivalent:

$$p \in q^{\bullet} \Leftrightarrow q \subseteq p \Leftrightarrow p^{\bullet} \subseteq q^{\bullet}$$

Furthermore, if $s \in |\mathcal{U}|$ then

$$p \in s \uparrow_{\Pi}^{\bullet} \Leftrightarrow s \in p.$$

(Recall that $s \uparrow_{\Pi} \in Pnt_{\Pi}$ by Definition 11.2.1 and Lemma 11.2.2.)

Proof. Unpacking Definition 11.7.1, $p \in q^{\bullet}$ implies $q \subseteq p$. It is a fact of sets that if $q \subseteq p$ then $p \subseteq p'$ implies $q \subseteq p'$, thus $p^{\bullet} \subseteq q^{\bullet}$. Finally, if $p^{\bullet} \subseteq q^{\bullet}$ then since $p \in p^{\bullet}$, also $p \in q^{\bullet}$.

Now consider $p \in s \uparrow_{\Pi}^{\bullet}$. By the previous paragraph this is if and only if $s \uparrow_{\Pi} \subseteq p$, and from condition 1 of Definition 11.1.3 this is if and only if $s \in p$. $\square$

COROLLARY 11.7.5. *The assignment $p \mapsto p^{\bullet}$ is injective from $|Pnt_{\Pi}|$ to $Nom(Pnt_{\Pi})$. As a corollary, $supp(p^{\bullet}) = supp(p)$.*

Proof. The first part follows using Lemma 11.7.4. The corollary follows by part 3 of Theorem 2.3.2. $\square$

Corollary 11.7.6 should remind us of condition 2 of Definition 7.9.1. We use it to get just that in Proposition 11.8.23:

COROLLARY 11.7.6. *Suppose $p, q \in |Pnt_{\Pi}|$ and $\forall b.(b a) \cdot p \in q^{\bullet}$. Then $p \in (\forall a.q)^{\bullet}$.*

Proof. Choose fresh b (so $b \# p, q$), so that $(b a) \cdot p \in q^{\bullet}$. We note two facts:

- By Proposition 2.3.4 $a \# (b a) \cdot p$.
- By Lemma 11.5.5 $a \# \forall a.q$, so that by Theorem 2.3.2 $a, b \# (\forall a.q)^{\bullet}$.

Now we reason as follows:

$$\begin{aligned}
 (b a) \cdot p \in q^{\bullet} &\Leftrightarrow q \subseteq (b a) \cdot p && \text{Lemma 11.7.4} \\
 &\Leftrightarrow \forall a.q \subseteq (b a) \cdot p && \text{Lemma 11.5.6(4)} \\
 &\Leftrightarrow (b a) \cdot p \in (\forall a.q)^{\bullet} && \text{Lemma 11.7.4} \\
 &\Leftrightarrow p \in (\forall a.q)^{\bullet} && \text{Corollary 2.3.5 } a, b \# (\forall a.q)^{\bullet}
 \end{aligned}
 \quad \square$$

The proof of Lemma 11.7.7 is simple given what we have proved so far, but it is important; for instance it is the final step in the proof of Completeness in Theorem 11.9.5.

LEMMA 11.7.7. *The following conditions are equivalent:*

$$s \uparrow_{\Pi}^{\bullet} \subseteq t \uparrow_{\Pi}^{\bullet} \Leftrightarrow s \uparrow_{\Pi} \in t^{\bullet} \Leftrightarrow t \uparrow_{\Pi} \subseteq s \uparrow_{\Pi} \Leftrightarrow t \in s \uparrow_{\Pi} \Leftrightarrow s \rightarrow_{\Pi} t \Leftrightarrow \Pi \vdash s \rightarrow t$$

Proof. We obtain $s \uparrow_{\Pi}^{\bullet} \subseteq t \uparrow_{\Pi}^{\bullet} \Leftrightarrow s \uparrow_{\Pi} \in t^{\bullet} \Leftrightarrow t \uparrow_{\Pi} \subseteq s \uparrow_{\Pi} \Leftrightarrow t \in s \uparrow_{\Pi}$ from Lemma 11.7.4. Also, $t \in s \uparrow_{\Pi} \Leftrightarrow s \rightarrow_{\Pi} t$ is direct from Definition 11.2.1. Finally, by Notation 10.3.9 $s \rightarrow_{\Pi} t$ is synonymous with $\Pi \vdash s \rightarrow t$. $\square$

COROLLARY 11.7.8. *$s \uparrow_{\Pi}^{\bullet} = t \uparrow_{\Pi}^{\bullet}$ if and only if $s =_{\Pi} t$.*

Proof. From Lemma 11.7.7. $\square$

11.7.2. Commutation properties

Recall $q[a \mapsto u]$ from Definition 11.3.1, $p[u \leftarrow a]$ from Definition 3.3.1, and (since by Corollary 11.2.5 Pnt_{Π} is an $\mathfrak{v}$ -algebra) $p^*[a \mapsto u]$ from Definition 3.4.1.

Theorem 11.7.9 is fairly easy to prove, but part 1 of it is key. It relates the natural σ -action $p^*[a \mapsto u]$ to the σ -action on points $p[a \mapsto u]$ from Proposition 11.4.6. Compare Theorem 11.7.9 with Lemma 6.4.1:

THEOREM 11.7.9. *Suppose $q \in |Pnt_{\Pi}|$ and $u \in |\mathcal{U}|$.*

1. $q^*[a \mapsto u] = (q[a \mapsto u])^*$.
As a corollary taking $p = s^*_{\Pi}$, $s^*_{\Pi}[a \mapsto t] = s[a := t]^*_{\Pi}$.
2. $\pi \cdot (q^*) = (\pi \cdot q)^*$.

Proof. Consider some p ; we wish to show that $p \in q^*[a \mapsto u] \Leftrightarrow p \in (q[a \mapsto u])^*$. By Lemmas 3.4.4 and 11.3.4(1) we may α -rename a in $q^*[a \mapsto u]$ and $q[a \mapsto u]$ to assume without loss of generality that $a \# u, p$. We reason as follows; we use part 2 of Proposition 3.4.3 because by Lemma 11.1.5(2) p and q have small support:

$$\begin{aligned}
 p \in q^*[a \mapsto u] &\Leftrightarrow p[u \leftarrow a] \in q^* && \text{Proposition 3.4.3(2) } a \# u, p \\
 &\Leftrightarrow q \subseteq p[u \leftarrow a] && \text{Definition 11.7.1} \\
 &\Leftrightarrow q[a \mapsto u] \subseteq p && \text{Proposition 11.3.3 } a \# u, p \\
 &\Leftrightarrow p \in (q[a \mapsto u])^* && \text{Definition 11.7.1}
 \end{aligned}$$

The corollary follows using Lemma 11.6.1.

The second part is proved by similar calculations, or directly from Theorem 2.3.2. $\square$

Corollary 11.7.10 describes how the σ -action interacts with strictly supported sets union:

COROLLARY 11.7.10. *Suppose $\mathcal{P} \subseteq |Pnt_{\Pi}|$ is strictly small-supported (in particular by Lemma 2.5.5 it suffices that $\mathcal{P}$ be finite). Then*

$$\left(\bigcup_{p \in \mathcal{P}} p^* \right) [a \mapsto u] = \bigcup_{p \in \mathcal{P}} (p[a \mapsto u])^*.$$

Proof. From Lemma 5.1.1(2) and Theorem 11.7.9. $\square$

COROLLARY 11.7.11. 1. $a \# p^*$ then $(p^*)[a \mapsto u] = p^*$.

2. If $b \# p^*$ then $((b \ a) \cdot p^*)[b \mapsto u] = p^*[a \mapsto u]$.

Proof. For the first part, suppose $a \# p^*$. By Corollary 11.7.5 $a \# p$. We use Theorem 11.7.9 and Lemma 11.4.1.

For the second part we reason similarly using Theorem 11.7.9 and Lemma 11.3.4. $\square$

PROPOSITION 11.7.12. 1. $p^* \cap q^* = (p \wedge q)^*$.

2. $\bigcap^{a} p^* = (\forall a. p)^*$ ($\forall a. p$ is from Definition 11.5.1).

3. $p^* \bullet q^* = (p \bullet q)^*$.

4. $q^* \multimap p^* = (q \multimap p)^*$.

Proof. We consider each case in turn; with what we have proved so far, the calculations are routine. u will range over elements of $|\mathcal{U}|$:

1. We reason as follows:

$$\begin{aligned} r \in p^\bullet \cap q^\bullet &\Leftrightarrow p \subseteq r \text{ and } q \subseteq r \\ &\Leftrightarrow p \cup q \subseteq r \\ &\Leftrightarrow p \in (p \wedge q)^\bullet \end{aligned}$$

Definition 11.7.1

Fact

Definition 11.7.1

2. We reason as follows:

$$\begin{aligned} \bigcap^{#a} p^\bullet &= \bigcap_{u \in |\mathcal{U}|} p^\bullet[a \mapsto u] \\ &= \bigcap_{u \in |\mathcal{U}|} (p[a \mapsto u])^\bullet \\ &= (\bigcup_{u \in |\mathcal{U}|} p[a \mapsto u])^\bullet \\ &= (\forall a. p)^\bullet \end{aligned}$$

Definition 5.2.1

Theorem 11.7.9

Fact of Def 11.7.1

Definition 11.5.1

3. We reason as follows:

$$\begin{aligned} r \in p^\bullet \bullet q^\bullet &\Leftrightarrow \exists p' \in p^\bullet, q' \in q^\bullet. r \in p' \circ q' \\ &\Leftrightarrow \exists p', q'. p \subseteq p' \wedge q \subseteq q' \wedge r \in p' \circ q' \\ &\Leftrightarrow \exists p', q'. p \subseteq p' \wedge q \subseteq q' \wedge p' \bullet q' \subseteq r \\ &\Leftrightarrow p \bullet q \subseteq r \\ &\Leftrightarrow r \in (p \bullet q)^\bullet \end{aligned}$$

Proposition 9.2.6

Definition 11.7.1

Definition 11.7.1

Fact

Definition 11.7.1

4. We reason as follows:

$$\begin{aligned} r \in q^\bullet \multimap p^\bullet &\Leftrightarrow \forall q' \in q^\bullet. r \circ q' \subseteq p^\bullet \\ &\Leftrightarrow \forall q'. q \subseteq q' \Rightarrow (r \bullet q')^\bullet \subseteq p^\bullet \\ &\Leftrightarrow \forall q'. q \subseteq q' \Rightarrow p \subseteq r \bullet q' \\ &\Leftrightarrow p \subseteq r \bullet q \\ &\Leftrightarrow q \multimap p \subseteq r \\ &\Leftrightarrow r \in (q \multimap p)^\bullet \end{aligned}$$

Proposition 9.2.6

Definition 11.7.1

Definition 11.7.1

Fact

Lemma 11.5.4

Definition 11.7.1 $\square$

Corollary 11.7.13 resembles Lemma 9.4.9 and is proved similarly:

COROLLARY 11.7.13. Suppose $p, q \in |\mathcal{Pnt}_\Pi|$ and $u \in |\mathcal{U}|$. Then $\bullet$ and $\multimap$ validate axioms $(\sigma \bullet)$ and $(\sigma \multimap)$ from Figure 4:

$$\begin{aligned} (p^\bullet \bullet q^\bullet)[a \mapsto u] &= p^\bullet[a \mapsto u] \bullet q^\bullet[a \mapsto u] \\ (b_{\Pi}^\bullet \multimap p^\bullet)[a \mapsto u] &= b_{\Pi}^\bullet \multimap p^\bullet[a \mapsto u] \end{aligned}$$

Proof. We reason as follows:

$$\begin{aligned} (p^\bullet \bullet q^\bullet)[a \mapsto u] &= (p \bullet q)^\bullet[a \mapsto u] \\ &= ((p \bullet q)[a \mapsto u])^\bullet \\ &= (p[a \mapsto u] \bullet q[a \mapsto u])^\bullet \\ &= (p[a \mapsto u])^\bullet \bullet (q[a \mapsto u])^\bullet \\ &= p^\bullet[a \mapsto u] \bullet q^\bullet[a \mapsto u] \end{aligned}$$

Part 3 of Proposition 11.7.12

Part 1 of Theorem 11.7.9

Corollary 11.6.3

Part 3 of Proposition 11.7.12

Part 1 of Theorem 11.7.9

$$\begin{aligned} (b_{\Pi}^\bullet \multimap p^\bullet)[a \mapsto u] &= (b_{\Pi}^\bullet \multimap p)^\bullet[a \mapsto u] \\ &= ((b_{\Pi}^\bullet \multimap p)[a \mapsto u])^\bullet \\ &= (b_{\Pi}^\bullet \multimap (p[a \mapsto u]))^\bullet \\ &= b_{\Pi}^\bullet \multimap (p[a \mapsto u])^\bullet \\ &= b_{\Pi}^\bullet \multimap p^\bullet[a \mapsto u] \end{aligned}$$

Part 4 of Proposition 11.7.12

Part 1 of Theorem 11.7.9

Proposition 11.6.4

Part 4 of Proposition 11.7.12

Part 1 of Theorem 11.7.9 $\square$

COROLLARY 11.7.14. If $s', s' \in |\mathcal{U}|$ then $s' \uparrow_{\Pi}^\bullet \bullet s \uparrow_{\Pi}^\bullet = (s' s) \uparrow_{\Pi}^\bullet$ and $\lambda a. (s \uparrow_{\Pi}^\bullet) = (\lambda a. s) \uparrow_{\Pi}^\bullet$.

