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NODAL INTERSECTIONS FOR RANDOM EIGENFUNCTIONS ON THE TORUS

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Abstract. We investigate the number of nodal intersections of random Gaussian Laplace eigenfunctions on the standard two-dimensional flat torus ("arithmetic random waves") with a fixed smooth reference curve with nonvanishing curvature. The expected intersection number is universally proportional to the length of the reference curve, times the wavenumber, independent of the geometry.

Our main result prescribes the asymptotic behaviour of the nodal intersections variance for smooth curves in the high energy limit; remarkably, it is dependent on both the angular distribution of lattice points lying on the circle with radius corresponding to the given wavenumber, and the geometry of the given curve. In particular, this implies that the nodal intersection number admits a universal asymptotic law with arbitrarily high probability.

1. Introduction

1.1. Background.

A number of recent papers studied the fine structure of nodal lines of eigenfunctions of the Laplacian, and in particular the number of intersections of these nodal lines with a fixed reference curve. Thus let $\mathcal{C} \subset M$ be a smooth curve on a (smooth) Riemannian surface M. Let F be a real-valued eigenfunction of the Laplacian on M with eigenvalue λ^2: $-\Delta F = \lambda^2 F$. We want to estimate the number of nodal intersections

$$\mathcal{Z}(F) = \#\{x : F(x) = 0\} \cap \mathcal{C} \tag{1.1}$$

that is the number of zeros of F on $\mathcal{C}$, as $\lambda \to \infty$.

It is expected that in many situations, there is an upper bound of the form $\mathcal{Z}(F) \ll \lambda$, and general criteria for this to happen exist [26, 12], though it is difficult to verify these criteria in most situations. As for lower bounds, nothing seems to be known in general, see [14] for results on Hecke eigenfunctions on hyperbolic surfaces (and [21] for analogous results on the sphere), and [16, 17] for results on density one subsequences for hyperbolic surfaces. Aronovich and Smilansky [2] studied the nodal intersections of random monochromatic waves on the plane [3] with various reference curves.

The one context where we have more information is for the standard flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\mathcal{C} \subset \mathbb{T}^2$ be a smooth curve. Bourgain and Rudnick [5] showed that if $\mathcal{C}$ is not a segment of a closed geodesic, then it is not part

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of the nodal line of any eigenfunction with $\lambda > \lambda_{\mathcal{C}}$ sufficiently large, hence $\mathcal{Z}(F) < \infty$ for λ sufficiently large. If the reference curve $\mathcal{C}$ has nowhere-zero curvature, they gave upper and lower bounds [6] on the intersection numbers

$$\lambda^{1-o(1)} \ll \mathcal{Z}(F) \ll \lambda. \tag{1.2}$$

The lower bound is strengthened in [7], and assuming a number theoretic conjecture takes the form $\mathcal{Z}(F) \gg \lambda$ and is thus optimal up to a constant multiple. Moreover the number theoretic condition is known to hold for "generic" eigenvalues hence we know that for almost all eigenvalues, *all* eigenfunctions in the eigenspace satisfy the lower bound $\mathcal{Z}(F) \gg \lambda$.

In this paper we show that in this setting, for "generic" toral eigenfunctions there is in fact an asymptotic law for these nodal intersection numbers. We will show that for most eigenspaces, we in fact have an asymptotic result for "almost all" eigenfunctions in that eigenspace, once we take a limit of large eigenspace dimension.

1.2. **Our setting.** Let

$$\mathcal{E} = \{\mu \in \mathbb{Z}^2 : |\mu|^2 = m\} \tag{1.3}$$

be the set of lattice points on the circle of radius $\sqrt{m}$, and denote

$$N_m = \#\mathcal{E}. \tag{1.4}$$

We will restrict ourselves to sequences of numbers $\{m\}$ such that $N_m \to \infty$; by the formula for N_m in terms of the prime decomposition of m, if we write $m = m_1 m_2$ with m_1 divisible only by primes $p \equiv 1 \mod 4$, and m_2 co-prime to m_1, then $N_m \to \infty$ is equivalent to $m_1 \to \infty$. We consider the random Gaussian toral eigenfunctions

$$F(x) = \frac{1}{\sqrt{N_m}} \sum_{\mu \in \mathcal{E}} a_\mu e^{2\pi i \langle \mu, x \rangle}, \tag{1.5}$$

with eigenvalue

$$\lambda^2 = 4\pi^2 m,$$

defined on the standard torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, where a_μ are standard complex Gaussian random variables (that is $\mathbb{E}(a_\mu) = 0$, $\mathbb{E}(|a_\mu|^2) = 1$), independent save for the relations $a_{-\mu} = \overline{a}_\mu$. The random functions F are called "arithmetic random waves" [18].

We define the probability measures on the unit circle $\mathcal{S}^1 \subseteq \mathbb{R}^2$

$$\tau_m = \frac{1}{N_m} \sum_{\mu \in \mathcal{E}} \delta_{\mu/\sqrt{m}}, \tag{1.6}$$

where δ_x is the Dirac delta function at x. It is well known that the lattice points $\mathcal{E}$ are equidistributed on $\mathcal{S}^1$ along generic subsequences of energy levels (see e.g. [13], Proposition 6) in the sense that

$$\tau_{m_j} \Rightarrow \frac{1}{2\pi} d\theta$$

along some density 1 sequence $\{m_j\}$ (relatively to the set of integers representable as sum of two squares), and thus, in particular,

$$\widehat{\tau_{m_j}}(4) \to 0,$$

where for a measure τ on $\mathcal{S}^1$, we denote by $\widehat{\tau}(k)$ its k-th Fourier coefficient. Below we will assume that $|\widehat{\tau_{m_j}}(4)| \le 1$ is bounded away from 1 (see the formulation of the main results); for a probability measure τ on $\mathcal{S}^1$, invariant w.r.t. rotation by $\pi/2$, we have

$$\widehat{\tau}(4) = \pm 1,$$

if and only if $\tau = \frac{1}{4}\left(\sum_{k=0}^{3} \delta_{k\pi/2}\right)$ or $\tau = \frac{1}{4}(\delta_{\pm\pi/4} + \delta_{\pm 3\pi/4})$ (thinking of the circle as $\mathcal{S}^1 \cong \mathbb{R}/2\pi\mathbb{Z}$). Hence we only exclude these two limiting probability measures (see section 7 for more discussion on the possible limiting angular measures, and the peculiarities of these two).

Given a curve $\mathcal{C} \subset \mathbb{T}^2$, we wish to study the statistics of the number of nodal intersections $\mathcal{Z}(F)$ for an arithmetic random wave F. We do this when the curve $\mathcal{C}$ is smooth, with nowhere zero curvature.

Theorem 1.1. *Let $\mathcal{C} \subset \mathbb{T}^2$ be a smooth curve on the torus, with nowhere-zero curvature, of total length L.*

i) The expected number of nodal intersections is precisely

$$\mathbb{E}[\mathcal{Z}] = \sqrt{2m}L = \frac{\lambda}{\pi\sqrt{2}}L. \tag{1.7}$$

ii) Let $\{m\}$ be a sequence s.t. $N_m \to \infty$ and the Fourier coefficients $\{\widehat{\tau_m}(4)\}$ do not accumulate at ± 1, i.e. no subsequence of $\{\widehat{\tau_m}(4)\}$ converges to $+1$ or -1. Then the variance is

$$\mathrm{Var}(\mathcal{Z}) \ll \frac{m}{N_m} \ll \frac{\lambda^2}{N_m}. \tag{1.8}$$

By Chebyshev's inequality we deduce that under the conditions of Theorem 1.1, we have with arbitrarily high probability

$$\mathcal{Z}(F) \sim \sqrt{2m}L \tag{1.9}$$

for eigenfunctions with eigenvalue $4\pi^2 m$. Our main result in fact prescribes the asymptotic form for the variance, which depends on the distribution of the lattice points $\mathcal{E}$ once projected to the unit circle.