Proof. We reason as follows:

$$\begin{aligned}
 s' \uparrow_{\Pi}^{\bullet} \bullet s \uparrow_{\Pi}^{\bullet} &= (s' \uparrow_{\Pi}^{\bullet} \bullet s \uparrow_{\Pi}^{\bullet})^{\bullet} && \text{Part 3 of Prop 11.7.12} \\
 &= (s' s)^{\bullet} \uparrow_{\Pi}^{\bullet} && \text{Lemma 11.5.9} \\
 \lambda a. (s \uparrow_{\Pi}^{\bullet}) &= \forall a. (a \uparrow_{\Pi}^{\bullet} \multimap s \uparrow_{\Pi}^{\bullet}) && \text{Notation 10.2.1} \\
 &= \forall a. (a \uparrow_{\Pi}^{\bullet} \multimap s \uparrow_{\Pi}^{\bullet})^{\bullet} && \text{Part 4 of Prop 11.7.12} \\
 &= (\forall a. (a \uparrow_{\Pi}^{\bullet} \multimap s \uparrow_{\Pi}^{\bullet}))^{\bullet} && \text{Part 2 of Prop 11.7.12} \\
 &= (\lambda a. s)^{\bullet} \uparrow_{\Pi}^{\bullet} && \text{Proposition 11.5.10} \quad \square
 \end{aligned}$$

We conclude with a small calculation on $\multimap$:

LEMMA 11.7.15. *Suppose $p \in |Pnt_{\Pi}|$ and $s, t \in |\mathcal{U}|$. Then the following are equivalent:*

$$p \in t \uparrow_{\Pi}^{\bullet} \multimap s \uparrow_{\Pi}^{\bullet} \quad \Leftrightarrow \quad s \in p \bullet t \uparrow_{\Pi} \quad \Leftrightarrow \quad \exists s' \in p. (s' t \rightarrow_{\Pi} s).$$

Proof. We reason as follows:

$$\begin{aligned}
 p \in t \uparrow_{\Pi}^{\bullet} \multimap s \uparrow_{\Pi}^{\bullet} &\Leftrightarrow p \in (t \uparrow_{\Pi}^{\bullet} \multimap s \uparrow_{\Pi}^{\bullet})^{\bullet} && \text{Part 4 of Proposition 11.7.12} \\
 &\Leftrightarrow t \uparrow_{\Pi}^{\bullet} \multimap s \uparrow_{\Pi}^{\bullet} \subseteq p && \text{Lemma 11.7.4} \\
 &\Leftrightarrow s \uparrow_{\Pi}^{\bullet} \subseteq p \bullet t \uparrow_{\Pi} && \text{Lemma 11.5.4} \\
 &\Leftrightarrow s \in p \bullet t \uparrow_{\Pi} && \text{Part 1 of Definition 11.1.3} \\
 &\Leftrightarrow \exists s' \in p. (s' t \rightarrow_{\Pi} s) && \text{Unpacking Definition 11.5.1} \quad \square
 \end{aligned}$$

11.8. A topology

11.8.1. Giving Pnt_{Π} a topology

DEFINITION 11.8.1. Make Pnt_{Π} into a nominal spectral space with $\circ$ (Definition 9.2.9):

1. The topology generated under small-supported unions by $\{p^{\bullet} \mid p \in |Pnt_{\Pi}|\}$.
2. The combination operation $p \circ q = (p \bullet q)^{\bullet}$ from Definition 11.7.1.
3. The pointwise actions from Definitions 3.3.1 (for π and $[u \leftarrow a]$) and 9.2.4 (for $\bullet$ and $\multimap$).
4. $\partial_{Pnt_{\Pi}} u = u \uparrow_{\Pi}^{\bullet}$ for $u \in |\mathcal{U}|$.

REMARK 11.8.2. So $U \in \text{Opn}(Pnt_{\Pi})$ (meaning that U is open) when there exists some small-supported $\mathcal{P} \subseteq |Pnt_{\Pi}|$ (Definition 2.5.2) with $U = \bigcup \{p^{\bullet} \mid p \in \mathcal{P}\}$.

PROPOSITION 11.8.3. *Definition 11.8.1 does indeed determine a nominal $\sigma \circ$ -topological space in the sense of Definition 9.2.2.*

Proof. We start with the conditions from Definition 7.1.1.

By Corollary 11.2.5 ($|Pnt_{\Pi}|, \cdot, \mathcal{U}, \circ$) is an $\circ$ -algebra.

Consider $U, V \in \text{Opn}(Pnt_{\Pi})$. As noted in Remark 11.8.2 $U = \bigcup \{p^{\bullet} \mid p \in \mathcal{P}\}$ and $V = \bigcup \{q^{\bullet} \mid q \in \mathcal{Q}\}$ for some small-supported $\mathcal{P}, \mathcal{Q} \subseteq |Pnt_{\Pi}|$. By Theorem 2.3.2 each U is small-supported. Furthermore:

1. $\emptyset$ and $|Pnt_{\Pi}|$ are open. $\emptyset$ (the empty set of points) is open by construction of the topology, and $|Pnt_{\Pi}| = \bigcup \{\varnothing^{\bullet}\}$ is open—we noted in Remark 11.1.4 that $\varnothing$ the empty set of phrases is a point. Then $\varnothing^{\bullet}$ is the set of all points and $\{\varnothing^{\bullet}\}$ is strictly small-supported (by $\varnothing$ the empty set of atoms).

2. If U and V are open then so are $U \cap V$ and $U \cup V$. It is a fact of sets that $U \cap V = \bigcup \{p^\bullet \cap q^\bullet \mid p \in \mathcal{P}, q \in \mathcal{Q}\}$. We use part 1 of Proposition 11.7.12 and Theorem 2.3.2.
- For $U \cup V$, we note that the union of two small-supported sets is small-supported.
3. If $\mathcal{U}$ is a small-supported set of open sets then $\bigcup \mathcal{U}$ is open. Using Theorem 2.3.2.

Finally, we consider Definition 9.2.2 and note that $\circ$ is a combination operator (that $\circ$ is equivariant follows immediately from Theorem 2.3.2). $\square$

11.8.2. The compact open sets of Pnt_Π

LEMMA 11.8.4. If $U \in \text{Opn}(Pnt_\Pi)$ then

$$p \in U \text{ if and only if } p^\bullet \subseteq U.$$

Proof. Suppose $p \in U$. By Definition 11.8.1 $U = \bigcup \{q^\bullet \mid q \in \mathcal{Q}\}$ for some small-supported $\mathcal{Q} \subseteq |Pnt_\Pi|$, and there exists $q \in \mathcal{Q}$ with $p \in q^\bullet$. By Lemma 11.7.4, $p^\bullet \subseteq q^\bullet$.

The reverse implication is easy since $p \in p^\bullet$ by construction in Definition 11.7.1. $\square$

LEMMA 11.8.5. If $p \in |Pnt_\Pi|$ then $p^\bullet$ is compact in Pnt_Π with the topology from Definition 11.8.1. In symbols: $p^\bullet \in \text{Cpct}(Pnt_\Pi)$.

As a corollary, if $s \in |\mathcal{U}|$ then $s^\bullet_\Pi \in \text{Cpct}(Pnt_\Pi)$.

Proof. Suppose $\mathcal{U}$ covers $p^\bullet$. Since $p \in p^\bullet$, also $p \in U$ for some $U \in \mathcal{U}$. By Lemma 11.8.4 $p^\bullet \subseteq U$ and so $p^\bullet$ is covered by $\{U\}$.

The corollary follows from Lemma 11.2.2. $\square$

LEMMA 11.8.6. If $U \in \text{Opn}(Pnt_\Pi)$ is compact then $U = \bigcup \{p^\bullet \mid p \in \mathcal{P}\}$ for some finite $\mathcal{P} \subseteq |Pnt_\Pi|$.

Proof. Suppose U is compact. By construction $U = \bigcup \mathcal{P}'$ for some small-supported (but not necessarily finite) $\mathcal{P}' \subseteq |Pnt_\Pi|$. By compactness this has a finite subcover $\{p_1^\bullet, \dots, p_n^\bullet\}$. We take $\mathcal{P} = \{p_1, \dots, p_n\}$. $\square$

REMARK 11.8.7. We have mentioned that the canonical model Pnt_Π is not just a replay of $F(\mathcal{D})$ from the duality proof (Definition 7.2.1). So compare

- Theorem 7.4.3, which identifies compacts in $F(\mathcal{D})$ with sets of points of the form $x^\bullet$, with
- Lemma 11.8.6, which identifies compacts in Pnt_Π with *finite unions* of sets of the form $p^\bullet$.

This is related to issues discussed in Remarks 11.5.2 and 11.4.8.

11.8.3. Interaction of $\bullet$ and $\rightarrow$ with $\cup$ and $\subseteq$

Lemmas 11.8.8 and 11.8.9 are useful for Corollary 11.8.10:

LEMMA 11.8.8. Suppose $p, q, q' \in |Pnt_{\Pi}|$ and $q \subseteq q'$. Then $p \bullet q \subseteq p \bullet q'$ and $p \circ q' \subseteq p \circ q$.

Proof. $p \bullet q$ is defined in Definition 11.5.1, and the result follows direct from the definition. By Definition 11.7.1 $p \circ q' = \{r \in Pnt_{\Pi} \mid p \bullet q' \subseteq r\}$ and $p \circ q = \{r \in Pnt_{\Pi} \mid p \bullet q \subseteq r\}$. We use Lemma 11.7.4. $\square$

LEMMA 11.8.9. If $X \in \text{Opn}(Pnt_{\Pi})$ and $p, q \in |Pnt_{\Pi}|$ then the following conditions are equivalent:

$$p \bullet q \in X \Leftrightarrow (p \bullet q)^{\bullet} \subseteq X \Leftrightarrow p \circ q \subseteq X \Leftrightarrow p \circ (q^{\bullet}) \subseteq X$$

Proof. By Lemma 11.8.4 $p \bullet q \in X$ if and only if $(p \bullet q)^{\bullet} \subseteq X$. By Definition 11.7.1 $(p \bullet q)^{\bullet} = p \circ q$.

Suppose $p \circ q \subseteq X$. Consider any $q' \in q^{\bullet}$, meaning by Definition 11.7.1 that $q \subseteq q'$. By Lemma 11.8.8 $p \bullet q' \subseteq p \circ q$. Since q' was arbitrary, it follows from Definition 9.2.4 that $p \circ (q^{\bullet}) \subseteq X$.

Conversely if $p \circ (q^{\bullet}) \subseteq X$ then since (from Definition 11.7.1) $q \in q^{\bullet}$, from Definition 9.2.4 $p \circ q \subseteq X$. $\square$

COROLLARY 11.8.10. If $X \in \text{Opn}(Pnt_{\Pi})$ and $r, q \in |Pnt_{\Pi}|$ then $r \in (q_1^{\bullet} \cup \dots \cup q_n^{\bullet}) \rightarrow X$ if and only if $r \bullet q_i \in X$ for $1 \leq i \leq n$.

Proof. By Proposition 9.2.6 $r \in (q_1^{\bullet} \cup \dots \cup q_n^{\bullet}) \rightarrow X$ if and only if $r \circ (q_1^{\bullet} \cup \dots \cup q_n^{\bullet}) \subseteq X$. From Definition 9.2.4 this is if and only if $r \circ q_i^{\bullet} \subseteq X$ for $1 \leq i \leq n$. We use Lemma 11.8.9. $\square$

COROLLARY 11.8.11. $r \in q^{\bullet} \rightarrow (p_1^{\bullet} \cup \dots \cup p_n^{\bullet})$ if and only if $r \in q^{\bullet} \rightarrow p_i^{\bullet}$ for some $1 \leq i \leq n$.

Proof. By Proposition 9.2.6 $r \in q^{\bullet} \rightarrow (p_1^{\bullet} \cup \dots \cup p_n^{\bullet})$ if and only if $r \circ q \subseteq p_1^{\bullet} \cup \dots \cup p_n^{\bullet}$. By Lemma 11.8.9 this is if and only if $r \bullet q \in p_1^{\bullet} \cup \dots \cup p_n^{\bullet}$. It is a fact of sets that this is if and only if $r \bullet q \in p_i^{\bullet}$ for some $1 \leq i \leq n$. Using Lemma 11.8.9 and Proposition 9.2.6 again, this is if and only if $r \in q^{\bullet} \rightarrow p_i^{\bullet}$ for some $1 \leq i \leq n$, as required. $\square$

COROLLARY 11.8.12. Suppose $\mathcal{P}, \mathcal{Q} \subseteq |Pnt_{\Pi}|$ are finite. Then

$$\bigcup \{q^{\bullet} \mid q \in \mathcal{Q}\} \rightarrow \bigcup \{p^{\bullet} \mid p \in \mathcal{P}\} = \bigcap \{ \bigcup \{(q \rightarrow p)^{\bullet} \mid p \in \mathcal{P}\} \mid q \in \mathcal{Q} \}.$$

Proof. We combine Corollaries 11.8.10 and 11.8.11 with part 4 of Proposition 11.7.12. $\square$

11.8.4. Interaction of $\bigcap^{\#a}$ with unions

Remarkably, $\bigcap^{\#a}$ commutes with certain unions. This is Lemma 11.8.13. The property is not valid in general in $\text{inDi}\mathbb{V}_\bullet$; indeed this is not generally true in logic: $\forall x.(\phi \vee \psi)$ is not normally logically equivalent to $(\forall x.\phi) \vee (\forall x.\psi)$. But, it holds in the canonical model Pnt_Π :

LEMMA 11.8.13. *Suppose $p_1, \dots, p_n \in Pnt_\Pi$. Then*

$$\bigcap^{\#a} \bigcup_i (p_i^\bullet) = \bigcup_i \bigcap^{\#a} (p_i^\bullet).$$

Proof. Suppose $q \in \bigcap^{\#a} \bigcup_i p_i^\bullet$. Using Lemma 5.2.4 rename to assume without loss of generality that $a \# q$.

By Definition 5.2.1 and Corollary 11.7.10 $q \in \bigcap^{\#a} \bigcup_i p_i^\bullet$ when $q \in \bigcup_i (p_i[a \mapsto u])^\bullet$ for every $u \in |\mathcal{U}|$. Choose fresh b (so $b \# q, p_1, \dots, p_n$). By facts of sets, there exists $1 \leq i \leq n$ such that $q \in p_i[a \mapsto b]^\bullet$. Using Lemma 11.4.2 $q \in ((b \ a) \cdot p_i)^\bullet$, so by Theorem 2.3.2 $(b \ a) \cdot q \in p_i^\bullet$ and by Corollary 11.7.6 $q \in (\forall a. p_i)^\bullet$. We use Proposition 11.7.12(2).

Conversely, it is easy to prove that $\bigcap^{\#a} p_i^\bullet \subseteq \bigcap^{\#a} \bigcup_i p_i^\bullet$, either using Lemma 4.1.7 and Theorem 5.2.5 since $p_i^\bullet \subseteq \bigcup_i p_i^\bullet$ —or by an easy direct calculation from Definition 5.2.1 and Lemma 5.1.1(4). $\square$

REMARK 11.8.14. Lemma 11.8.13 depends on q being small-supported (Lemma 11.1.5(2)) and on the p_i having a σ -action (Definition 11.3.1). This makes the canonical model Pnt_Π powerfully well-behaved; neither property holds generally in $\mathfrak{v}$ -algebras.