Theorem 1.2. *Let $\mathcal{C} \subset \mathbb{T}^2$ be a smooth curve on the torus, with nowhere-zero curvature, of total length L, and $\{m\}$ a sequence s.t. $N_m \to \infty$ and the Fourier coefficients $\{\widehat{\tau_m}(4)\}$ do not accumulate at ± 1. Then*

$$\mathrm{Var}(\mathcal{Z}) = \left(4B_{\mathcal{C}}(\mathcal{E}) - L^2\right) \cdot \frac{m}{N_m} + O\left(\frac{m}{N_m^{3/2}}\right) \tag{1.10}$$

where

$$B_{\mathcal{C}}(\mathcal{E}) := \int_{\mathcal{C}}\int_{\mathcal{C}} \frac{1}{N_m}\sum_{\mu\in\mathcal{E}} \left\langle \frac{\mu}{|\mu|}, \dot\gamma(t_1)\right\rangle^2 \cdot \left\langle \frac{\mu}{|\mu|}, \dot\gamma(t_2)\right\rangle^2 dt_1 dt_2 \tag{1.11}$$

with $\gamma : [0, L] \to \mathcal{C}$ *a unit speed parameterization.*

Theorem 1.2 immediately implies the second part of Theorem 1.1. In Section 7 we discuss the possible partial limits of $B_{\mathcal{C}}(\mathcal{E})$ as $m \to \infty$: there is no unique limit, similar to what happens for the variance of the length of nodal lines in this model [18]. The leading constant

$$0 \le 4B_{\mathcal{C}}(\mathcal{E}) - L^2 \le L^2$$

is always non-negative and bounded (see Proposition 7.1); it can however vanish, for instance when $\mathcal{C}$ is a full circle, see § 7.2.

1.3. **About the proof and plan of the paper.** First, we may restrict F along $\mathcal{C}$; this reduces computing the nodal intersections $\mathcal{Z}$ to counting zeros of a random process f defined on an interval. The Kac-Rice formula (see e.g. [10] or [1, Theorems 11.2.1, 11.5.1]) is a standard tool for studying the expected number of zeros of a process and its higher moments by expressing the k-th (factorial) moment in terms of a certain k-dimensional integral.

For the expected value of $\mathcal{Z}$ we do this in § 2. For the second moment, the Kac-Rice formula would state

$$\mathbb{E}[\mathcal{Z}^2] = \iint\limits_{\mathcal{C}\times\mathcal{C}} K_2(t_1, t_2)dt_1dt_2 + \mathbb{E}[\mathcal{Z}], \tag{1.12}$$

where K_2 is the suitably defined "2-point correlation function", that is, provided that we justify its use. Unfortunately, to our best knowledge, all the available references impose a certain non-degeneracy condition on f and its derivative f', which is far from being satisfied. In fact, it is easy to construct an example where the Kac-Rice integral in (1.12) is off from computing the second (factorial) moment: one checks that the functions (1.5) satisfy that $F(x) = 0$ if and only if $F(x + (1/2, 1/2)) = 0$. Hence if $\mathcal{C}$ is a simple closed curve, invariant w.r.t. the translation

$$\phi : x \mapsto \left(\frac{1}{2}, \frac{1}{2}\right) + x,$$

i.e. $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_i$ are the maximal subsets of $\mathcal{C}$ so that $\phi(\mathcal{C}_1) = \mathcal{C}_2$, then the total number of nodal intersections $\mathcal{Z}$ is twice the number of intersections with $\mathcal{C}_1$ (so that the variance is multiplied by 4); however the linear part on the RHS of (1.12) is not invariant, and therefore the precise Kac-Rice formula as stated in (1.12) is in general wrong.

To cope with this situation we develop an *approximate* form of the Kac-Rice for the second moment of the number of zeros of a random eigenfunction along a smooth curve, which is sufficient for our purposes. This is quite delicate and takes up all of sections 3, 4, and Appendix A; we believe that the developed techniques are of independent interest, and could be used in a variety of situations where Kac-Rice is not directly applicable (e.g. [11]). In our situation the result gives the variance of $\mathcal{Z}$ in terms of the second moments of the covariance function (also referred to as covariance *kernel*) $r(t_1,t_2) = \mathbb{E}\{F(\gamma(t_1))F(\gamma(t_2))\}$ and its derivatives $r_j = \partial r/\partial t_j$, $r_{ij} = \partial^2 r/\partial r_i \partial t_j$ along the curve:

Proposition 1.3. *Fix $\epsilon_0 > 0$. Then for all m such that $|\widehat{\tau_m}(4)| < 1-\epsilon_0$ one has the following approximate Kac-Rice formula,*

(1.13)
$$\mathrm{Var}(\mathcal{Z}) = m \iint_0^L \left(r^2 - \left(\frac{r_1}{\sqrt{2\pi^2 m}}\right)^2 - \left(\frac{r_2}{\sqrt{2\pi^2 m}}\right)^2 + \left(\frac{r_{12}}{2\pi^2 m}\right)^2 \right) dt_1 dt_2 + O\left(\frac{m}{N_m^{3/2}}\right),$$

where the implied constant depends only on ϵ_0.

In the proof of Proposition 1.3 we also have to control the fourth moment of r and its derivatives; this is done in § 6.

Proposition 1.3 reduces our problem to evaluating the second moment of the covariance function and its various derivatives along the given curve. To this end, we eventually encounter an arithmetic problem, which is to show that

$$\sum_{\mu \neq \mu' \in \mathcal{E}} \frac{1}{|\mu - \mu'|} = o(N_m). \tag{1.14}$$

This is done in § 5, appealing among other things to a theorem of Mordell [22] about representing a binary quadratic form as a sum of two squares, in other words counting the number of pairs of distinct vectors $(\mu, \mu') \in \mathcal{E} \times \mathcal{E}$ with a given inner product. The 3-dimensional version of the quantity (1.14) is essentially the electrostatic energy of point charges placed at the integer points at on the sphere of radius $\sqrt{m}$ and is analyzed in [8].

The term $B_{\mathcal{C}}(\mathcal{E})$ in (1.11), which determines the leading term of the variance, arises from the asymptotics of the second moment $\iint (r_{12})^2$. In § 7 we analyze $B_{\mathcal{C}}(\mathcal{E})$ and determine when it attains its minimum value (this is equivalent to vanishing of the leading term in (1.10)), and study its limiting value distribution when $N_m \to \infty$, as a function of the curve $\mathcal{C}$.

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2. The expected number of nodal intersections

2.1. **Kac-Rice formula for computing the expected number of zeros.** Let $f : I \to \mathbb{R}$ be a centered Gaussian random function ("process"), a.s. smooth (e.g. C^2), with the parameter space I some nice subset of $\mathbb{R}$, e.g. a closed interval or a finite collection of closed intervals, and let

$$r(t_1, t_2) = r_f(t_1, t_2) := \mathbb{E}[f(t_1)f(t_2)]$$

be the covariance function of f. Denote $\mathcal{Z}$ to be the number of zeroes of f on I. For $t \in I$ define $K_1(t) = K_{1;f}(t)$ to be the Gaussian expectation

$$K_1(t) = \phi_{f(t)}(0) \cdot \mathbb{E}[|f'(t)| \big| f(t) = 0],$$

where $\phi_{f(t)}$ is the probability density function of the random variable $f(t)$. The latter involves the centered Gaussian vector $(f(t), f'(t))$ with covariance matrix

$$\Gamma(t) = \Gamma_f(t) = \begin{pmatrix} r(t,t) & \partial_{t_1} r(t_1,t_2)|_{(t,t)} \\ \partial_{t_1} r(t_1,t_2)|_{(t,t)} & \partial_{t_1}\partial_{t_2} r(t_1,t_2)|_{(t,t)} \end{pmatrix}.$$