One corollary of small support in particular is that the canonical model cannot support classical negation, for if we had classical negation then we could reason as follows (this is not intended to be fully formal, but it could be made so):

$$p \in \bigcap^{\#a} q^\bullet \Leftrightarrow \mathcal{M}b.(b \ a) \cdot p \in q^\bullet \Leftrightarrow \mathcal{M}b.(b \ a) \cdot p \notin Pnt_\Pi \setminus q^\bullet \Leftrightarrow p \in \bigcup^{\#a} q^\bullet.$$

In other words $\forall = \exists$. Informally, this suggests we can have any two, but not all three, of the following qualities:

1. $\forall$ decomposed into $\mathcal{M}$, as in condition 4 of Definition 6.1.2 or condition 2 of Definition 7.9.1 or Corollary 11.7.6.
2. Filters with small support.
3. Classical negation (so that prime filters are ultrafilters).

In this paper we build models of the untyped λ -calculus in an ambient lattice which is a nominal distributive lattice with $\forall$. It is natural to ask whether we could promote this lattice to be a Boolean Algebra—in other words, “Can we add negation?”. We think the answer is probably “No, at least for the paper in its current form.”: negation would either cost us condition 4 of Definition 6.1.2 (and all that depends on it), or it would cost us Lemma 11.8.13, with undesirable consequences for the proofs to follow, starting with Lemma 11.8.16.

11.8.5. Proof that Pnt_Π is coherent

We saw in Proposition 11.8.3 that Pnt_Π is a nominal $\sigma \circ$ -topological space in the sense of Definition 9.2.2. We now show that it is coherent (Definitions 7.5.1 and 9.2.7) and sober (Definition 7.9.1).

LEMMA 11.8.15. *Suppose U and V are compact in Pnt_{Π} and suppose $u \in |\mathcal{U}|$. Then:*

- $|Pnt_{\Pi}|$ is compact.
- $U \cap V$ is compact.
- $U[a \mapsto u]$ is compact.

Proof. By Lemma 11.8.6 we may assume $U = \bigcup \{p^* \mid p \in \mathcal{P}\}$ and $V = \bigcup \{q^* \mid q \in \mathcal{Q}\}$ for finite $\mathcal{P}, \mathcal{Q} \subseteq |Pnt_{\Pi}|$. We consider each part in turn:

- $|Pnt_{\Pi}| = \emptyset^*$ (which is a point, as noted in Remark 11.1.4). We use Lemma 11.8.5.
- Using part 1 of Proposition 11.7.12 $U \cap V = \bigcup \{(p \wedge q)^* \mid p \in \mathcal{P}, q \in \mathcal{Q}\}$. By Lemma 11.8.5 each $(p \wedge q)^*$ is compact, and a finite union of compact sets is compact.
- By Lemma 5.1.1(2) and Theorem 11.7.9 $U[a \mapsto u] = \bigcup \{(p[a \mapsto u])^* \mid p \in \mathcal{P}\}$. By Lemma 11.8.5 each $(p[a \mapsto u])^*$ is compact, and a finite union of compact sets is compact. $\square$

LEMMA 11.8.16. *Suppose U is compact in Pnt_{Π} . Then continuing Lemma 11.8.15:*

- $\bigcap^{\#a} U$ is compact.

Proof. By Lemma 11.8.6 we may assume $U = \bigcup \{p^* \mid p \in \mathcal{P}\}$ for finite $\mathcal{P} \subseteq |Pnt_{\Pi}|$. By Lemma 11.8.13 and by part 2 of Proposition 11.7.12 $\bigcap^{\#a} U = \bigcup \{(\forall a.p)^* \mid p \in \mathcal{P}\}$. By Lemma 11.8.5 each $(\forall a.p)^*$ is compact, and a finite union of compact sets is compact. $\square$

LEMMA 11.8.17. *Suppose U and V are compact in Pnt_{Π} . Then continuing Lemma 11.8.16:*

- $U \bullet V$ is compact.
- $V \multimap U$ is compact.

Proof. By Lemma 11.8.6 we may assume $U = \bigcup \{p^* \mid p \in \mathcal{P}\}$ and $V = \bigcup \{q^* \mid q \in \mathcal{Q}\}$ for finite $\mathcal{P}, \mathcal{Q} \subseteq |Pnt_{\Pi}|$. We consider each part in turn:

- By Lemma 9.2.11 and part 3 of Proposition 11.7.12 $U \bullet V = \bigcup \{(p \bullet q)^* \mid p \in \mathcal{P}, q \in \mathcal{Q}\}$. By Lemma 11.8.5 each $(p \bullet q)^*$ is compact, and a finite union of compact sets is compact.
- By Corollary 11.8.12 $V \multimap U = \bigcap_{q \in \mathcal{Q}} \bigcup_{p \in \mathcal{P}} (q \multimap p)^*$. By Lemma 11.8.5 each $(q \multimap p)^*$ is compact; by Lemma 11.8.15 a finite intersection of compact sets is compact; and a finite unions of compact sets is compact. $\square$

PROPOSITION 11.8.18. *Pnt_{Π} is a coherent nominal $\sigma \circ$ -topological space.*

Proof. Pnt_{Π} is a nominal $\sigma \circ$ -topological space by Proposition 11.8.3. It remains to check the additional coherence conditions of Definitions 7.5.1 and 9.2.7.

We check the conditions of Definition 7.5.1. Suppose U and V are compact in Pnt_{Π} . By Lemma 11.8.17 $U[a \mapsto u]$, $|Pnt_{\Pi}|$, and $U \cap V$ are compact. By Lemma 11.8.13 $\bigcap^{\#a} U$ is compact.

By construction in Definition 11.8.1 every open set is a small-supported union of sets of the form p^* for $p \in |Pnt_{\Pi}|$. We note by Lemma 11.8.5 that p^* is compact, so every open set is also a small-supported union of compact sets.

We check the conditions of Definition 9.2.7. Suppose U and V are compact. By Lemma 11.8.17 $U \bullet V$ and $V \dashv U$ are compact.

We now check conditions 6 and 7 of Definition 9.2.7 (validity of $(\sigma \bullet)$ and $(\sigma \dashv)$). By Lemma 11.8.6 we may assume $U = \bigcup \{p^\bullet \mid p \in \mathcal{P}\}$ and $V = \bigcup \{q^\bullet \mid q \in \mathcal{Q}\}$ for finite $\mathcal{P}, \mathcal{Q} \subseteq |Pnt_\Pi|$. We reason as follows:

$$\begin{aligned}
 (U \bullet V)[a \mapsto u] &= \bigcup \{(p^\bullet \bullet q^\bullet)[a \mapsto u] \mid p \in \mathcal{P}, q \in \mathcal{Q}\} && \text{Lemmas 9.2.11 \& 5.1.1(2)} \\
 &= \bigcup \{p^\bullet[a \mapsto u] \bullet q^\bullet[a \mapsto u] \mid p \in \mathcal{P}, q \in \mathcal{Q}\} && \text{Corollary 11.7.13} \\
 &= \bigcup \{p^\bullet[a \mapsto u] \mid p \in \mathcal{P}\} \bullet \bigcup \{q^\bullet[a \mapsto u] \mid q \in \mathcal{Q}\} && \text{Lemma 9.2.11} \\
 &= U[a \mapsto u] \bullet V[a \mapsto u] && \text{Lemmas 9.2.11 \& 5.1.1(2)} \\
 (\partial_{Pnt_\Pi} b \dashv U)[a \mapsto u] &= (\bigcup_{p \in \mathcal{P}} b^\bullet \dashv p^\bullet)[a \mapsto u] && \text{Def 11.8.1 \& Cor 11.8.12} \\
 &= \bigcup_{p \in \mathcal{P}} (b^\bullet \dashv p^\bullet)[a \mapsto u] && \text{Lemma 5.1.1(2)} \\
 &= \bigcup_{p \in \mathcal{P}} b^\bullet \dashv p^\bullet[a \mapsto u] && \text{Corollary 11.7.13} \\
 &= \partial_{Pnt_\Pi} b \dashv (U[a \mapsto u]) && \text{Cor 11.8.12 \& Lem 5.1.1(2)}
 \end{aligned}$$

□

11.8.6. Proof that Pnt_Π is sober, thus spectral

Suppose $p \in |Pnt_\Pi|$. Recall $p^\bullet$ from Definition 7.9.4; given a point we form the prime filter of compact open sets containing it. Lemma 11.8.19 is in the same spirit as Corollary 11.7.5:

LEMMA 11.8.19. *The map $p \in |Pnt_\Pi| \mapsto p^\bullet \in |FG(Pnt_\Pi)|$ is injective.*

Proof. Suppose $p^\bullet = q^\bullet$ and suppose $s \in |\mathcal{U}|$. We note the following:

$$\begin{aligned}
 s \in p &\Leftrightarrow p \in s^\bullet_\Pi && \text{Lemma 11.7.4} \\
 &\Leftrightarrow s^\bullet_\Pi \in p^\bullet && \text{Definition 7.9.4}
 \end{aligned}$$

Similarly $s \in q \Leftrightarrow s^\bullet_\Pi \in q^\bullet$. It follows that $s \in p \Leftrightarrow s \in q$, so that $p = q$. □

LEMMA 11.8.20. *Suppose $\mathcal{P} \subseteq |Pnt_\Pi|$. Then $\bigcap \{p^\bullet \mid p \in \mathcal{P}\} = (\bigcup \mathcal{P})^\bullet$. (By Proposition 11.1.7 $\bigcup \mathcal{P} \in |Pnt_\Pi|$.)*

Proof. By sets calculations using Lemma 11.7.4. □

LEMMA 11.8.21. *Suppose $\mathcal{U} \subseteq Cpct(Pnt_\Pi)$ is a prime filter. Then for every $U \in \mathcal{U}$ there exists $p \in |Pnt_\Pi|$ such that $p^\bullet \subseteq U$ and $p^\bullet \in \mathcal{U}$.*

Proof. By Lemma 11.8.6 there exists a finite set of points $\mathcal{P} \subseteq |Pnt_\Pi|$ such that $U = \bigcup \{p^\bullet \mid p \in \mathcal{P}\}$ and by Lemma 11.8.5 $p^\bullet$ is compact for every $p \in \mathcal{P}$. We assumed that $\mathcal{U}$ is a prime filter (Definition 6.1.16) and it follows that there exists some $p \in \mathcal{P}$ such that $p^\bullet \in \mathcal{U}$. □

COROLLARY 11.8.22. *Suppose $\mathcal{U} \subseteq Cpct(Pnt_\Pi)$ is a prime filter. Then there exists $q \in |Pnt_\Pi|$ such that $q^\bullet = \bigcap \mathcal{U}$ and $\mathcal{U} = q^\bullet$.*

Proof. Write $\mathcal{P} = \{p \mid p^\bullet \in \mathcal{U}\}$ and $\mathcal{P}^\bullet = \{p^\bullet \mid p \in \mathcal{P}\}$. From Lemma 11.8.21 $\bigcap \mathcal{U} = \bigcap \mathcal{P}^\bullet$. By Lemma 11.8.20 $\bigcap \mathcal{P}^\bullet = (\bigcup \mathcal{P})^\bullet$. We take $q = \bigcup \mathcal{P}$. □

PROPOSITION 11.8.23. Pnt_{Π} is sober (Definition 7.9.1).

Proof. By Lemma 7.9.7 it suffices to check two conditions:

1. The map $p \in |Pnt_{\Pi}| \mapsto p^* \in |FG(Pnt_{\Pi})|$ is a bijection.
2. If $p \in |Pnt_{\Pi}|$ and $U \in Cpct(Pnt_{\Pi})$ then $\forall b.p \in (b a) \cdot U$ implies $p \in \bigcap^{#a} U$.

Condition 1 is Lemma 11.8.19 and Corollary 11.8.22.

For condition 2, by Lemma 11.8.6 $U = \bigcup \{q^* \mid q \in \mathcal{Q}\}$ for some finite $\mathcal{Q} \subseteq |Pnt_{\Pi}|$. Choose fresh b (so $b\#U$ and $b\#p$ ⁴⁶ and $\forall q \in \mathcal{Q}. b\#q$). It is a fact of sets that $p \in (b a) \cdot q^*$ for one $q \in \mathcal{Q}$, so that by Theorem 2.3.9 also $\forall b.p \in (b a) \cdot q^*$. By Corollary 11.7.6 $p \in (\forall a.q)^* \stackrel{P11.7.12(2)}{=} \bigcap^{#a} q^*$. Now $q^* \subseteq U$ and it follows that $p \in \bigcap^{#a} U$. $\square$

THEOREM 11.8.24. Pnt_{Π} from Definition 11.8.1 is indeed a nominal spectral space with $\circ$.

Proof. By Proposition 11.8.3 Pnt_{Π} is a nominal $\sigma\circ$ -topological space. By Proposition 11.8.18 it is coherent and by Proposition 11.8.23 it is sober. It remains to check that it is impredicative; that is, that $\partial_{Pnt_{\Pi}} = \cdot^*$ is a morphism of σ -algebras (Definition 4.3.5) from $\mathcal{U}$ to $Cpct(Pnt_{\Pi})$. This is just Theorem 11.7.9. $\square$

11.9. Logical properties of the topology, and completeness

Recall that at the start of this section we fixed an idiom $\mathcal{U}$ (Definition 10.3.1) and a λ -reduction theory Π over $\mathcal{U}$ (Definition 10.3.7).

REMARK 11.9.1. By Proposition 11.8.18 Pnt_{Π} is coherent. Thus from Definition 7.8.1 and Theorem 7.8.3 we have that $G(Pnt_{\Pi})$ is an nominal distributive lattice with $\forall$; it consists of compact open sets in Pnt_{Π} ordered by subset inclusion. By Lemma 11.8.6, each compact open set is a finite union of sets of the form p^* for $p \in |Pnt_{\Pi}|$.

NOTATION 11.9.2. We call $G(Pnt_{\Pi})$ the **canonical model**.

We now set about proving Theorem 11.9.5, which uses $G(Pnt_{\Pi})$ to prove completeness—the converse direction to soundness from Theorem 10.4.7.

Recall from Definition 10.4.1 the definition of $\llbracket s \rrbracket^{G(Pnt_{\Pi})}$ and recall from Definition 10.1.2 that $LmTm$ is the set of λ -terms. Suppose $\mathcal{U} = LmTm$.