The function $K_1(t)$ is the *zero density* (or first intensity) of f; it may be computed explicitly in terms of the entries of the matrix $\Gamma(t)$, and in our case the expression is especially simple, as $\Gamma(t)$ is diagonal and independent of t (a consequence of the fact that our process is induced from an underlying 2-dimensional *stationary* field restricted on a curve), see below. By the Kac-Rice formula, if for all $t \in I$ the matrix $\Gamma(t)$ is nonsingular, then [10]

$$\mathbb{E}[\mathcal{Z}] = \int\limits_I K_1(t)dt.$$

2.2. **Zero density for nodal intersections.** The random field $F(x)$ is centered Gaussian with covariance function

$$r_F(x,y) := \mathbb{E}[F(x) \cdot F(y)] = \frac{1}{N_m} \sum_{\mu \in \mathcal{E}} \cos(2\pi \langle \mu, y - x \rangle)$$

for $x, y \in \mathbb{T}^2$; it is *stationary* in the sense that $r_F(x,y) = r_F(y-x)$ depends on $y - x$ only (by the well-accepted abuse of notation). Let $\gamma(t) : [0, L] \to \mathbb{T}^2$ be the arc-length parameterization of $\mathcal{C}$; it induces the process

$$f(t) = F(\gamma(t)) \tag{2.1}$$

on $I := [0, L]$ with the covariance function

$$r(t_1, t_2) = r_F(\gamma(t_1) - \gamma(t_2));$$

the process f is unit variance. Let $\mathcal{Z}$ be the number of zeros of f (on I); it equals the number of nodal intersections of F with $\mathcal{C}$.

Lemma 2.1. *The zero density of f is*

$$K_1(t) = K_{1;m}(t) \equiv \sqrt{2} \cdot \sqrt{m}.$$

In particular,

$$\mathbb{E}[\mathcal{Z}] = \sqrt{2}\sqrt{m} \cdot L.$$

To facilitate the computation of the zero density we formulate the following lemma whose proof will be given in a moment. It is probably well-known, but nevertheless we give it here as we didn't find a direct reference.

Lemma 2.2. *If f is unit variance, then for every $t \in [0, L]$, $f(t)$ is independent of $f'(t)$.*

Proof of Lemma 2.1 assuming Lemma 2.2. We are to compute the zero density of $f(t)$:

$$K_1(t) = \frac{1}{\sqrt{2\pi}} \mathbb{E}[|f'(t)| \big| f(t) = 0], \tag{2.2}$$

thus we are to compute the covariance matrix of $(f(t), f'(t))$. Since f is unit variance, by Lemma 2.2, the covariance matrix is

$$A_m = \begin{pmatrix} 1 & \\ & \alpha \end{pmatrix},$$

where

$$\alpha = \alpha_m(t) = \frac{\partial^2}{\partial t_1 \partial t_2} r|_{(t,t)}, \tag{2.3}$$

and, upon computing the Gaussian expectation (2.2) explicitly (see e.g. [10]), we obtain

$$K_{1;m}(x) = \frac{1}{\pi}\sqrt{\alpha}. \tag{2.4}$$

Now (chain rule)

$$\partial_{t_1} r(t_1, t_2) = \nabla r_F(\gamma(t_1) - \gamma(t_2)) \cdot \dot{\gamma}(t_1) \tag{2.5}$$

and

$$\alpha = -\dot{\gamma}(t_2)^t \cdot H_{r_F}(\gamma(t_1) - \gamma(t_2)) \cdot \dot{\gamma}(t_1)|_{(t,t)},$$

where H_{r_F} is the Hessian of r_F (thought of as $r_F(x) = r_F(x_1, x_2)$. The Hessian $H_{r_F}(0)$ was computed to be a scalar matrix [25]

$$H_{r_F}(0) = -2\pi^2 m \cdot I_2,$$

so that universally

$$\alpha = 2\pi^2 m \|\dot{\gamma}(t)\|^2 = 2\pi^2 m, \tag{2.6}$$

since we assumed that t is the arc-length parameter of $\mathcal{C}$ (i.e. $\|\dot{\gamma}(t)\| = 1$), and the zero density is

$$K_1(t) = \sqrt{2} \cdot \sqrt{m}.$$

□

Proof of Lemma 2.2. The correlation between $f(t)$ and $f'(t)$ is given by

$$\mathbb{E}[f(t)f'(t)] = \frac{\partial}{\partial t_1} r|_{(t,t)}.$$

Since we know that

$$r(t,t) = 1,$$

upon differentiating,

$$0 = \left(\frac{\partial}{\partial t_1} r + \frac{\partial}{\partial t_2} r\right)|_{(t,t)} = 2\frac{\partial}{\partial t_1} r|_{(t,t)}$$

by the symmetry. □

Remark 2.3. In fact, the proof above shows that the covariance of the underlying stationary field F satisfies $\nabla r_F(0) = 0$, as $r_F(x,x) \equiv 1$.

3. The 2-point correlation function

3.1. **Kac-Rice formula for computing the second moment of the number of zero crossings.** Let f and $\mathcal{Z}$ be as in section 2.1. We define the 2-point correlation function $K_2 = K_{2;f} : I \times I \to \mathbb{R}$ (also called the second intensity) in the following way: for $t_1 \neq t_2$ we define it as the conditional Gaussian expectation

$$K_2(t_1,t_2) = \phi_{t_1,t_2}(0,0) \cdot \mathbb{E}[|f'(t_1)| \cdot |f'(t_2)||f(t_1) = f(t_2) = 0]$$

where ϕ_{t_1,t_2} is the probability density function of the random Gaussian vector $(f(t_1), f(t_2))$. The function K_2 admits a continuation to a smooth function on the whole of $I \times I$ (see section 4.4), though its values at the diagonal are of no significance for our purposes. We will find an explicit expression for $K_2(t_1,t_2)$ in terms of r and its derivatives (see Lemma 3.1 below); finding such an expression involves studying the centered Gaussian vector $(f(t_1), f(t_2), f'(t_1), f'(t_2))$ with the covariance matrix $\Sigma = \Sigma_{4\times 4}(t_1,t_2)$ as in (3.6).

It is known [10] that under the assumption that for all $t_1 \neq t_2$ the matrix $\Sigma(t_1,t_2)$ is nonsingular (i.e. the Gaussian distribution of

$$(f(t_1), f(t_2), f'(t_1), f'(t_2))$$

is nondegenerate), the factorial second moment of $\mathcal{Z}$ is

$$\mathbb{E}[\mathcal{Z}^2 - \mathcal{Z}] = \iint\limits_{I\times I} K_2(t_1,t_2)dt_1dt_2,$$

so that accordingly

$$\mathrm{Var}(\mathcal{Z}) = \int\limits_{I\times I} \left(K_2(t_1,t_2) - K_1(t_1)\cdot K_1(t_2)\right) dt_1dt_2 + \mathbb{E}[\mathcal{Z}]; \tag{3.1}$$

note that the "extra" $\mathbb{E}[\mathcal{Z}]$ manifests the degeneracy of the matrix $\Sigma(t_1,t_2)$ on the diagonal $t_2 = t_1$.

Moreover, if $I_1, I_2 \subseteq I$ are *disjoint* nice sets (e.g. intervals), and the degeneracy assumption holds for all $(t_1, t_2) \in I_1 \times I_2$, then if for $J \subseteq I$ we denote $\mathcal{Z}_J$ to be the number of zero crossing of f in J, then (either employing the proof in [10] or using [1, Theorems 11.2.1, 11.5.1] on $I_1 \cup I_2$, whence we will need to make the non-degeneracy assumption for all $(t_1, t_2) \in (I_1 \cup I_2)^2$)

$$\mathbb{E}[\mathcal{Z}_{I_1} \cdot \mathcal{Z}_{I_2}] = \int\limits_{I_1 \times I_2} K_2(t_1, t_2) dt_1 dt_2,$$

so that

$$\mathrm{Cov}[\mathcal{Z}_{I_1} \cdot \mathcal{Z}_{I_2}] = \int\limits_{I_1 \times I_2} \left(K_2(t_1, t_2) - K_1(t_1) K_1(t_2) \right) dt_1 dt_2. \tag{3.2}$$

However, the non-degeneracy assumption is not satisfied in the case of f as in (2.1), and we may construct examples of curves, where the Kac-Rice formula as stated is wrong. However, in a situation like this we will be able to write an *approximate* Kac-Rice formula, prescribing the same order of magnitude for the fluctuations of the nodal intersections as the *precise* Kac-Rice (see Proposition 1.3). We will see in section 3.3 (Proposition 3.2) that under certain conditions on r (namely that $|r|$ is bounded away from 1) we will be able to approximate the 2-point correlation function in terms of powers of r and its derivatives; this will allow us to write the approximate Kac-Rice formula of Proposition 1.3 in terms of the relevant moments of r and its derivatives rather than in terms of the integral of 2-point correlation function. We will prove the approximate Kac-Rice formula of Proposition 1.3 in section 4 assuming the preparatory work in section 3.3, and some upper bounds for the 4-th moments of r and its derivatives along the relevant curve in section 6 (Lemma 6.1).