LEMMA 11.9.3. $\llbracket s \rrbracket^{G(Pnt_{\Pi})} = s \uparrow_{\Pi}^*$ and as a corollary $\llbracket s \rrbracket^{G(Pnt_{\Pi})} \in |G(Pnt_{\Pi})|$.

Proof. By a routine induction on λ -terms:

- $\llbracket a \rrbracket^{G(Pnt_{\Pi})} \stackrel{\text{Def 10.4.1}}{=} \partial_{G(Pnt_{\Pi})} a \stackrel{\text{Def 11.8.1}}{=} a \uparrow_{\Pi}^*$.
- $\llbracket s' s \rrbracket^{G(Pnt_{\Pi})} \stackrel{\text{Def 10.4.1}}{=} \llbracket s' \rrbracket^{G(Pnt_{\Pi})} \bullet \llbracket s \rrbracket^{G(Pnt_{\Pi})} \stackrel{\text{ind hyp}}{=} s' \uparrow_{\Pi}^* \bullet s \uparrow_{\Pi}^* \stackrel{\text{Cor 11.7.14}}{=} (s' s) \uparrow_{\Pi}^*$.
- $\llbracket \lambda a.s \rrbracket^{G(Pnt_{\Pi})} \stackrel{\text{Def 10.4.1}}{=} \lambda a. \llbracket s \rrbracket^{G(Pnt_{\Pi})} \stackrel{\text{ind hyp}}{=} \lambda a. (s \uparrow_{\Pi}^*) \stackrel{\text{Cor 11.7.14}}{=} (\lambda a.s) \uparrow_{\Pi}^*$.

The corollary follows from Lemma 11.8.5, since $|G(Pnt_{\Pi})|$ is by definition the set of compact open sets of Pnt_{Π} . $\square$

⁴⁶Recall from Lemma 11.1.5(2) that $p \in |Pnt_{\Pi}|$ has small support here.

Recall from Notation 10.3.9 and Definition 10.4.1 the notations $\Pi \vdash s \rightarrow t$ and $G(Pnt_\Pi) \models \Pi$.

PROPOSITION 11.9.4. $G(Pnt_\Pi) \models \Pi$.

Proof. Unpacking Definition 10.4.1, we must show that $(s \rightarrow t) \in \Pi$ implies $\llbracket s \rrbracket^{G(Pnt_\Pi)} \leq \llbracket t \rrbracket^{G(Pnt_\Pi)}$, where $\leq$ means $\subseteq$.

So suppose $(s \rightarrow t) \in \Pi$. By Notation 10.3.9 this means precisely $s \rightarrow_\Pi t$ and it follows by Lemma 11.7.7 that $s^\star_\Pi \subseteq t^\star_\Pi$, so by Lemma 11.9.3 $\llbracket s \rrbracket^{G(Pnt_\Pi)} \subseteq \llbracket t \rrbracket^{G(Pnt_\Pi)}$ as required. $\square$

Recall $\mathcal{T} \vdash s \rightarrow t$ from Notation 10.3.9 and $\mathcal{T} \models s \leq t$ from Definition 10.4.1:

THEOREM 11.9.5 (Completeness). *Suppose $\mathcal{T}$ is a set of reduction axioms. Then:*

$$\mathcal{T} \vdash s \rightarrow t \quad \text{if and only if} \quad \mathcal{T} \models s \leq t.$$

Proof. The left-to-right implication is Theorem 10.4.7.

Now suppose $\mathcal{T} \models s \leq t$ and write $\Pi = \text{rew}(\mathcal{T})$ (Definition 10.3.8). By Proposition 11.9.4 $G(Pnt_\Pi) \models \Pi$ so (since $\mathcal{T} \subseteq \Pi$) $\llbracket s \rrbracket^{G(Pnt_\Pi)} \subseteq \llbracket t \rrbracket^{G(Pnt_\Pi)}$. By Lemma 11.9.3 $\llbracket s \rrbracket^{G(Pnt_\Pi)} = s^\star_\Pi$ and $\llbracket t \rrbracket^{G(Pnt_\Pi)} = t^\star_\Pi$. By Lemma 11.7.7 $\Pi \vdash s \rightarrow t$, thus by Lemma 10.3.10 $\mathcal{T} \vdash s \rightarrow t$ as required. $\square$

11.10. Interlude: an interesting disconnect

The duality theorem from Theorem 9.6.6 is more general than the completeness theorem needs it to be. The completeness result of Theorem 11.9.5 is based on $Pnt_\Pi \in \text{inSpect}\mathbb{V}_\bullet$ from Definition 11.1.3. Although Pnt_Π is a spectral space (Theorem 11.8.24) it has much more structure than that. For instance:

- It is replete (Definition 10.4.3).
- It has an existential quantifier, as we note later in Definition B.2.1.
- Perhaps most usefully, points in Pnt_Π have small support (Lemma 11.1.5(2)). As we noted in Remarks 3.2.3 and 6.1.6, the full generality of our duality theorem encompasses spaces whose points do not necessarily have small support.

Could we obtain a more specific duality result for structures that have more of the structure apparent in Pnt_Π ?

We probably could. However, we do not do it in this paper. $\text{inDi}\mathbb{V}_\bullet/\text{inSpect}\mathbb{V}_\bullet$ have the minimal structure we need to interpret the λ -calculus and to carry out a filter-based duality proof. The less structure we impose, the more general our duality result,⁴⁷ and we have more representations.

But in the completeness result we are happy if the canonical model has more structure, since it suggests more programming and reasoning constructs; an existential quantifier, for example, suggests that the ambient meta-logic implicit in Pnt_Π permits unconstrained

⁴⁷Broadly speaking, within a given class of structures, the less structure we assume the more challenging the duality result is to prove. If duality theory were a competitive sport then it would be like golf: the lower your score the better your game.

search. We do not care about any other structures because we have built one *particular* concrete structure and having built it, we want to obtain as many bells and whistles from it for free as possible.⁴⁸

So on the one hand we have a world where less structure is good, because fewer assumptions means stronger theorems that are valid for a larger class of entities (provided we can still build the things we want to in those entities, i.e. interpret the λ -calculus, which we can), and on the other hand we have a world where more structure is good, because it gives us more tools to actually do things.

The apparent disconnect comes from a difference between two styles, each of which is optimised for its own purpose.

The general trend in this paper is a progression from the abstract and general, like $\text{inSpect}\forall_\bullet$, to the relatively more concrete and specific, like Pnt_Π .

12. Conclusions

The semantics of this paper has the moderately *unusual* feature of being *absolute*, meaning that variables are interpreted directly in the denotation and there is no (Tarski-style) valuation.

The reader may find this takes some getting used to, but it is actually simple and natural.

What corresponds to valuations is the σ -action, which allows us to take some x and ‘evaluate’ a to u in x by forming $x[a \mapsto u]$. This is an abstract nominal algebraic property of x ; it is characterised by axioms (Figure 1). We do not necessarily have access to the internal structure of x .

However, we can certainly build concrete σ -algebras: Two examples are λ -term syntax LmTm from Definition 10.1.1 and the canonical model Pnt_Π from Subsection 11.9. Another example is how we move from τ -algebra structure to σ -algebra structure (and back) using nominal powersets (Definition 3.4.1). Nominal powersets have intersections, unions, and complements, and by combining all of these things we can interpret $\forall$ (Definition 5.2.1).

In fact, it turns out that with a little more effort and just a bit more structure we can interpret application and λ too. This brings us on to another unusual feature of our topological semantics: it is *purely* sets-based.

Algebraic (dually: topological) semantics for the λ -calculus exist, but our semantics is this in a different and stronger sense than usual, because *everything* is interpreted algebraically (dually: topologically), including variables, substitution, and λ -abstraction.

This paper gives a panoramic view of the interaction between nominal foundations and the λ -calculus. This gives us something that shorter papers might not do so well: a feel for the overall point of view, and how the parts of the puzzle fit together.

⁴⁸So canonical models are like tennis: a higher score is better.

12.1. Related work

12.1.1. Algebraic semantics

Algebraic semantics for logics or calculi with binding include polyadic algebras [Hal06, Part II], cylindric algebras [HMT85], and Lambda Abstraction Algebras [Sal00]. As far as we know, what is done in this paper has not been done in any of these (but see below).

We can suggest technical reasons for this. Consider for instance the treatment of substitution in this paper.

For us substitution exists independently from β -reduction—this is the notion of σ -algebra from Subsection 3.1. This is important for our constructions to work. For instance, Subsections 3.3 and 3.4 do not assume λ and application, they only assume σ and τ . This is reflected in commutation results like Lemma 6.4.1 and Theorem 11.7.9.

The commutations for λ are later, and much harder: Lemmas 9.4.9 and 9.4.10 for $\text{inDi}\forall_\bullet$ and Corollary 11.7.14 for Pnt_Π .

It not obvious how substitution *on its own* could be axiomatised without permutations and freshness side-conditions, i.e. without nominal algebra. LAAs do not do this, neither do cylindric algebras. Polyadic algebras *assume* a monoid of substitutions. This is tantalisingly close to finite permutations, but without their invertibility.

By enriching the foundation with names and binding, nominal techniques allow us to express new kinds of algebraic structures—such as substitution and its dual amgis (Figure 1), and universal quantification (Figure A.6). We exploited that this paper to break constructions up into more manageable parts:

- we split λ into $\forall$ and $\rightarrow$ (Notation 10.2.1),
- $\forall$ into $\mathbb{I}$ (see the discussion of condition 4 of Definition 6.1.2 in Subsection 1.4),
- β -reduction into $\bullet$, $\rightarrow$, and σ (Proposition 10.2.4), and then we split
- σ (substitution) itself into permutation and freshness (Definitions 3.1.4 or 3.4.1).

Consequences of this include that the theory of $\forall$ becomes partially independent of the theory of σ .⁴⁹ So breaking constructions up into more manageable parts does not mean here that familiar proofs are carried out in smaller steps: it means that new kinds of proofs are made possible by assembling these parts into new kinds of structures with usefully different proof-dependencies to those we are used to seeing.

Representation theorems exist for cylindric algebras; for instance [Mon61] gives a representation theorem for cylindric algebras, and [PS95] gives one for LAAs. In both cases, an algebra is represented concretely as a set of valuations on the variables (on the indexes; the things that correspond to atoms in this paper).

⁴⁹We see this in various places in this paper, and since this paper is large we point two of them out:

Condition 4 of Definition 6.1.2 imposes a condition on filters relating $\forall$ to $\mathbb{I}$ and not to σ . This is not the condition that one would expect from reading Definition 5.2.1, but it is necessary, as discussed in Remark 6.1.7.

Also, Figure 1 does not mention $\forall$ and Figure A.6 (the algebraic presentation of $\forall$; the body of the paper uses nominal lattices instead, but this is equivalent) does not mention σ . There is a connection between $\forall$ and σ of course, which is expressed explicitly as a further axiom: see the notion of *compatibility* in Definition 4.2.1, whose validity requires non-trivial work to check in Lemma 5.2.2 and Proposition 5.2.3.

This is typical. Representation theorems for cylindric-algebra-style systems do all seem to use something corresponding to a set of valuations. It works, but it is a soundness and completeness proof with respect to Tarski-style semantics. There is no duality result.

The only duality results we know of for logic were undertaken by Forssell [For07] and by the first author [Gab14b]. See [Gab14b] for a comparison of the two.

A *Stone Representation* has been given for Lambda Abstraction Algebras. This is a factorisation result in the style of the HSP theorem (also known as Birkhoff's theorem): every LAA can be factored as a product of irreducible LAAs. The factorisations are identified by *central elements* of the algebra [MS10]. The LAA is never represented as anything resembling a Stone space, and there is no duality.

In passing, we note that the HSP part of the LAA result also follows in this paper (for $\text{inDiV}_\bullet$) off-the-shelf, by the *nominal* HSPA theorem [Gab09a, Gab13]. In other words, we get some of [MS10] for free, just by virtue of being nominal and using nominal algebra. The HSPA factorisation is slightly better than the HSP factorisation (because it has an A in it: for atoms-abstraction). Investigating any extra power this gives what that might mean for (nominal) LAAs, is an open problem.

We mention also [KP10]. This is an attempt to encode what makes nominal techniques work using many-sorted universal algebra. Equivariance, however, gets lost in the translation; a similar phenomenon was noted in [DG12] translating permissive-nominal logic (a first-order generalisation of nominal algebra) to higher-order logic.

12.1.2. Absolute semantics

Absolute semantics have appeared before. Lambda-abstraction algebras (for the λ -calculus) and cylindric algebras and polyadic algebras (for first-order logic) are absolute. Selinger made a case for using absolute semantics for the λ -calculus in [Sel02] (see Subsection 2.2); a line of thought echoed by the first author with Mulligan in [GM11].

Yet absolute semantics have not caught on. We are inclined to believe that this is because the mathematical foundations to support it were not in place before, but they are now. Now that we have nominal techniques we can make a lot of things work that would not work before.

Without nominal techniques things we use repeatedly in the current paper, like small support, freshness side-conditions, equivariance, the $\mathcal{V}$ -quantifier, and even α -equivalence, become challenging in various technical fiddly ways, and even the statements of some properties become practically impossible to even write out. For instance, how we might we render condition 4 of Definition 6.1.2, or condition 2 of Definition 11.1.3, or Definition 11.3.1—to choose three out of many possible examples—without small support, freshness, and the $\mathcal{V}$ -quantifier? We would probably have to invent them first.

We also mention Kit Fine's *arbitrary objects* [Fin85] as an instance of a similar impulse towards absolute semantics, coming from philosophy.

There are precedents for this paper in the first author's work; indeed this paper is based on them. The nominal semantics and duality results for first-order logic in [Gab14b] and [Gab16] are absolute, and are very much in the style and research programme of this paper.

Nominal algebra has helped us to reduce mathematical overhead and to simplify some technical manipulations that are otherwise all too easy to get bogged down in. This

is just what any good mathematical toolbox or foundation should do.

12.1.3. η -expansion

In Proposition 10.2.4 we saw β -reduction and η -expansion appear spontaneously as corollaries of adjoint properties. So our notion of λ -reduction theory is more general than an extensional λ -equality theory because reductions can go one way and not the other, but it is also more specific than just any set of reductions because it must contain η -expansion.

The reader used to seeing η as a contraction rule in rewrite systems might be interested in a thread of publications by Barry Jay and Neil Ghani, which argues in favour of η -expansion from the point of view of rewriting, for better confluence and other properties which they list. See [JG95] and [Gha97].

For us too, expansion rather than contraction seems to be the natural primitive.