3.2. **An explicit expression for the 2-point correlation function.** Let $K_2(t_1, t_2) = K_{2;m}(t_1, t_2)$ be the 2-point correlation function of our process f as in (2.1), i.e. for $t_2 \neq t_1$

$$K_2(t_1, t_2) = \phi_{t_1, t_2}(0, 0) \cdot \mathbb{E}[|f'(t_1)| \cdot |f'(t_2)| | f(t_1) = f(t_2) = 0],$$

where ϕ_{t_1, t_2} is the probability density function of the random Gaussian vector $(f(t_1), f(t_2))$. The following lemma gives an explicit expression for K_2 in terms of r_f and its derivatives; recall the definition (2.3) for α and its explicit value $\alpha = 2\pi^2 m$.

Lemma 3.1. *We have explicitly*

$$K_2 = K_{2;m}(t_1, t_2) = \frac{1}{\pi^2 (1 - r^2)^{3/2}} \cdot \mu \cdot (\sqrt{1 - \rho^2} + \rho \arcsin \rho), \tag{3.3}$$

where

$$\mu = \mu_m(t_1, t_2) = \sqrt{\alpha(1 - r^2) - r_1^2} \cdot \sqrt{\alpha(1 - r^2) - r_2^2}, \tag{3.4}$$

and

$$\rho = \rho_m(t_1,t_2) = \frac{r_{12}(1-r^2)+rr_1r_2}{\sqrt{\alpha(1-r^2)-r_1^2}\cdot\sqrt{\alpha(1-r^2)-r_2^2}}, \tag{3.5}$$

is the correlation between the derivatives $f'(t_1)$ and $f'(t_2)$, conditioned on both values vanishing (thus satisfying $|\rho|\le 1$).

Proof. The covariance matrix for $(f_m(t_1), f_m(t_2), f_m'(t_1), f_m'(t_2))$ is

$$\Sigma = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \tag{3.6}$$

where

$$A = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \frac{\partial r}{\partial t_2} \\ \frac{\partial r}{\partial t_1} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha & \frac{\partial^2 r}{\partial t_1 \partial t_2} \\ \frac{\partial^2 r}{\partial t_1 \partial t_2} & \alpha \end{pmatrix}.$$

We abbreviate

$$r_1 := \frac{\partial r}{\partial t_1}, \quad r_2 := \frac{\partial r}{\partial t_2}, \quad r_{12} := \frac{\partial^2 r}{\partial t_1 \partial t_2}.$$

The covariance matrix for the conditional distribution of $f_m'(t_1), f_m'(t_2)$ conditioned on $f_m(t_1)=f_m(t_2)=0$ is

(3.7)

$$\Omega = \Omega_m(t_1,t_2) = C - B^tA^{-1}B = \begin{pmatrix} \alpha & r_{12} \\ r_{12} & \alpha \end{pmatrix} - \frac{1}{1-r^2}\begin{pmatrix} r_1^2 & -rr_1r_2 \\ -rr_1r_2 & r_2^2 \end{pmatrix}$$
$$= \frac{1}{1-r^2}\begin{pmatrix} \alpha(1-r^2)-r_1^2 & r_{12}(1-r^2)+rr_1r_2 \\ r_{12}(1-r^2)+rr_1r_2 & \alpha(1-r^2)-r_2^2 \end{pmatrix}.$$

The two-point correlation function is then given by

$$K_{2;m}(t_1,t_2) = \frac{1}{2\pi\sqrt{\det A}}\mathbb{E}[|W_1W_2|],$$

where

$$(W_1,W_2) \sim N(0,\Omega)$$

are centered Gaussian with covariance Ω. By normalizing the random variables

$$(W_1,W_2) = \left(\frac{\sqrt{\alpha(1-r^2)-r_1^2}}{\sqrt{1-r^2}}Y_1, \frac{\sqrt{\alpha(1-r^2)-r_2^2}}{\sqrt{1-r^2}}Y_2\right)$$

we write $K_{2;m}$ as

$$K_{2;m} = \frac{1}{2\pi(1-r^2)^{3/2}}\cdot\mu\cdot\mathbb{E}[|Y_1Y_2|], \tag{3.8}$$

where μ is given by (3.4), $(Y_1,Y_2)\sim N(0,\Delta(\rho))$ with

$$\Delta(\rho) = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \tag{3.9}$$

and ρ is given by (3.5).

It remains to evaluate

$$G(\rho) = \mathbb{E}[|Y_1 Y_2|]$$

with $(Y_1, Y_2) \sim N(0, \Delta(\rho))$. We may compute G explicitly to be equal to (see e.g. [4]),

$$G(\rho) = \frac{2}{\pi}\left(\sqrt{1-\rho^2} + \rho \arcsin \rho\right), \tag{3.10}$$

which finally yields the explicit formula (3.3) via (3.8). □

3.3. **Asymptotics for the 2-point correlation function.**

Proposition 3.2. *For every $\epsilon_2 > 0$, the two point correlation function satisfies, uniformly for $|r| < 1 - \epsilon_2$:*

$$\begin{aligned} K_2(t_1, t_2) = \frac{\alpha}{\pi^2}&\left(1 + \frac{1}{2}r^2 - \frac{1}{2}(r_1/\sqrt{\alpha})^2 - \frac{1}{2}(r_2/\sqrt{\alpha})^2 + \frac{1}{2}(r_{12}/\alpha)^2\right) \\ &+ \alpha \cdot O\left(r^4 + (r_1/\sqrt{\alpha})^4 + (r_2/\sqrt{\alpha})^4 + (r_{12}/\alpha)^4\right). \end{aligned} \tag{3.11}$$

Bearing in mind (2.4)*, we may equivalently write*

$$\begin{aligned} K_2(t_1, t_2) - K_1(t_1)K_1(t_2) = \frac{\alpha}{2\pi^2}&\left(r^2 - \left(\frac{r_1}{\sqrt{\alpha}}\right)^2 - \left(\frac{r_2}{\sqrt{\alpha}}\right)^2 + \left(\frac{r_{12}}{\alpha}\right)^2\right) \\ &+ \alpha \cdot O\left(r^4 + \left(\frac{r_1}{\sqrt{\alpha}}\right)^4 + \left(\frac{r_2}{\sqrt{\alpha}}\right)^4 + \left(\frac{r_{12}}{\alpha}\right)^4\right) \end{aligned}$$

with constants involved in the 'O'-notation depending on ϵ_2 only.

Proof. Note that if r, $\frac{r_1}{\sqrt{m}}$, $\frac{r_2}{\sqrt{m}}$, and $\frac{r_{12}}{m}$ are *small*, then ρ is small too. We may then expand ρ and μ about $r = 0$, $\frac{r_1}{\sqrt{\alpha}} = 0$, $\frac{r_2}{\sqrt{\alpha}} = 0$, $\frac{r_{12}}{\alpha} = 0$:

$$\begin{aligned} \rho &= \frac{r_{12}}{\alpha} \cdot \left(1 - \left(r^2 + (r_1/\sqrt{\alpha})^2\right)\right)^{-1/2} \cdot \left(1 - \left(r^2 + (r_2/\sqrt{\alpha})^2\right)\right)^{-1/2} \\ &\quad + O\left(r^3 + (r_1/\sqrt{\alpha})^3 + (r_2/\sqrt{\alpha})^3 + (r_{12}/\alpha)^3\right) \\ &= \frac{r_{12}}{\alpha} + O\left(r^3 + (r_1/\sqrt{\alpha})^3 + (r_2/\sqrt{\alpha})^3 + (r_{12}/\alpha)^3\right), \end{aligned} \tag{3.12}$$

Next we need to Taylor expand the function $G(\rho)$ as in (3.10) about $\rho = 0$:

$$G(\rho) = \frac{2}{\pi}\left(1 + \frac{1}{2}\rho^2\right) + O(\rho^4).$$

Substituting (3.12), we obtain

$$G(\rho) = \frac{2}{\pi}\left(1 + \frac{1}{2}(r_{12}/\alpha)^2\right) + O\left(r^4 + (r_1/\sqrt{\alpha})^4 + (r_2/\sqrt{\alpha})^4 + (r_{12}/\alpha)^4\right).$$

Next,

$$\begin{aligned}\mu &= \alpha\sqrt{1-(r^2+(r_1/\sqrt{\alpha})^2)}\cdot\sqrt{1-(r^2+(r_2/\sqrt{\alpha})^2)}\\ &= \alpha\left(1-r^2-\frac{1}{2}(r_1/\sqrt{\alpha})^2-\frac{1}{2}(r_2/\sqrt{\alpha})^2\right)\\ &+\alpha O\left(r^4+(r_1/\sqrt{\alpha})^4+(r_2/\sqrt{\alpha})^4+(r_{12}/\alpha)^4\right),\end{aligned}$$

and

$$\frac{1}{(1-r^2)^{3/2}}=1+\frac{3}{2}r^2+O(r^4).$$

Finally, substituting all the estimates above into (3.3) we obtain

$$\begin{aligned}K_{2;m}(t_1,t_2) &= \frac{1}{2\pi}\cdot\left(1+\frac{3}{2}r^2\right)\cdot\alpha\left(1-r^2-\frac{1}{2}\left(\frac{r_1}{\sqrt{\alpha}}\right)^2-\frac{1}{2}\left(\frac{r_2}{\sqrt{\alpha}}\right)^2\right)\\ &\cdot\frac{2}{\pi}\left(1+\frac{1}{2}(\frac{r_{12}}{\alpha})^2\right)+\alpha O\left(r^4+\left(\frac{r_1}{\sqrt{\alpha}}\right)^4+\left(\frac{r_2}{\sqrt{\alpha}}\right)^4+\left(\frac{r_{12}}{\alpha}\right)^4\right)\\ &= \frac{\alpha}{\pi^2}\left(1+\frac{1}{2}r^2-\frac{1}{2}\left(r_1/\sqrt{\alpha}\right)^2-\frac{1}{2}\left(r_2/\sqrt{\alpha}\right)^2+\frac{1}{2}\left(r_{12}/\alpha\right)^2\right)\\ &+\alpha O\Big(r^4+(r_1/\sqrt{\alpha})^4+(r_2/\sqrt{\alpha})^4+(r_{12}/\alpha)^4\Big).\end{aligned}$$

An inspection of each step reveals that all the expansions are valid under the assumption that $|r|$ is bounded away from 1. $\square$

4. Approximate Kac-Rice for computing the variance of nodal intersections

This section is entirely dedicated to proving Proposition 1.3. Throughout the present section we assume that $\epsilon_0>0$ is fixed, and m satisfies $|\widehat{\tau_m}(4)|<1-\epsilon_0$.

4.1. **Nodal intersections on short arcs.** Let $c_0>0$ be a small number (depending on ϵ_0), and divide our curve into short arcs of size roughly $\frac{c_0}{\sqrt{m}}$. More precisely, let $K=K_m=\left\lfloor L\cdot\frac{\sqrt{m}}{c_0}\right\rfloor+1$,

$$\delta_0=\delta_{0;m}=\frac{L}{K}\le\frac{c_0}{\sqrt{m}},$$

and define the partition $I=\bigcup\limits_{i=1}^{K}I_i$ of $I=[0,L]$ into short intervals

$$I_i:=[(i-1)\cdot\delta_0,i\cdot\delta_0],$$

$i=1,\dots,K$, disjoint save for the overlaps at the endpoints. We will eventually choose c_0 sufficiently small so that the Kac-Rice formula will hold on the short intervals (see Lemma 4.3), and the value of r or of one of its derivatives in a "singular cube" will be bounded away from 0 (see Definition 4.5 and Lemma 4.6).

For the future we record that, as $c_0 > 0$ is constant,

$$\delta_0 \asymp \frac{1}{\sqrt{m}}. \tag{4.1}$$

For $1 \leq i \leq K$, let $\mathcal{Z}_i$ be the number of nodal intersections of F_m with $\gamma(I_i)$, that is $\mathcal{Z}_i$ is the number of zeros of f on I_i. We have a.s.

$$\mathcal{Z} = \sum_{i=1}^{K} \mathcal{Z}_i,$$

so that

$$\mathbb{E}[\mathcal{Z}^2] = \sum_{i=1}^{K} \mathbb{E}[\mathcal{Z}_i^2] + 2\sum_{i<j} \mathbb{E}[\mathcal{Z}_i \cdot \mathcal{Z}_j]; \tag{4.2}$$

equivalently

$$\mathrm{Var}(\mathcal{Z}) = \sum_{i=1}^{K} \mathrm{Var}(\mathcal{Z}_i) + 2\sum_{i<j} Cov\left(\mathcal{Z}_i, \mathcal{Z}_j\right). \tag{4.3}$$

Later we will apply Kac-Rice (3.1) to "most" of the summands in (4.3) (see section 4.3) and bound the contribution of the rest of the summands; integrating and summing these up will eventually establish the statement of Proposition 1.3.

4.2. **Nodal intersections variance on short arcs.** As a first goal, we will establish an estimate on the variance $\mathrm{Var}(\mathcal{Z}_i)$ of nodal intersections with a short arc of γ; with the help of the latter we will be able to control the contribution of any individual summand in (4.3), via Cauchy-Schwartz (Corollary 4.2).

Proposition 4.1. *For every $0 < \epsilon_0 < 1$ we can choose $c_0 = c_0(\epsilon_0)$ sufficiently small, such that for any m with $|\widehat{\tau_m}(4)| < 1 - \epsilon_0$, we have*

$$\mathrm{Var}(\mathcal{Z}_i) = O(1),$$

uniformly for $i \leq K$, where the constant involved in the "O"-notation depends on ϵ_0 and c_0 only.

Before proving Proposition 4.1 we draw the following corollary, as announced above.

Corollary 4.2. *For every $0 < \epsilon_0 < 1$ we can choose $c_0 = c_0(\epsilon_0)$ sufficiently small, such that for any m with $|\widehat{\tau_m}(4)| < 1 - \epsilon_0$, we have*

$$|Cov(\mathcal{Z}_i, \mathcal{Z}_j)| = O(1),$$

uniformly for $i, j \leq K$, where the constant involved in the "O"-notation depends on ϵ_0 and c_0 only.

Proof of Corollary 4.2. Applying Cauchy-Schwartz we have

$$|Cov(\mathcal{Z}_i, \mathcal{Z}_j)| \le \sqrt{\mathrm{Var}(\mathcal{Z}_i) \cdot \mathrm{Var}(\mathcal{Z}_j)} = O(1),$$

by Proposition 4.1. □

To prove Proposition 4.1 we will need Lemma 4.3 and Proposition 4.4 stated below.