We believe that we can remove η -expansion at some cost in complexity in the models. We do this by considering *two* application operations $\bullet'$ and $\bullet$; one intensional and one extensional; a detailed development is the topic of [GG14].

12.1.4. Surjective pairing

To extend the λ -calculus with **surjective pairing** we add constants $pair$, $proj_1$, and $proj_2$ and equations $proj_1(pair\ s_1\ s_2) = s_1$ and $proj_2(pair\ s_1\ s_2) = s_2$ and *surjectivity* $pair((proj_1\ s)(proj_2\ s)) = s$ as axioms to the untyped λ -calculus.⁵⁰

Surjective pairing is of interest because the categorial semantics of the untyped λ -calculus naturally generates semantics for the λ -calculus with surjective pairing; nice discussions are in [Sco00, subsection 2.5] and (for a more detailed exposition) [LS86, Part I section 17].

It seems straightforward to add $pair$, $proj_1$, and $proj_2$ with axioms as above to $\text{inDiV}_\bullet$ (Definitions 4.4.1 and 9.1.1). Determining what corresponding additional structure this would translate to in the dual category $\text{inSpectV}_\bullet$ (Definition 9.2.9) is future work.

12.1.5. Previous treatment of λ -calculus by the authors

In [GM06a, GM06b] the first author and Mathijssen developed *nominal algebra* and axiomatised substitution and first-order logic, with completeness proofs. Journal versions are [GM08a, GM08c].

An axiomatisation, again with completeness proofs, for the λ -calculus followed in [GM08b, GM10].

So the σ -axioms which appear in Figure 1 are taken from [GM06a], and the axioms for β and η are descended from [GM08b, GM10].

In [Gab14b] we applied duality theory to in nominal sets to the axiomatisation of [GM06b, GM08c]. The main conceptual challenge (aside from the inherent difficulty of

⁵⁰*Pairing* can be implemented definitionally as $\lambda a.\lambda b.\lambda f.f\ a\ b$ with first and second projections $\lambda a.a\ \text{true}$ and $\lambda a.a\ \text{false}$ where $\text{true} = \lambda a.\lambda b.a$ and $\text{false} = \lambda a.\lambda b.b$ —but this is not *surjective*; not every lambda term is a pair. Surjective pairing is surjective by the surjectivity axiom above. This is a proper extension of the λ -calculus [Bar74].

duality proofs) was to invent $\top$ -algebras.⁵¹ The $\top$ -axioms of Figure 1 are from [Gab14b]. We have taken this further in [Gab16].

This paper carries out a similar project to [Gab14b], but for the λ -calculus. This has been a tougher target than first-order logic, which is unsurprising. The main conceptual difficulty of this paper over the previous work is the treatment of application and λ using adjoints and the logical quantifier. The ideas for this are from [GG10] (see Figure 2, where $\multimap$ is written $\triangleright$). The similarity with [GG10] is somewhat hidden just because it was written in a ‘modal logic’ style. That style has been replaced in this paper by the nominal foundations.

In summary, and at least in principle, this paper just combines [GM06a], [GM08b], and [GG10] with [Gab14b]. (What could be simpler or more natural?)

12.1.6. No conflict with topological incompleteness results

The best-known models of the untyped λ -calculus are Scott’s domain models and generalisations: graph semantics; filter semantics; stable semantics; strongly stable semantics; and so on. An excellent discussion with references—an annotated bibliography and survey, in fact—appears in [Sal01] between Theorems 4.5 and 4.6.

These are all ordered structures, and this is key, since the idea is to reduce the function space using continuity conditions.

These semantics are all incomplete. That is, domains-based denotational semantics proved the λ -calculus consistent, but results like [Sal03, Theorems 3.5 and 4.9] proved that this is not the whole story: see also [Sal01].⁵²

The reader familiar with this literature and who has seen e.g. Theorem 3.5 of [Sal03] might be puzzled by Theorem 11.9.5: the former states that no semantics in terms of partially-ordered models with a bottom element can be complete, whereas the latter claims to prove completeness for a semantics based on $\text{inDi}\mathbb{V}_\bullet$, and an object of $\text{inDi}\mathbb{V}_\bullet$ is a lattice and has a bottom element $\perp$.

However, nothing insists that $\perp$ should be a program. That is, in the notation of Notation 4.4.3, it is perfectly possible that $\mathcal{D} \in \text{inDi}\mathbb{V}_\bullet$ and $\perp \notin \partial\mathcal{D}$.

This illustrates that $\mathcal{D}$ is a logical structure—its dual is topological—and just a *subset* $\partial\mathcal{D}$ is deemed to be ‘computational’. The models of the λ -calculus live in $\partial\mathcal{D} \subseteq \mathcal{D}$, and $\partial\mathcal{D}$ need not be closed under meets or joins.

The formal sense in which this is intended is just that programs are the things that can be substituted for by the σ -action; so intuitively atoms in $\mathcal{D} \in \text{inDi}\mathbb{V}_\bullet$ ‘range over’ programs (more on this in Subsection 12.2.4). In the light of this reading of the definitions, Definition 10.4.3 calls $\mathcal{D}$ *replete* when its programs are Turing complete.

It remains to discover whether there exists a λ -equality theory such that if $\mathcal{D} \in \text{inDi}\mathbb{V}_\bullet$ is a model of that theory then it can have no non-trivial order on its *programs* (so if $x, y \in |\partial\mathcal{D}|$ then if $x \leq y$ then $y \leq x$).

⁵¹This took a couple of years: once the first author understood that for a duality result, a *dual* to σ was needed, the paper was easy to write. At least, for a certain highly technical value of ‘easy’.

⁵²Page 2 of [Sal03] includes a brief but comprehensive history of such results. The first incompleteness result was given in [HDR92] for the continuous semantics (Scott’s construction). This was followed by several generalisations. Salibra’s treatment has the benefit of covering a range of semantics in a uniform way.

12.1.7. Game models

A game semantics for the untyped λ -calculus is given in [Ker00, KNO02] and is complete (unlike the models discussed in Subsection 12.1.6), and *also* abstract.

Let us briefly compare and contrast the following three complete semantics for the untyped λ -calculus: terms quotiented by equivalence, the games semantics, and the topological duality models of this paper. Of these three, the last two are abstract, and informally speaking they are listed above in *increasing* order of size—that is, a term model is a fairly small entity, a games semantics is only slightly larger, and the duality models may be very large indeed (see Subsection 12.2.4). Also broadly speaking, if we wanted to compute on a λ -term then a term or games model would probably be the most efficient—if we wanted to investigate logical non-computational properties of models, then the models of this paper might be helpful.

Key technical moments in [Ker00] are Theorem 1.5.4 on page 13, the definition of an *effectively almost-everywhere copycat* (EAC) strategy on page 57, and the discussion opening Subsection 5.2 on page 106. These suggest that the game semantics works essentially by identifying the abstract properties that Böhm trees representing λ -terms possess such that these trees look game semantics.

The precise relationship with the models in this paper is unclear, except perhaps to observe that the games model is ‘bottom-up’, building semantics by abstracting from concrete tree structures, whereas our semantics is more ‘top-down’ and algebraic, building semantics by using axioms and topologies to carve out well-behaved subspaces of huge powersets. If there is any point of contact, the natural place to look for it would be in Section 11, where we concretely build models Pnt_{Π} out of syntax (see Subsection 12.2.1).

12.1.8. Sheaves

We impose a topology on a set to reduce the size of the function-space by restricting to continuous functions. Sheaves do much the same thing, but in more generality.

Nominal sets form a category which admits a sheaf presentation (a discussion specific to nominal techniques is in [Gab11b]). Simplifying a little, this amounts to observing that equivariance (commuting with the permutation action) can be represented as a generalised ‘continuity’ condition. There is no need to stop there. We could try to make ‘continuity’ represent, for instance, *compatibility* conditions such as $(\sigma\bullet)$ from Figure 4.

This is what is done by the *Topological representation of the λ -calculus* considered in [Awo00]. Examining equation (15) of the paper we see that, essentially, an open set is a set of substitution instances of evaluations from variables to terms. (The calculations are given only for the simply-typed λ -calculus.) Continuity ensures that function application commutes with substitution, i.e. $(\sigma\bullet)$.

The closest thing to the construction of [Awo00] is the construction of Pnt_{Π} in Definition 11.1.3.

Both are representations of the (simply-typed) λ -calculus, and both are topological, but beyond that we see little resemblance between the two constructions. Our consistency conditions are axiomatic, and we use the topology to do logic and so to break λ -down into $\forall$ and $\rightarrow\bullet$. Substitution σ is managed by axioms.

12.1.9. Proof theory

The design of the *combination operator* $\circ$ from Definitions 9.2.1 and 9.2.2 goes back to the Kripke-style models of the untyped λ -calculus from [GG14, Gab14a]. A discussion with specific references is in Remark 9.2.3.

These Kripke-style models were developed from a proof-theory [Gab11a] which views λ -reduction as a logical derivation rule, and gives it a cut-free sequent-style derivation system. Thus this paper comes (in some sense) full circle when in Notation 10.2.1 we interpret λ semantically in a way that explicitly contains a logical quantifier $\forall$.

The current paper, extensive as it is, is also embedded in and consistent with a broader research context.

12.1.10. In what universe does this paper take place?

The points built in Theorem 6.1.23 do not have small support, and in Definition 3.2.1 we assume a set with a permutation action but not necessarily a nominal set. Thus, this paper does not take place entirely in the topos of nominal sets; we do whatever is convenient to get the results we need and do not commit to any specific logic when we get them, even though our main results can be stated entirely in the nominal sets universe. In this we are being typical mathematicians, reasoning freely in English about informally but precisely specified mathematical objects.⁵³

12.1.11. Discussion of some decisions

While formulating the mathematics in this paper we faced certain high-level design decisions. For the reader's convenience we briefly mention some of these decisions along with our reasons for making them:

1. *When building a nominal topological space it is natural that open sets (representing predicates) should be small-supported. Should points also be small-supported?*
Not in general. Intuitively this is because we want Zorn-style arguments building maximally consistent sets of predicates to be possible, such as appear in Theorem 6.1.23 in this paper and in Theorem 7.1.20 of [Gab16]. See Remark 3.2.3. However, the points of the canonical model Pnt_{Π} (Subsection 11.9) are small supported. See the discussion in 11.8.14.
2. *When considering compactness, should covering sets be small-supported? Or should they be strictly small-supported?*
The more convenient notion seems to be 'small-supported'; see Definition 7.4.1. However, Proposition 7.3.4 notes that in one important special case, the two are equivalent.
3. *What is the correct notion of 'point' in the presence of a nominal universal quantifier?*

This paper contains two distinct notions of 'point': one based on filters which is designed for the duality construction; the other based on sets of syntax closed under reduction which is designed for the canonical model. Both require an

⁵³Something similar happens in category theory when we talk about 'the category of all sets'; what does that live in? This is usually left unspecified, which is generally fine, or at least, is generally not objected to.

non-evident condition to account for $\forall$. See Definition 6.1.2 condition 4, and Definition 11.1.3 condition 2.

4. *In the presence of an amgis-action, what is an appropriate notion of freshness?*
The two notions of point give distinct answers to this question. In Definition 6.1.2 the question is ill-formed because points are not necessarily small-supported. In Definition 11.1.3 we assume small-supported, and the answer is expressed by Definition 11.1.1, Definition 11.1.3 condition 2, and Proposition 11.1.6.

12.2. Future work

12.2.1. Fine structure of the canonical model

In Subsection 11.10 we noted that the canonical model Pnt_{Π} has plenty of structure. It remains to explore that structure: Pnt_{Π} is a lattice and so contains a logic. We know this has interesting structure, investigated from Subsection 11.1. That does not exhaust the possibilities: Appendix B.2 explores an existential quantifier on Pnt_{Π} , and this invites us to ask what the full logic of the canonical model Pnt_{Π} is, and might we use it to investigate the λ -calculus.

Proposition 11.4.6 notes that Pnt_{Π} also has a σ -action, which we characterise in different ways in Subsection 11.3.2; the characterisation in Lemma 11.3.9 seems particularly appealing. As we note in the body of the paper, there is probably a general theory here: a way of, given a σ -action on $\mathcal{X}$, building a σ -action on the nominal powerset of $\mathcal{X}$. Such a theory was already undertaken in [Gab09b], where constructions were applied to models of Fraenkel-Mostowski set theory; thus generating a huge class of huge σ -algebras, since there are many models of FM sets and many sets in each model. The construction in Proposition 11.4.6 suggests the possibility of a cleaner and/or alternative development of similar ideas.⁵⁴

Pnt_{Π} is a remarkable mathematical object which may be somewhat overshadowed in this paper by the rest of the material. The proofs in Section 11 have ‘coincidences’ (two of several examples are the existence of the σ -action mentioned in the previous paragraph, and Lemma 11.8.13), and some strong hints towards general theories (such as that noted in Subsection 11.3.3 of ‘nominal adjoints’, that is; adjoint relations subject to freshness side-conditions). These suggest richnesses of behaviour which may merit further examination.

12.2.2. Weaken the axioms

We have used nominal lattices to give semantics to the λ -calculus. Part of our axiomatisation includes the conditions of Figures 1 and 4.

We can consider weakening these axioms, to obtain more general classes of structures. For instance:

1. $(\sigma\alpha)$ from Figure 1 expresses that σ is a binder in the sense that the a in $x[a \rightarrow u]$ may be α -converted. If we remove this condition then we obtain a notion of

⁵⁴...and this is exciting. Most of this paper works by building various σ -algebras over relatively simple nominal algebraic structures like sets of points. What more could be achieved if we gave ourselves an entire mathematical foundation structure to play with?

‘fusion’ between names and values, but without then binding the name. This was touched on in Subsection 3.4.3; a concrete model is obtained by a simplified variant of Definition 3.4.1 in which the $\forall c$ quantification is removed (so that we lose Lemma 3.4.4).

2. The axiom $(\sigma\bullet)$ expresses that substitution commutes with application.

We discussed in detail in Remark 9.1.10 how this might be weakened in one direction to model an explicit substitution.

It might also be removed entirely, to get an environment which would allow functions to detect atoms (that is, variable symbols) in their arguments.

Meta-programming is a large field which has proven resistant so far even to precise categorisation. Generalisations of $\text{inDiV}_\bullet$ without $(\sigma\bullet)$ might be one place to start looking for mathematical semantics.