Lemma 4.3. *For every $0 < \epsilon_0 < 1$ we can choose $c_0 = c_0(\epsilon_0)$ sufficiently small, such that for any m with $|\widehat{\tau_m}(4)| < 1-\epsilon_0$, the matrix $\Sigma(t_1, t_2)$, defined in (3.6), is nonsingular for all $t_1, t_2 \in [0, L]^2$ with*

$$0 < |t_2 - t_1| < \frac{c_0}{\sqrt{m}}.$$

The proof of Lemma 4.3 is quite long and technical, and is thereupon relegated to Appendix A.

Proposition 4.4. *For $t_1 \in [0, L]$ and $|t_2 - t_1| < \frac{c_0}{\sqrt{m}}$ one has the uniform estimate*

$$K_2(t_1, t_2) = O(m)$$

with constant depending on c_0 only.

The proof of Proposition 4.4 is deferred to section 4.4.

Proof of Proposition 4.1 assuming Lemma 4.3 and Proposition 4.4. Thanks to Lemma 4.3 the covariance matrix $\Sigma(t_1, t_2)$ is nonsingular for all $(t_1, t_2) \in I_i^2$ with $t_2 \neq t_1$, so, by the discussion in section 3.1 above we may apply Kac-Rice (3.1) to $I_i \subseteq I$ to write

$$\mathrm{Var}(\mathcal{Z}_i) = \int\limits_{I_i \times I_i} (K_2(t_1, t_2) - K_1(t_1)K_1(t_2))dt_1 dt_2 + \mathbb{E}[\mathcal{Z}_i]. \tag{4.4}$$

Applying Proposition 4.4 and the Kac-Rice formula (2.1) for computing the expected number of zeros on I_i

$$\mathbb{E}[\mathcal{Z}_i] = \int\limits_{I_i} K_1(t)dt \ll \sqrt{m} \cdot \delta_0$$

(see Lemma 2.1) to (4.4) yields

$$\mathrm{Var}(\mathcal{Z}_i) \ll m \cdot \delta_0^2 + \sqrt{m} \cdot \delta_0 \ll 1,$$

bearing in mind (4.1). This concludes the proof of the present proposition. □

4.3. **Proof of Proposition 1.3.** Recalling the notation from section 4.1 we now divide the domain of the integration, namely, the cube $S := I^2 = [0, L]^2$ into small cubes $S_{ij} = I_i \times I_j$ of side δ_0; some of the latter will be designated as "singular" and the rest as "nonsingular". Let $\epsilon_1 > 0$ be a small number that will be fixed till the end (e.g. $\epsilon_1 = \frac{1}{100}$ is sufficient).

Definition 4.5. *(Singular and nonsingular cubes and sets.)*

(i) *We call a point* $(t_1, t_2) \in [0, L]^2$ singular *if either* $|r(t_1, t_2)| > \epsilon_1$ *or* $|r_1(t_1, t_2)| > \epsilon_1 \cdot \sqrt{m}$ *or* $|r_2(t_1, t_2)| > \epsilon_1 \cdot \sqrt{m}$ *or* $|r_{12}(t_1, t_2)| > \epsilon_1 \cdot m$.

(ii) *Let*

$$S_{ij} = I_i \times I_j = [i\delta_0, (i+1)\delta_0] \times [j\delta_0, (j+1)\delta_0]$$

be a cube in $[0, L]^2$*. We say that* S_{ij} *is a singular cube if it contains a singular point.*

(iii) *The union of all the singular cubes is the singular set*

$$B = B_m = \bigcup_{S_{ij} \text{ singular}} S_{ij}.$$

Note that outside the singular set $\Sigma(t_1, t_2)$ is nonsingular (provided that ϵ_1 is chosen sufficiently small) by (3.4), (3.5) and (3.9); we are thereupon allowed to apply the Kac-Rice formula on $S \setminus B$; in particular for all i, j with $S_{i,j} \cap Int(B) = \emptyset$ (this implies $i \neq j$):

$$\mathbb{E}[\mathcal{Z}_i \mathcal{Z}_j] = \int_{S_{ij}} K_2(t_1, t_2) dt_1 dt_2.$$

We plan to approximate the 2-point correlation function as the corresponding sum of powers of r and its derivatives; by Proposition 3.2 we are allowed to do so unless r is big, and we will bound the contribution of the domain where it is.

Lemma 4.6. *If* $S_{ij} \subseteq B$ *is singular, then for all* $(t_1, t_2) \in S_{ij}$ *either* $r(t_1, t_2) > \epsilon_1/2$ *or the analogous statement holds for one of the derivatives in the definition of singular point (Definition 4.5 (i)).*

Proof. The statement for c_0 sufficiently small follows from the fact that the scaled covariance function $r_F(y/\sqrt{m})$ of the ambient field F and its derivatives are Lipschitz with a universal constant (independent of m) (as it is easy to check, first for the individual function $x \mapsto \cos(2\pi\langle \mu, x \rangle)$, and then for their average), and thus the same holds for r. □

Lemma 4.7. *The total area of the singular set is*

$$\text{meas}(B) = O\left(N_m^{-3/2}\right).$$

Proof. We apply the Chebyshev-Markov inequality on the measure of B. Lemma 4.6 shows that it is bounded from above by

$$\operatorname{meas}(B) \ll \int_0^L \int_0^L \left(r(t_1,t_2)^4 + \frac{1}{m^2} r_1(t_1,t_2)^4 + \frac{1}{m^2} r_2(t_1,t_2)^4 + \frac{1}{m^4} r_{12}(t_1,t_2)^4 \right) dt_1 dt_2,$$

which is small by Lemma 6.1 (which is independent of the arguments of the present section). □

Recall that B consists of cubes of side length $\delta \asymp \frac{1}{\sqrt{m}}$ (see (4.1)). Lemma 4.7 implies that the number of singular cubes is $\ll \frac{m}{N_m^{3/2}}$ and, teamed with Corollary 4.2, yields the following estimate on the total contribution of the singular domain B.

Corollary 4.8. *The total contribution of the singular set is:*

$$\left| \sum_{S_{ij} \text{ singular}} \operatorname{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) \right| = O(m \cdot N_m^{-3/2}).$$

Proof of Proposition 1.3. Consider the equality (4.3) and apply Kac-Rice on every nonsingular cube (i.e. use (3.2) for those I_i and I_j such that S_{ij} is not lying in B, bearing in mind that for all $(t_1,t_2) \in S_{ij}$, $\Sigma(t_1,t_2)$ is nonsingular). We then obtain

$$\begin{aligned}\operatorname{Var}(\mathcal{Z}) &= \int_{S\setminus B} (K_2(t_1,t_2) - K_1(t_1)K_1(t_2))dt_1dt_2 + \sum_{S_{ij} \text{ singular}} Cov(\mathcal{Z}_i, \mathcal{Z}_j) \\ &= \int_{S\setminus B} (K_2(t_1,t_2) - K_1(t_1)K_1(t_2))dt_1dt_2 + O(m \cdot N_m^{-3/2}),\end{aligned}$$

by Corollary 4.8. We finally use the expansion in Proposition 3.2 for K_2 valid outside of B (the latter of the two equivalent forms), and use Lemma 6.1 again for bounding the contribution of the error term in (3.11), together with the everywhere boundedness of the integrand on the rhs of (1.13) to conclude the proof. □

4.4. **Proof of Proposition 4.4.**

Proof. From Lemma 3.1, since $\frac{2}{\pi} \le G \le 1$,

$$K_2(t_1,t_2) \ll \frac{1}{(1-r^2)^{3/2}} \cdot \mu \ll \frac{1}{(1-r)^{3/2}} \sqrt{\alpha(1-r^2) - r_1^2} \cdot \sqrt{\alpha(1-r^2) - r_2^2}.$$

Note that

$$\begin{aligned}
&\frac{1}{(1-r^2)^{3/2}}\sqrt{\alpha(1-r^2)-r_1^2}\cdot\sqrt{\alpha(1-r^2)-r_2^2}\\
&=\frac{\alpha}{\sqrt{1-r^2}}\sqrt{1-\frac{r_1^2}{\alpha(1-r^2)}}\cdot\sqrt{1-\frac{r_2^2}{\alpha(1-r^2)}} \qquad (4.5)\\
&\ll\frac{\alpha}{\sqrt{1-r}}\sqrt{1-\frac{r_1^2}{\alpha(1-r^2)}}\cdot\sqrt{1-\frac{r_2^2}{\alpha(1-r^2)}}\leq\frac{\alpha}{\sqrt{1-r}}.
\end{aligned}$$

The diagonal cube $S = S_{ij}$ contains a point of the form (t_1, t_1). We may Taylor expand the integrand $K_2(t_1, t_2)$ for $(t_1, t_2) \in S$ about (t_1, t_2) as a function of t_2, t_1 *fixed*, and assuming WLOG $t_2 > t_1$.

To expand r we differentiate and evaluate the derivatives at the diagonal $t_2 = t_1$: The first derivative $r_2 = \partial r/\partial t_2$ is

$$r_2 = -\nabla r_{F_m}(\gamma(t_1)-\gamma(t_2))\cdot\dot{\gamma}(t_2),$$

and on the diagonal

$$r_2(t,t) = 0. \qquad (4.6)$$

The second derivative $r_{22} = \partial^2 r/\partial t_2^2$ is

$$r_{22} = \dot{\gamma}(t_2)^t\cdot H_{r_{F_m}}(\gamma(t_1)-\gamma(t_2))\cdot\dot{\gamma}(t_2) - \nabla r_{F_m}(\gamma(t_1)-\gamma(t_2))\cdot\ddot{\gamma}(t_2),$$

on the diagonal $r_{22}(t,t) = -\alpha$. The third derivative is

(4.7)

$$\begin{aligned}
r_{222} &= \frac{\partial}{\partial t_2}\left[\dot{\gamma}(t_2)^t\cdot H_{r_{F_m}}(\gamma(t_1)-\gamma(t_2))\cdot\dot{\gamma}(t_2)\right]\\
&\quad+\dot{\gamma}(t_2)\cdot H_{r_{F_m}}(\gamma(t_1)-\gamma(t_2))\cdot\ddot{\gamma}(t_2) - \nabla r_{F_m}(\gamma(t_1)-\gamma(t_2))\cdot\dddot{\gamma}(t_2)\\
&= \dot{\gamma}(t_2)^t\cdot\frac{\partial}{\partial t_2}\left[H_{r_{F_m}}(\gamma(t_1)-\gamma(t_2))\right]\cdot\dot{\gamma}(t_2)\\
&\quad+3\dot{\gamma}(t_2)^t\cdot H_{r_{F_m}}(\gamma(t_1)-\gamma(t_2))\cdot\ddot{\gamma}(t_2) - \nabla r_{F_m}^t(\gamma(t_1)-\gamma(t_2))\cdot\dddot{\gamma}(t_2),
\end{aligned}$$

and on the diagonal

$$r_{222}(t_1,t_1) = -3\alpha\dot{\gamma}(t_2)^t\cdot\ddot{\gamma}(t_2) = 0, \qquad (4.8)$$

since the acceleration is always orthogonal to the velocity (t is the arc-length parameter). Moreover, the Hessian satisfies $H \ll m$ and $\partial H/\partial t_2 \ll m^{3/2}$ everywhere, so that we have

$$r_{222}(t_1,t_2) = O(m^{3/2})$$

everywhere.

The expansion of $r(t_1, t_2)$ around the diagonal $t_2 = t_1$, valid for $0 < t_2 - t_1 \leq \frac{c_0}{\sqrt{m}}$ with c_0 sufficiently small, is

$$r = 1 - \frac{\alpha}{2}(t_2-t_1)^2 + O(m^{3/2}(t_2-t_1)^3),$$

and

(4.9)

$$1-r^2 = (1-r)(1+r)$$
$$= \left[\frac{\alpha}{2}(t_2-t_1)^2 + O\left(m^{3/2}(t_2-t_1)^3\right)\right]\left[2-\frac{\alpha}{2}(t_2-t_1)^2 + O\left(m^{3/2}(t_2-t_1)^3\right)\right]$$
$$= \alpha(t_2-t_1)^2\left(1+O(\sqrt{m}(t_2-t_1))\right),$$

$$r_2^2 \approx r_1^2 = \alpha^2(t_2-t_1)^2\left(1+O\left(m^{1/2}(t_2-t_1)\right)\right),$$

thus

$$\frac{r_1^2}{\alpha(1-r^2)} = 1+O\left(m^{1/2}(t_2-t_1)\right),$$

and hence

$$0 \le 1-\frac{r_1^2}{\alpha(1-r^2)} = O\left(m^{1/2}(t_2-t_1)\right),$$

and the same estimate holds for

$$1-\frac{r_2^2}{\alpha(1-r^2)}.$$

Consolidating all the estimates we conclude that (4.5) is uniformly bounded by

$$\frac{\alpha}{\sqrt{1-r}}\sqrt{1-\frac{r_1^2}{\alpha(1-r^2)}}\cdot\sqrt{1-\frac{r_2^2}{\alpha(1-r^2)}} \ll \frac{\alpha}{m^{1/2}(t_2-t_1)}\cdot O(m^{1/2}(t_2-t_1)) = O(m),$$

recalling that $\alpha = 2\pi^2 m$. □

5. Asymptotics for the second moments of the covariance function and its derivatives

Recall that r is the covariance function restricted to the curve $\mathcal{C}$:

$$r(t_1,t_2) = r(\gamma(t_1),\gamma(t_2)) \tag{5.1}$$

Proposition 5.1. *If $\mathcal{C}\subset\mathbb{T}^2$ is a (smooth) curve with nowhere vanishing curvature, then for all $\epsilon>0$*

$$\int_{\mathcal{C}}\int_{\mathcal{C}} r^2 = \int_0^L\int_0^L r(t_1,t_2)^2dt_1dt_2 = \frac{L^2}{N_m}+O\left(\frac{1}{N_m^{2-\epsilon}}\right) \tag{5.2}$$

$$\int_{\mathcal{C}}\int_{\mathcal{C}}\left|\frac{1}{2\pi\sqrt{m}}\frac{\partial r}{\partial t_1}\right|^2 = \frac{L^2}{2N_m}+O\left(\frac{1}{N_m^{2-\epsilon}}\right) \tag{5.3}$$

and

$$\int_{\mathcal{C}}\int_{\mathcal{C}}\left|\frac{1}{4\pi^2 m}\frac{\partial^2 r}{\partial t_1\partial t_2}\right|^2 = \frac{B_{\mathcal{C}}(\mathcal{E})}{N_m}+O\left(\frac{1}{N_m^{2-\epsilon}}\right) \tag{5.4}$$

where

$$B_{\mathcal{C}}(\mathcal{E}) := \int_{\mathcal{C}}\int_{\mathcal{C}} \frac{1}{N_m} \sum_{\mu\in\mathcal{E}} \left\langle \frac{\mu}{|\mu|}, \dot\gamma(t_1)\right\rangle^2 \cdot \left\langle \frac{\mu}{|\mu|}, \dot\gamma(t_2)\right\rangle^2 dt_1 dt_2. \tag{5.5}$$

Before proceeding with the proof, we can conclude the proof of Theorem 1.2: Use Proposition 1.3 to write an approximate integral formula for the nodal intersections number variance and substitute the result of Proposition 5.1 in place of the main term of (1.3). □

5.1. **Main terms.** Squaring out, we have (on isolating the diagonal pairs $\mu = \mu'$)

$$|r(t_1,t_2)|^2 = \frac{1}{N_m} + \frac{1}{N_m^2} \sum_{\substack{\mu,\mu'\in\mathcal{E}\\ \mu\neq\mu'}} e^{2\pi i\langle \mu-\mu', \gamma(t_1)-\gamma(t_2)\rangle} \tag{5.6}$$

and hence integrating we find

$$\iint |r(t_1,t_2)|^2 dt_1 dt_2 = \frac{L^2}{N_m} + \frac{1}{N_m^2} \sum_{\substack{\mu,\mu'\in\mathcal{E}\\ \mu\neq\mu'}} \left| \int_0^L e^{2\pi i\langle \mu-\mu', \gamma(t)\rangle} dt\right|^2. \tag{5.7}$$