3. Further examples are easy to generate; it suffices to choose an axiom and modify or delete it. For instance, relaxing $(\sigma\#)$ permits structures that are sensitive to associating a value to a fresh name (like a valuation context, a memory state, or indeed like the τ -algebras in this paper), and relaxing $(\sigma\sigma)$ permits structures that are sensitive to the order of association (like a process listening on a pair of ports), and so forth.

12.2.3. More structure in the existing axioms

On a related note, in [GG10] we noted that λ has a dual construction, $\forall a_1 \dots a_n.(t \multimap s)$, of *pattern-matching* (i.e. it applies to points in the ‘pattern’ t and outputs the same points in the pattern s). Operational semantics taking pattern-matching as fundamental include for instance Jay’s *pattern calculus* [Jay04], and there may be more to discover about what logics and calculi—other than the application to untyped λ -calculus contained in this paper—the structures in this paper can help develop.

The structures $\text{inDiV}_\bullet$ and Pnt_Π are not just for the λ -calculus. Far from it: it is clear that they are rich and interesting environments, combining computational and logical structures. We have used them as a bridge between lattice-theory and λ -calculus, but we would go so far as to suggest that for some people this bridge might be just as interesting as the λ -calculus itself, and could be studied in its own right.

12.2.4. Duality and cardinality

We discussed in Remarks 4.4.2 and 6.1.26 that we insist that $\mathcal{D}$ be no larger than $\mathbb{A}$ (see Definitions 4.4.1 and 7.7.1). This condition is needed just once in this paper, in the proof of Theorem 6.1.23, which is key to the duality proof.

Traditionally nominal techniques take $\mathbb{A}$ to be countable, though the theory works perfectly well for larger sets of atoms, as we have seen in this paper. Provided $\mathbb{A}$ is infinite, the proofs work, and this paper is parametric in the choice of size made in Definition 2.1.1.

An argument can be made that we intend λ -calculus models to represent computable objects so they should anyway be countable. Then we can take $\text{size}(\mathbb{A})$ in Definition 2.1.1 to be $\aleph_0$ the first infinite cardinal (and the cardinality of $\mathbb{N}$), and we can take α in the proof of Theorem 6.1.23 to be ω_0 the first infinite ordinal.

We also recall the Löwenheim-Skolem theorems [Hod93, Corollaries 3.1.5 and 6.1.4] (specifically, that in suitable conditions a model exists if and only if a countable model does). This suggests intuitively that countable models already have ‘all interesting structure’.

And indeed: this is reflected in the proofs. Soundness (Theorem 10.4.7) does not care if $\text{size}(\mathbb{A}) < \text{size}(\mathcal{D})$, and completeness depends only on the syntactic idiom from Example 10.3.2 (the canonical model is constructed from λ -syntax and is countable if its set of atoms is countable).

So for soundness and completeness we can take $\mathbb{A}$ to be countable, even if $\mathcal{D} \in \text{inDiV}$ is uncountable. The proofs still work and even the text of the proofs remains unchanged: we just replace ‘small support’ with ‘finite support’ and $\text{size}(\mathbb{A})$ with ‘countable’.

Only if we want a *duality* result do we care that atoms be as large as the model as a whole. We could simply insist that models be countable, but we prefer to allow a set of atoms of any size; it is the more general choice.

All this suggests a slogan: under duality, *names are the dual to programs*. This seems intuitively reasonable, and pursuing the maths here further is future work.

The Löwenheim-Skolem theorems also suggest future work to see if we can transform a model constructed over a set of atoms with one cardinality, into one with more (or even fewer) atoms. Methods for doing this in nominal algebra (see Appendix A.1) already exist [Gab12a].

12.2.5. Size of permutations

We noted in Remark 2.1.7 that we take permutations in Definition 2.1.3 to be finite, not small. This may seem like a mismatch between permutations (which are finite) and support (which is small, but may be infinite). However, what governs the size of the permutations required is how powerful our *binders* are, not how many *free atoms* (i.e. how large a support) we need to allow. λ -syntax is finite and only nests finitely many binders, so we only ever need to rename finitely many atoms at a time. We allow potentially infinite support so we can use atoms to ‘name’ elements, as discussed in Subsection 12.2.4.

It may be worth sparing a few words for the generalisation to infinite permutations.⁵⁵ Suppose the set of atoms $\mathbb{A}$ is uncountable, and suppose permutations and support are both taken to be countable. Consider the set $\mathbb{L}$ of countable streams of distinct atoms; we can model this as $\mathbb{N} \hookrightarrow \mathbb{A}$ the set of injections from natural numbers to atoms. Consider the relation $\sim \subseteq \mathbb{L} \times \mathbb{L}$ such that $l \sim l'$ when $\exists n \in \mathbb{N}. \forall n' \geq n. l(n') = l'(n')$; by a convenient abuse of notation write this as $\forall n. l(n) = l'(n)$, and we call l and l' **asymptotically equal**.

Write $[l]_{\sim} \in \mathbb{L}/\sim$ for the equivalence class of l under $\sim$. So $l' \in [l]_{\sim}$ when l' is asymptotically equal to l . Note that $\mathbb{L}/\sim$ inherits the pointwise permutation action from $\mathbb{L}$, and $A \subseteq \mathbb{A}$ supports $[l]_{\sim}$ when (again, abusing notation) $\forall n. l(n) \in A$.

Following Definition 2.1.9 write $\text{supp}([l]_{\sim})$ for the intersection of all small A supporting $[l]_{\sim}$ and write $a\#[l]_{\sim}$ for $a \notin \text{supp}([l]_{\sim})$. It is clear that $(a' a) \cdot [l]_{\sim} = [l]_{\sim}$ and $\text{supp}([l]_{\sim}) = \emptyset$ and $a\#[l]_{\sim}$ for every pair of atoms a and a' .

⁵⁵Thanks to an anonymous referee for noting this subtle issue.

Fix two streams of distinct atoms $l, l' \in \mathbb{L}$. Write $(l' l)$ for the (countable) permutation that swaps $l(n)$ with $l'(n)$ for every $n \in \mathbb{N}$. It is not hard to check that $(l' l) \cdot l = l'$ and $(l' l) \cdot [l]_{\sim} = [l']_{\sim} \neq [l]_{\sim}$. Note that $\text{nontriv}((l' l)) \cap \text{supp}([l]_{\sim}) = \emptyset$.

Thus if π is an infinite permutation, it is not in general the case that $\forall a \in \mathbb{A}. a \# x$ implies $\pi \cdot x = x$ [Gab07, Lemma 21]. This implication only holds in general if π is *finite*. This is investigated in [Gab07, §6.2] where it is called *fuzzy support*. See also the discussion at the end of [Gab13, Subsection 2.4].

Infinite binding, and more specifically $\mathbb{L}$, permutations like $(l' l)$, and infinite atoms-abstraction $[l]x$, are interesting: they arise from the study of name-generating processes running in finite but unbounded time; they arise independently in nominal semantics for nominal terms syntax [Gab12a, Gab12b]; and they are interesting in their own right. So in future work we might want to generalise parts of this paper to admit infinite permutations and infinite atoms-binding.

We believe that much, and perhaps all, of the underlying nominal machinery admits this generalisation. Making this speculation into a theorem is, thankfully, not required for this paper. If we choose to do so in future work then we would pay the following price in complexity and convenience: we either surrender the convenience of talking about a *single* unique least supporting set $\text{supp}(x)$; or we insist by definition that $\mathcal{X}$ is only a ‘nominal set’ when its elements have a unique least small supporting set of atoms, which excludes examples like $\mathbb{L}/\sim$ but incurs an extra proof-obligation on the $\mathcal{X}$ we construct.

References

- [ACCL91] Martín Abadi, Luca Cardelli, Pierre-Louis Curien, and Jean-Jacques Lévy, *Explicit substitutions*, Journal of Functional Programming **1** (1991), no. 4, 375–416.
- [Awo00] Steve Awodey, *Topological representation of the lambda-calculus*, Mathematical Structures in Computer Science **10** (2000), no. 1, 81–96.
- [Bar74] Henk P. Barendregt, *Pairing without conventional constraints*, Zeitschrift für mathematischen Logik und Grundlagen der Mathematik **20** (1974), 289–306.
- [BBKL12] Mikolaj Bojanczyk, Laurent Braud, Bartek Klin, and Slawomir Lasota, *Towards nominal computation*, Proceedings of the 39th ACM SIGACT-SIGPLAN Symposium on Principles of Programming Languages (POPL 2012), ACM Press, January 2012, pp. 401–412.
- [BS81] Stanley N. Burris and H. P. Sankappanavar, *A course in universal algebra*, Graduate texts in mathematics, Springer, 1981.
- [DG12] Gilles Dowek and Murdoch J. Gabbay, *PNL to HOL: from the logic of nominal sets to the logic of higher-order functions*, Theoretical Computer Science **451** (2012), 38–69.
- [DP02] B. A. Davey and Hilary A. Priestley, *Introduction to lattices and order*, 2 ed., Cambridge University Press, 2002.
- [End77] Herbert B. Enderton, *Elements of set theory*, Academic Press, 1977.
- [FG07] Maribel Fernández and Murdoch J. Gabbay, *Nominal rewriting (journal version)*, Information and Computation **205** (2007), no. 6, 917–965.
- [FGM04] Maribel Fernández, Murdoch J. Gabbay, and Ian Mackie, *Nominal Rewriting Systems*, Proceedings of the 6th ACM SIGPLAN symposium on Principles and Practice of Declarative Programming (PPDP 2004), ACM Press, August 2004, pp. 108–119.

- [Fin85] Kit Fine, *Reasoning with arbitrary objects*, Blackwell, 1985.
- [For07] Henrik Forssell, *First-order logical duality*, Ph.D. thesis, Carnegie Mellon University, December 2007.
- [Gab01] Murdoch J. Gabbay, *A Theory of Inductive Definitions with alpha-Equivalence*, Ph.D. thesis, University of Cambridge, UK, March 2001.
- [Gab07] ———, *A General Mathematics of Names*, Information and Computation **205** (2007), no. 7, 982–1011.
- [Gab09a] ———, *Nominal Algebra and the HSP Theorem*, Journal of Logic and Computation **19** (2009), no. 2, 341–367.
- [Gab09b] ———, *A study of substitution, using nominal techniques and Fraenkel-Mostowski sets*, Theoretical Computer Science **410** (2009), no. 12-13, 1159–1189.
- [Gab11a] Michael Gabbay, *A proof-theoretic treatment of λ -reduction with cut-elimination: λ -calculus as a logic programming language*, Journal of Symbolic Logic **76** (2011), no. 2, 673–699.
- [Gab11b] Murdoch J. Gabbay, *Foundations of nominal techniques: logic and semantics of variables in abstract syntax*, Bulletin of Symbolic Logic **17** (2011), no. 2, 161–229.
- [Gab12a] ———, *Finite and infinite support in nominal algebra and logic: nominal completeness theorems for free*, Journal of Symbolic Logic **77** (2012), no. 3.
- [Gab12b] ———, *Meta-variables as infinite lists in nominal terms unification and rewriting*, Logic Journal of the IGPL **20** (2012), no. 6, 967–1000.
- [Gab13] ———, *Nominal terms and nominal logics: from foundations to meta-mathematics*, Handbook of Philosophical Logic, vol. 17, Kluwer, 2013, pp. 79–178.
- [Gab14a] Michael Gabbay, *A simple model theory for untyped lambda-equality*, HOWARD-60. A Festschrift on the Occasion of Howard Barringer’s 60th Birthday, Easychair books, 2014, pp. 165–177.
- [Gab14b] Murdoch J. Gabbay, *Stone duality for First-Order Logic: a nominal approach*, HOWARD-60. A Festschrift on the Occasion of Howard Barringer’s 60th Birthday, Easychair books, 2014, pp. 178–209.
- [Gab16] ———, *Semantics out of context: nominal absolute denotations for first-order logic and computation*, Journal of the ACM **63** (2016), no. 3, 1–66.
- [GC11] Murdoch J. Gabbay and Vincenzo Ciancia, *Freshness and name-restriction in sets of traces with names*, Foundations of software science and computation structures, 14th International Conference (FOSSACS 2011), Lecture Notes in Computer Science, vol. 6604, Springer, 2011, pp. 365–380.
- [GG10] Michael J. Gabbay and Murdoch J. Gabbay, *A simple class of Kripke-style models in which logic and computation have equal standing*, International Conference on Logic for Programming Artificial Intelligence and Reasoning (LPAR 2010), Lecture Notes in Computer Science, vol. 6355, Springer, 2010, pp. 231–254.
- [GG14] ———, *A simple and complete model theory for intensional and extensional untyped lambda-equality*, IFCoLog Journal of Logic and its Applications **1** (2014), no. 2.
- [Gha97] Neil Ghani, *Eta-expansions in dependent type theory—the calculus of constructions*, Typed Lambda Calculi and Applications (1997), 164–180.
- [GLP11] Murdoch J. Gabbay, Tadeusz Litak, and Daniela Petrişan, *Stone duality for nominal Boolean algebras with NEW*, Proceedings of the 4th international conference on algebra and coalgebra in computer science (CALCO 2011), Lecture Notes in Computer Science, vol. 6859, Springer, 2011, pp. 192–207.
- [GM05] Dov M. Gabbay and Larisa Maksimova, *Interpolation and definability: Modal and intuitionistic logics*, Oxford Logic Guides, Oxford Science Publications,

- 2005.
- [GM06a] Murdoch J. Gabbay and Aad Mathijssen, *Capture-avoiding Substitution as a Nominal Algebra*, ICTAC 2006: Theoretical Aspects of Computing (Berlin), Lecture Notes in Computer Science, vol. 4281, Springer, November 2006, pp. 198–212.
 - [GM06b] ———, *One-and-a-halfth-order logic*, Proceedings of the 8th ACM-SIGPLAN International Symposium on Principles and Practice of Declarative Programming (PPDP 2006), ACM, July 2006, pp. 189–200.
 - [GM07] ———, *A Formal Calculus for Informal Equality with Binding*, WoLLIC’07: 14th Workshop on Logic, Language, Information and Computation, Lecture Notes in Computer Science, vol. 4576, Springer, July 2007, pp. 162–176.
 - [GM08a] ———, *Capture-Avoiding Substitution as a Nominal Algebra*, Formal Aspects of Computing **20** (2008), no. 4–5, 451–479.
 - [GM08b] ———, *The lambda-calculus is nominal algebraic*, Reasoning in simple type theory: Festschrift in Honour of Peter B. Andrews on his 70th Birthday (Christoph Benzmüller, Chad Brown, Jörg Siekmann, and Rick Statman, eds.), Studies in Logic and the Foundations of Mathematics, College Publications, December 2008.
 - [GM08c] ———, *One-and-a-halfth-order Logic*, Journal of Logic and Computation **18** (2008), no. 4, 521–562.
 - [GM09a] ———, *Nominal universal algebra: equational logic with names and binding*, Journal of Logic and Computation **19** (2009), no. 6, 1455–1508.
 - [GM09b] ———, *Nominal universal algebra: equational logic with names and binding*, Journal of Logic and Computation **19** (2009), no. 6, 1455–1508.
 - [GM10] ———, *A nominal axiomatisation of the lambda-calculus*, Journal of Logic and Computation **20** (2010), no. 2, 501–531.
 - [GM11] Murdoch J. Gabbay and Dominic P. Mulligan, *Nominal Henkin Semantics: simply-typed lambda-calculus models in nominal sets*, Proceedings of the 6th International Workshop on Logical Frameworks and Meta-Languages (LFMTP 2011), EPTCS, vol. 71, September 2011, pp. 58–75.
 - [GP01] Murdoch J. Gabbay and Andrew M. Pitts, *A New Approach to Abstract Syntax with Variable Binding*, Formal Aspects of Computing **13** (2001), no. 3–5, 341–363.
 - [GTL89] Jean-Yves Girard, Paul Taylor, and Yves Lafont, *Proofs and types*, Cambridge University Press, 1989.
 - [Hal06] Paul R. Halmos, *Algebraic logic*, AMS Chelsea Publishing, 2006.
 - [HDR92] Furio Honsell and Simona Ronchi Della Rocca, *An approximation theorem for topological lambda models and the topological incompleteness of lambda calculus*, Journal of Computer and System Sciences **45** (1992), no. 1, 49–75.
 - [HMT85] Leon Henkin, J. Donald Monk, and Alfred Tarski, *Cylindric algebras*, North Holland, 1971 and 1985, Parts I and II.
 - [Hod93] Wilfrid Hodges, *Model theory*, Cambridge University Press, 1993.
 - [Jay04] C. Barry Jay, *The pattern calculus*, ACM Transactions in Programming Language Systems **26** (2004), no. 6, 911–937.
 - [JG95] C. Barry Jay and Neil Ghani, *The virtues of eta-expansion*, Journal of Functional Programming **5** (1995), no. 2, 135–154.
 - [Joh86] Peter T. Johnstone, *Stone spaces*, vol. 3, Cambridge University Press, 1986.
 - [Joh03] ———, *Sketches of an elephant: A topos theory compendium*, Oxford Logic Guides, vol. 43 and 44, OUP, 2003.
 - [Ker00] Andrew D. Ker, *Innocent game models of untyped lambda-calculus*, Ph.D. thesis, Oxford University, 2000.
 - [KNO02] Andrew D. Ker, Hanno Nickau, and C.-H. Luke Ong, *Innocent game models of*

- untyped λ -calculus*, Theoretical Computer Science **272** (2002), no. 1–2, 247–292, Theories of Types and Proofs 1997.
- [KP10] Alexander Kurz and Daniela Petrişan, *On universal algebra over nominal sets*, Mathematical Structures in Computer Science **20** (2010), 285–318.
 - [KW96] Edward Keenan and Dag Westerståhl, *Generalized quantifiers in linguistics and logic*, Handbook of Logic and Language (J. Van Benthem and A. Ter Meulen, eds.), Elsevier, 1996, pp. 837–894.
 - [LS86] J. Lambek and P.J. Scott, *Introduction to higher order categorical logic*, Cambridge University Press, 1986.
 - [Mac71] Saunders Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, vol. 5, Springer, 1971.
 - [MM92] Saunders Mac Lane and Ieke Moerdijk, *Sheaves in geometry and logic: A first introduction to topos theory*, Universitext, Springer, 1992.
 - [Mon61] Donald Monk, *On the representation theory for cylindric algebras*, Pacific Journal of Mathematics **11** (1961), no. 4, 1447–1457.
 - [MS10] Giulio Manzonetto and Antonino Salibra, *Applying universal algebra to lambda calculus*, Journal of Logic and Computation **20** (2010), no. 4, 877–915.
 - [Pit13] Andrew M. Pitts, *Nominal sets: Names and symmetry in computer science*, Cambridge University Press, May 2013.
 - [PS95] Don Pigozzi and Antonino Salibra, *Lambda abstraction algebras: representation theorems*, Theoretical Computer Science **140** (1995), no. 1, 5–52.
 - [Sal00] Antonino Salibra, *On the algebraic models of lambda calculus*, Theoretical Computer Science **249** (2000), no. 1, 197–240.
 - [Sal01] ———, *A continuum of theories of lambda calculus without semantics*, Proceedings of the 16th IEEE Symposium on Logic in Computer Science (LICS 2001), IEEE, 2001, pp. 334–343.
 - [Sal03] ———, *Topological incompleteness and order incompleteness of the lambda calculus*, ACM Transactions on Computational Logic **4** (2003), no. 3, 379–401.
 - [Sco00] Philip J. Scott, *Some aspects of categories in computer science*, Handbook of algebra (M. Hazewinkel, ed.), vol. 2, Elsevier, 2000, pp. 3–77.
 - [Sel02] Peter Selinger, *The lambda calculus is algebraic*, Journal of Functional Programming **12** (2002), no. 6, 549–566.
 - [UPG04] Christian Urban, Andrew M. Pitts, and Murdoch J. Gabbay, *Nominal Unification*, Theoretical Computer Science **323** (2004), no. 1–3, 473–497.
 - [Wes89] Dag Westerståhl, *Quantifiers in formal and natural languages*, Handbook of Philosophical Logic, Synthèse, vol. 4, Reidel, 1989, pp. 1–131.

A. More on fresh-finite limits

A.1. Nominal algebra axiomatisation of fresh-finite limits

A key definition in this paper has been Definition 4.3.3; this is ‘poset-flavoured’, in the sense that $\wedge$ and $\forall$ were characterised using fresh-finite limits (Definition 4.1.2).

It is interesting to ask whether fresh-finite limits can be rephrased in the syntax of nominal algebra, using equalities subject to freshness side-conditions.

This has implications, because if this can be done then $\text{inDi}\forall$ and $\text{inDi}\forall_\bullet$ are algebraic varieties, and satisfy the nominal HSP theorems from [Gab09a, Gab13].⁵⁶ This gives us

⁵⁶The nominal HSPA theorem states that every nominal algebra model is a subobject of a homomorphic

$(\forall\alpha)$	$b\#x \Rightarrow$	$\forall b.(b\ a) \cdot x = x$
$(\forall\wedge)$		$\forall a.(x\wedge y) = (\forall a.x)\wedge(\forall a.y)$
$(\forall\vee)$	$a\#y \Rightarrow$	$\forall a.(x\vee y) = (\forall a.x)\vee y$
$(\forall\leq)$		$\forall a.x \leq x$

Figure A.6: Nominal algebra axioms for $\forall$

off-the-shelf factorisation theorems similar to those considered in [MS10] (see especially Theorem 14), and in general, it is useful to know when a class of structures is an algebraic variety.

DEFINITION A.1.1. A **bounded lattice** in nominal sets is a tuple

$$\mathcal{L} = (|\mathcal{L}|, \cdot, \wedge, \vee, \perp, \top)$$

where:

- $(|\mathcal{L}|, \cdot)$ is a nominal set which we may just write $\mathcal{L}$,
- $\perp \in |\mathcal{L}|$ and $\top \in |\mathcal{L}|$ are equivariant **bottom** and **top** elements,
- $\wedge, \vee : (\mathcal{L} \times \mathcal{L}) \rightarrow \mathcal{L}$ are equivariant functions, such that $\wedge$ and $\top$ form an idempotent monoid and $\vee$ and $\perp$ form an idempotent monoid,

$$\begin{array}{llll} (x\wedge y)\wedge z = x\wedge(y\wedge z) & x\wedge y = y\wedge x & x\wedge x = x & x\wedge \top = x \\ (x\vee y)\vee z = x\vee(y\vee z) & x\vee y = y\vee x & x\vee x = x & x\vee \perp = x \end{array}$$

- and $\wedge$ and $\vee$ satisfy *absorption*

$$x\wedge(x\vee y) = x \qquad x\vee(x\wedge y) = x.$$

Here, x, y, z range over elements of $|\mathcal{L}|$.

A bounded lattice is a poset by taking $x \leq y$ to mean $x\wedge y = x$ or $x\vee y = y$ (the two conditions are provably equivalent). Definition A.1.1 is the usual definition of a bounded lattice, but over a nominal set; but we have not done anything with it yet.

Definition A.1.2 exploits the nominal set structure to algebraise the universal quantifier:

DEFINITION A.1.2. Suppose $\mathcal{L}$ is a nominal poset.

A **(nominal) universal quantifier** $\forall$ on $\mathcal{L}$ is an equivariant map $\forall : (\mathbb{A} \times \mathcal{L}) \rightarrow \mathcal{L}$ satisfying the equalities $(\forall\alpha)$ to $(\forall\leq)$ in Figure A.6.

LEMMA A.1.3. $a\# \forall a.x$.

image of a cartesian product of atoms-abstractions of free algebras (atoms-abstraction for nominal algebras is defined in [Gab09a], as is ‘free algebra’ and so on). In spirit, this is like the factorisation of natural numbers into primes and similar factorisation theorems. Such factorisation results are useful because they constrain the structure of models, and this is one of the applications of abstract algebraic techniques.

The result proved in [Gab09a] considered an untyped syntax, but we expect it to generalise unproblematically to the typed case, if necessary.

Proof. Choose fresh b (so $b\#x$). By Proposition 2.3.4 $a\#(b\ a)\cdot x$. It follows by Theorem 2.3.2 that $a\#\forall b.(b\ a)\cdot x$. By $(\forall\alpha)\ \forall b.(b\ a)\cdot x = \forall a.x$. $\square$

PROPOSITION A.1.4. *Suppose $\mathcal{L}$ is a bounded lattice with $\forall$ and $x \in |\mathcal{L}|$. Then $a\#x$ implies $\forall a.x = x$*

Proof. We reason as follows:

$$\forall a.x = \forall a.(x\forall x) \stackrel{(\forall\forall)}{=} (\forall a.x)\forall x$$

So $x \leq \forall a.x$. Furthermore by $(\forall\leq)\ \forall a.x \leq x$, and we are done. $\square$

COROLLARY A.1.5. *Suppose $\mathcal{L}$ is a bounded lattice and suppose $\mathcal{L}$ has a nominal universal quantifier $\forall$. Then $\forall a.x$ is the $a\#$ limit for $\{x\}$.*

Proof. By $(\forall\leq)\ \forall a.x \leq x$ and by Lemma A.1.3 $a\#\forall a.x$.

Suppose $z \in |\mathcal{L}|$ and $a\#z$ and $z \leq x$. It follows using $(\forall\wedge)$ that $\forall a.z \leq \forall a.x$ and by Proposition A.1.4 $\forall a.z = z$. $\square$

PROPOSITION A.1.6. *The notion of a nominal distributive lattice with $\forall$ from Definition 4.3.3 is characterised in nominal algebra as:*

a bounded lattice (Definition A.1.1) with a nominal universal quantifier (Definition A.1.2) that is distributive (Definition 4.3.1) and has a compatible σ -action (Definition 4.2.1).

Proof. By routine calculations. The interesting part is to check that the characterisation of $\forall a$ from Definition 4.1.2 as a fresh-finite limit coincides with the algebraic characterisation of $\forall a$ in Figure A.6. The meat of this is handled by Proposition A.1.4. $\square$

A.2. More on fresh-finite limits

The quantifier $\forall a.x$ from Definition 4.1.2 is greatest in the set $\{x' \mid x' \leq x \wedge a\#x'\}$, but we have seen other ways to characterise quantification too: Definition 5.2.1 and Proposition 6.1.10 characterise the quantifier in σ -powersets (of filters); Propositions 4.2.5 and 4.2.6 do something similar in the abstract; Lemma 11.5.6 does it again, for points.

A higher-level view is possible of some general principles behind these results. Propositions 4.2.5 and 4.2.6 are part of this higher-level picture, and we can also observe:

- Definition A.2.1 and Proposition A.2.3 characterise $\forall a.x$ as a limit of a strictly small-supported set ($\{x' \mid x' \leq x \wedge a\#x'\}$ is small-supported but not *strictly* small-supported).
- Definition A.2.5 and Proposition A.2.8 characterise $\forall a.x$ as a limit of a permutation orbit and using the $\mathbb{N}$ -quantifier. This ties $\forall a.x$ to Proposition 6.1.10 and Remark 4.2.7 and to the opening discussion of Subsection 6.1.

See also [Gab09b, GC11] and computational studies such as [BBKL12], with their emphasis on studying nominal sets in terms of their permutation orbits.

DEFINITION A.2.1. Suppose $\mathcal{L}$ is a nominal poset and $x \in |\mathcal{L}|$. Consider the set

$$B = \{x' \in |\mathcal{L}| \mid \text{supp}(x') \subseteq S \wedge x' \leq x\}.$$

Then write $\bigwedge^{\subseteq S} x$ for the $\leq$ -greatest element of B , if this exists. We call this the **S -strict limit** of x .

REMARK A.2.2. So:

- $\forall a.x$ is the greatest x' beneath x such that $a \notin \text{supp}(x')$.
- $\bigwedge^{\subseteq \text{supp}(x) \setminus \{a\}} x$ is greatest x' beneath x such that $\text{supp}(x') \subseteq \text{supp}(x) \setminus \{a\}$.

It is not *a priori* evident that these two notions must coincide. However, they often do, as we will now show.

PROPOSITION A.2.3. If $\forall a.x$ exists then so does $\bigwedge^{\subseteq \text{supp}(x) \setminus \{a\}} x$ and they are equal.

Proof. Suppose $\forall a.x$ exists. By construction $\text{supp}(\forall a.x) \subseteq \text{supp}(x) \setminus \{a\}$ and $\forall a.x \leq x$. Therefore $\forall a.x \in B$ (notation from Definition A.2.1). Also by construction $x' \leq \forall a.x$ for every $x' \in B$, since if $\text{supp}(x') \subseteq \text{supp}(x) \setminus \{a\}$ then certainly $a \# x'$. It follows that $\forall a.x$ is greatest in B . $\square$

REMARK A.2.4. It is easy to prove that if $x \leq y$ then $\bigwedge^{\subseteq S} x \leq \bigwedge^{\subseteq S} y$ (so $\bigwedge^{\subseteq S}$ is monotone or *functorial*). Another plausible definition for $\forall a.x$ is to set

$$\forall a.x = \text{the } \leq\text{-greatest element of } \{x' \mid \text{supp}(x') \subseteq \text{supp}(x) \setminus \{a\} \wedge x' \leq x\}. \quad (\text{A.1})$$

However, monotonicity is then not so obvious, though in view of Proposition A.2.3 monotonicity does hold of (A.1), for the cases we care about.

Recall from Notation 2.1.4 the definition of fix .

DEFINITION A.2.5. Following [Gab09b, GC11] define x^{fix} by

$$\begin{aligned} x^{\text{fix}} &= \{\pi \cdot x \mid \pi \in \text{fix}(\text{supp}(x) \setminus \{a\})\}. \\ &= \{x\} \cup \{(b \ a) \cdot x \mid b \# x\} \end{aligned}$$

Write $\bigwedge x^{\text{fix}}$ for the $\leq$ -greatest lower bound of x^{fix} , if this exists.

REMARK A.2.6. So we can rewrite this as follows:

- $\forall a.x$ is the greatest x' beneath x such that $a \notin \text{supp}(x')$.
- Definition A.2.5 specifies the greatest x' beneath x and beneath $(b \ a) \cdot x$ for every $b \# x$.

LEMMA A.2.7. $a \# x^{\text{fix}}$.

Proof. By the pointwise action $\pi' \cdot x^{\text{fix}} = \{(\pi' \circ \pi) \cdot x \mid \pi \in \text{fix}(\text{supp}(x) \setminus \{a\})\}$. We choose a fresh b (so $b \# x$, that is, $b \notin \text{supp}(x)$) and use Corollary 2.3.5(3) and routine calculations. $\square$

PROPOSITION A.2.8. Suppose $a \in \mathbb{A}$ and $x \in |\mathcal{L}|$. Then:

1. If $\forall a.x$ exists then so does $\bigwedge x^{\text{fix}}$, and they are equal.

2. Suppose $\mathcal{L}$ has a monotone σ -action (Definition 4.2.1). Then if $\bigwedge x^{\leq a}$ exists, then so does $\forall a.x$, and they are equal.

Proof. 1. By Definition 4.1.1 $\forall a.x \leq x$ and $a\# \forall a.x$. It follows by equivariance of $\leq$ and Corollary 2.3.5 that $\forall a.x \leq \pi \cdot x$ for every $\pi \in \text{fix}(\text{supp}(x) \setminus \{a\})$. Therefore $\forall a.x$ is a lower bound for $x^{\leq a}$.

Now consider z some other lower bound for $x^{\leq a}$, so that $\forall \pi \in \text{fix}(\text{supp}(x) \setminus \{a\}). z \leq \pi \cdot x$. Choose fresh b (so $b\#x, z$); by Theorem 2.3.2 $\forall b.(b\ a) \cdot x$ exists (and by Lemma 4.1.3 it is equal to $\forall a.x$). It follows by equivariance of $\leq$ and Corollary 2.3.5 (since $a, b\#z$) that $z \leq (b\ a) \cdot x$ so $z \leq \forall b.(b\ a) \cdot x \stackrel{4.1.3}{=} \forall a.x$.

2. By Lemma A.2.7 and Theorem 2.3.2 $a\# \bigwedge x^{\leq a}$ so $\bigwedge x^{\leq a}$ is an $a\#$ lower bound for x .

Consider any other z such that $z \leq x$ and $a\#z$. By $(\sigma\#) z[a \mapsto n] = z$ for every $n \in \mathbb{A}$. By monotonicity $z = z[a \mapsto n] \leq x[a \mapsto n]$ for every $n \in \mathbb{A}$. It follows using Lemma 3.2.4 that $z \leq \bigwedge x^{\leq a}$. $\square$

REMARK A.2.9. Nominal posets offer a richness of what we can collectively term ‘fresh limits’:

- Fresh-finite limits from Definition 4.1.2.
- *Strict* fresh-finite limits from Definition A.2.1.
- Limits of permutation orbits.
- If a σ -algebra structure is present, then we have even more: limits of all substitution instances (Proposition 4.2.5; see also Definition 5.2.1) and substitution for all names or all fresh names (see Propositions 4.2.6 and 6.1.10).

For a nominal poset with a monotone σ -action, these are all equal where they exist.

We return briefly to Proposition A.2.3, which stated that in a nominal poset $\mathcal{L}$ if $\forall a.x$ exists then so does $\bigwedge_{\subseteq \text{supp}(x) \setminus \{a\}} x$ and they are equal. What can we say about the other way around? When does the existence of $\bigwedge^{\subseteq S} x$ imply the existence of $\forall a.x$?

We can note Proposition A.2.12 and Remark A.2.13:

LEMMA A.2.10. $\text{supp}(\bigwedge^{\subseteq S} x) \subseteq S$.

Proof. By Lemma 2.5.3 $\text{supp}(B) \subseteq S$ (B is from Definition A.2.1). We use Theorem 2.3.2. $\square$

DEFINITION A.2.11. Say $\mathcal{L}$ has **support interpolation** when for every x and y , if $x \leq y$ then there exists a z such that

$$x \leq z \quad \text{and} \quad z \leq y \quad \text{and} \quad \text{supp}(z) \subseteq \text{supp}(x) \cap \text{supp}(y).$$

(*Interpolation* is used here by analogy with the concept in logic [GM05]; here we interpolate on the set of names.)

PROPOSITION A.2.12. Suppose $\mathcal{L}$ has support interpolation and suppose $\bigwedge^{\subseteq S} x$ exists for all x and $S \subseteq \mathbb{A}$.

Then $\forall a.x$ exists for all x and a and is equal to $\bigwedge_{\subseteq \text{supp}(x) \setminus \{a\}} x$.

Proof. Suppose all strict fresh-finite limits $\bigwedge^{\subseteq S} x$ exist. Consider some a and x .

- We show that if $x' \leq x$ and $a \# x'$ then $x' \leq \bigwedge^{\subseteq \text{supp}(x) \setminus \{a\}} x$.
Suppose $x' \leq x$ and $a \# x'$. By support interpolation there exists a z with $x' \leq z \leq x$ and $\text{supp}(z) \subseteq \text{supp}(x) \cap \text{supp}(x')$. By assumption $z \leq \bigwedge^{\subseteq \text{supp}(x) \setminus \{a\}} x$ and so $x' \leq \bigwedge^{\subseteq \text{supp}(x) \setminus \{a\}} x$.
- We show that $\bigwedge^{\subseteq \text{supp}(x) \setminus \{a\}} x$ is least with this property.
By Lemma A.2.10 $\text{supp}(\bigwedge^{\subseteq \text{supp}(x) \setminus \{a\}} x) \subseteq \text{supp}(x) \setminus \{a\}$. Thus in particular $a \# \bigwedge^{\subseteq \text{supp}(x) \setminus \{a\}} x$. Also by construction $\bigwedge^{\subseteq \text{supp}(x) \setminus \{a\}} x \leq x$.
Thus for any other greatest element y in $\{x' \mid x' \leq x \wedge a \# x'\}$, $\bigwedge^{\subseteq \text{supp}(x) \setminus \{a\}} x \leq y$.

Thus, $\forall a. x$ exists and is equal to $\bigwedge^{\subseteq \text{supp}(x) \setminus \{a\}} x$. $\square$

REMARK A.2.13. In the absence of support interpolation Proposition A.2.12 may fail. For instance, consider a nominal poset consisting of singleton atoms $\{a\}$ and unordered pairs of atoms $\{a, b\}$ and an element $*$, such that:

- $\{a, b\} \leq \{a\}$ for all (distinct) a and b .
- $*$ $\leq \{a\}$ for all a .

We assume the natural pointwise permutation actions and $\pi \cdot * = *$, so that $\text{supp}(\{a, b\}) = \{a, b\}$ and $\text{supp}(\{a\}) = \{a\}$ and $\text{supp}(*) = \emptyset$.

Then $*$ $= \bigwedge^{\subseteq \emptyset} \{a\}$ but $\forall a. \{a\}$ does not exist since $*$ $\not\leq \{a, b\}$.

B. Additional properties of the canonical model

We noted in Subsection 11.6 that Pnt_{Π} has structure above and beyond being in $\text{inDi}\mathbf{V}_{\bullet}$. We conclude with some further reflection on this.

These properties were not needed for our main results, but they seem striking enough to merit a note in an Appendix.

B.1. λ commutes with unions

We consider λ from Notation 10.2.1 in Pnt_{Π} and prove Corollary B.1.3: λ commutes with sets union (cf. a similar property for $\forall$ proved in Lemma 11.8.13).

This property is not valid in general in $\text{inDi}\mathbf{V}_{\bullet}$, because $\neg \bullet$ and $\forall$ do not commute with $\vee$ in general,⁵⁷ but it does hold in the canonical model Pnt_{Π} .

LEMMA B.1.1. $q \circ p^* = (q \bullet p)^*$.

Proof. We reason as follows:

$$\begin{aligned} q \circ p^* &= \bigcup \{q \circ p' \mid p \subseteq p'\} && \text{Defs 11.7.1 \& 9.2.4} \\ &= \bigcup \{(q \bullet p')^* \mid p \subseteq p'\} && \text{Definition 11.7.1} \\ &= (q \bullet p)^* && \text{Fact of Def 11.7.1} \end{aligned} \quad \square$$

LEMMA B.1.2. $p^* \neg \bullet (\bigcup_i p_i^*) = \bigcup_i (p^* \neg \bullet p_i^*)$.

⁵⁷The closest we get to this in the general case is $(\neg \bullet \forall)$ from Figure 4 and $(\forall \forall)$ from Figure A.6

Proof. We reason as follows:

$$\begin{aligned}
 q \in p^* \bullet (\bigcup_i p_i^*) &\Leftrightarrow q \circ p^* \subseteq \bigcup_i p_i^* && \text{Proposition 9.2.6} \\
 &\Leftrightarrow (q \bullet p)^* \subseteq \bigcup_i p_i^* && \text{Lemma B.1.1} \\
 &\Leftrightarrow \forall r. (q \bullet p \subseteq r \Rightarrow \exists i. p_i \subseteq r) && \text{Definition 11.7.1} \\
 &\Leftrightarrow \exists i. p_i \subseteq q \bullet p && \text{Suffices to take } r = q \bullet p \\
 &\Leftrightarrow \exists i. (q \bullet p)^* \subseteq p_i^* && \text{Definition 11.7.1, fact} \\
 &\Leftrightarrow \exists i. q \circ p \subseteq p_i^* && \text{Definition 11.7.1} \\
 &\Leftrightarrow \exists i. q \in p^* \bullet p_i^* && \text{Proposition 9.2.6} \\
 &\Leftrightarrow p \in \bigcup_i (p^* \bullet p_i^*) && \text{Fact} \quad \square
 \end{aligned}$$

COROLLARY B.1.3. $\lambda a. \bigcup_i p_i^* = \bigcup_i \lambda a. p_i^*$.

Proof. We recall λ from Notation 10.2.1 and using Lemmas B.1.2 and 11.8.13 deduce that $\lambda a. \bigcup_i p_i^* = \bigcup_i \lambda a. p_i^*$. $\square$

B.2. An existential quantifier

Pnt_{Π} from Definition 11.1.3 has a universal quantifier, defined in Definition 11.5.1. Proposition 11.1.7 hints that an existential quantifier might exist.⁵⁸ And indeed, it is not hard to construct as the natural dual to $\forall a. p$ from Definition 11.5.1:

DEFINITION B.2.1. Suppose $p \in |Pnt_{\Pi}|$ and $A \subseteq \mathbb{A}$. Define $\exists a. p$ by:

$$\exists a. p = \bigcap_{u \in |\mathbb{U}|} p[a \mapsto u]$$

LEMMA B.2.2. If $p \in |Pnt_{\Pi}|$ then $\exists a. p \in |Pnt_{\Pi}|$.

Proof. From Proposition 11.1.7. $\square$

A dual version of Lemma 11.5.6 exists for $\exists$ instead of $\forall$, and is easy to prove. We omit details.

The universal and existential quantifiers $\forall$ and $\exists$ interact with $\bullet$ similarly to how we saw $\forall$ interact with $\vee$ in $(\forall\vee)$ of Figure A.6 or $(\text{distrib}\forall)$ of Definition 4.3.1; this is Lemmas B.2.4 and B.2.5.

We need Lemma B.2.3 as a simple technical lemma:

LEMMA B.2.3. If $p \subseteq p'$ then $p \bullet q \subseteq p' \bullet q$.

Proof. Routine from Definition 11.5.1. $\square$

LEMMA B.2.4. If $a \# q$ then $\forall a. (p \bullet q) \subseteq (\forall a. p) \bullet q$.

Proof. By Theorem 2.3.2 and Lemma 11.5.5, $a \# (\forall a. p) \bullet q$. Thus by part 4 of Lemma 11.5.6, $\forall a. (p \bullet q) \subseteq (\forall a. p) \bullet q$ if and only if $p \bullet q \subseteq (\forall a. p) \bullet q$. By part 1 of Lemma 11.5.6 and Lemma B.2.3, $p \bullet q \subseteq (\forall a. p) \bullet q$ is true. $\square$

⁵⁸Why? Because the proof of Proposition 11.1.7 would be simple and direct if we assume that $\mathcal{P}$ is strictly small-supported. The fact that it works for *all* small-supported $\mathcal{P}$ suggests there might be some way of removing atoms from the support of $p \in \mathcal{P}$ while also making p smaller as a set. This is exactly what an existential quantifier on points would do (recall that everything is inverted/dual in Pnt_{Π}).

LEMMA B.2.5. *If $a \# q$ then $(\exists a.p) \bullet q \subseteq \exists a.(p \bullet q)$.*

Proof. Suppose $v \in (\exists a.p) \bullet q$. So there exist s' and t such that $\forall u. s'[a:=u] \in p \wedge t \in q \wedge s't \rightarrow_{\Pi} v$. In particular, $s't \rightarrow_{\Pi} v$. It follows by condition 3 of Definition 10.3.7 that $s'[a:=u](t[a:=u]) \rightarrow_{\Pi} v[a:=u]$ for every u . But $a \# q$ so by Proposition 11.1.6 $a \#_{\sigma} q$ so $t[a:=u] \in q$ for every u . Thus, $v \in \exists a.(p \bullet q)$. $\square$