For the second moment of the derivative $r_1 = \partial/\partial t_1$ we compute

$$\frac{1}{2\pi i\sqrt{m}}\frac{\partial r}{\partial t_1}(t_1,t_2) = \frac{1}{N_m}\sum_{\mu} \left\langle \frac{\mu}{|\mu|}, \dot\gamma(t_1)\right\rangle e^{2\pi i\langle \mu, \gamma(t_1)-\gamma(t_2)\rangle} \tag{5.8}$$

and setting

$$A_{\mu,\mu'}(t) = \left\langle \frac{\mu}{|\mu|}, \dot\gamma(t)\right\rangle \left\langle \frac{\mu'}{|\mu'|}, \dot\gamma(t)\right\rangle \tag{5.9}$$

we find

$$\begin{aligned}\iint \left|\frac{1}{2\pi\sqrt{m}}\frac{\partial r}{\partial t_1}(t_1,t_2)\right|^2 dt_1 dt_2 &= \frac{1}{N_m^2}\sum_{\mu}\int_0^L A_{\mu,\mu}(t_1)dt_1 \int_0^L 1 dt_2 \\ &+ \frac{1}{N_m^2} \sum_{\substack{\mu,\mu'\in\mathcal{E}\\ \mu\neq\mu'}} \int_0^L A_{\mu,\mu'}(t_1) e^{2\pi i\langle \mu-\mu', \gamma(t_1)\rangle} dt_1 \int_0^L e^{2\pi i\langle \mu'-\mu, \gamma(t_2)\rangle} dt_2.\end{aligned} \tag{5.10}$$

Similarly,

$$\begin{aligned}\iint \left|\frac{1}{4\pi^2 m}\frac{\partial^2 r}{\partial t_1\partial t_2}(t_1,t_2)\right|^2 dt_1 dt_2 &= \frac{1}{N_m^2}\sum_{\mu}\iint A_{\mu,\mu}(t_1)A_{\mu,\mu}(t_2) dt_1 dt_2 \\ &+ \frac{1}{N_m^2} \sum_{\substack{\mu,\mu'\in\mathcal{E}\\ \mu\neq\mu'}} \left|\int_0^L A_{\mu,\mu'}(t) e^{2\pi i\langle \mu-\mu', \gamma(t)\rangle} dt\right|^2.\end{aligned} \tag{5.11}$$

For $\partial r/\partial t_1$ we use (see [25, Lemma 2,3]) that for any $v \in \mathbb{R}^2$,

$$\frac{1}{N_m} \sum_{\mu\in\mathcal{E}} \langle \mu, v\rangle^2 = \frac{m}{2}||v||^2 \tag{5.12}$$

and applying it for $v = \dot\gamma(t)$ which has unit length we get that

$$\frac{1}{N_m} \sum_{\mu} A_{\mu,\mu}(t) = \frac{1}{2}||\dot\gamma(t)||^2 = \frac{1}{2}. \tag{5.13}$$

Integrating over t_1 and t_2 shows that the diagonal term in (5.10) is $L^2/2N_m$.

For $\partial^2 r/\partial t_1\partial t_2$ the diagonal term in (5.11) is

$$\frac{1}{N_m}\iint \frac{1}{N_m}\sum_{\mu} \left\langle \frac{\mu}{|\mu|}, \dot\gamma(t_1)\right\rangle^2 \cdot \left\langle \frac{\mu}{|\mu|}, \dot\gamma(t_2)\right\rangle^2 dt_1 dt_2 = \frac{B_{\mathcal{C}}(\mathcal{E})}{N_m}. \tag{5.14}$$

5.2. **Off-diagonal terms.** To handle the off-diagonal terms $\mu \neq \mu'$, we need the following consequence of van der Corput's lemma (see [7]): For each $0 \neq \xi \in \mathbb{R}^2$ define a phase function on the curve $\mathcal{C}$ by

$$\phi_\xi(t) = \left\langle \frac{\xi}{|\xi|}, \gamma(t)\right\rangle. \tag{5.15}$$

Let $A \in C^\infty(0,L)$ be a smooth amplitude and for k real, set

$$I(k) = \int A(t) e^{ik\phi_\xi(t)} dt. \tag{5.16}$$

Lemma 5.2. *Assume $\mathcal{C}$ has nowhere vanishing curvature. Then for $|k| \geq 1$,*

$$|I(k)| \ll \frac{1}{|k|^{1/2}} \left\{ ||A||_\infty + ||A'||_1 \right\}, \tag{5.17}$$

the implied constant depending only on the curve $\mathcal{C}$ (independent of ξ).

Applying Lemma 5.2, we see that for $\mu \neq \mu'$,

$$\int_0^L e^{2\pi i \langle \mu-\mu', \gamma(t)\rangle} dt \ll_{\mathcal{C}} \frac{1}{|\mu-\mu'|^{1/2}}. \tag{5.18}$$

Moreover, $|A_{\mu,\mu'}| \leq 1$ and $|A'_{\mu,\mu'}| \leq 2K_{\max}$ where $K_{\max}$ is the maximum value of the curvature on $\mathcal{C}$, because

(5.19)

$$A'_{\mu,\mu'} = \left\langle \frac{\mu}{|\mu|}, \ddot\gamma(t)\right\rangle \left\langle \frac{\mu'}{|\mu'|}, \dot\gamma(t)\right\rangle + \left\langle \frac{\mu}{|\mu|}, \dot\gamma(t)\right\rangle \left\langle \frac{\mu'}{|\mu'|}, \ddot\gamma(t)\right\rangle =$$
$$= \kappa(t)\left(\left\langle \frac{\mu}{|\mu|}, \nu(t)\right\rangle \cdot \left\langle \frac{\mu'}{|\mu'|}, \dot\gamma(t)\right\rangle + \left\langle \frac{\mu}{|\mu|}, \dot\gamma(t)\right\rangle \left\langle \frac{\mu'}{|\mu'|}, \nu(t)\right\rangle \right),$$

where $\ddot\gamma = \kappa\nu$ with κ the curvature and ν the unit normal to the curve. Therefore we likewise find

$$\int_0^L A_{\mu,\mu'}(t) e^{2\pi i \langle \mu-\mu', \gamma(t)\rangle} dt \ll_{\mathcal{C}} \frac{1}{|\mu-\mu'|^{1/2}}. \tag{5.20}$$

Hence we find that

$$\iint |r(t_1,t_2)|^2 dt_1 dt_2 = \frac{L^2}{N_m} + O\left(\frac{1}{N_m^2}\sum_{\substack{\mu,\mu'\in\mathcal{E}\\ \mu\neq\mu'}}\frac{1}{|\mu-\mu'|}\right) \tag{5.21}$$

and for $j = 1, 2$

$$\iint \left|\frac{1}{2\pi\sqrt{m}}\frac{\partial r}{\partial t_j}(t_1,t_2)\right|^2 dt_1 dt_2 = \frac{L^2}{2N_m} + O\left(\frac{1}{N_m^2}\sum_{\substack{\mu,\mu'\in\mathcal{E}\\ \mu\neq\mu'}}\frac{1}{|\mu-\mu'|}\right), \tag{5.22}$$

and finally

$$\iint \left|\frac{1}{4\pi^2 m}\frac{\partial^2 r}{\partial t_1 \partial t_2}(t_1,t_2)\right|^2 dt_1 dt_2 = \frac{B_{\mathcal{C}}(\mathcal{E})}{N_m} + O\left(\frac{1}{N_m^2}\sum_{\substack{\mu,\mu'\in\mathcal{E}\\ \mu\neq\mu'}}\frac{1}{|\mu-\mu'|}\right). \tag{5.23}$$

Proposition 5.1 hence follows from

Proposition 5.3.

$$\sum_{\substack{\mu,\mu'\in\mathcal{E}\\ \mu\neq\mu'}}\frac{1}{|\mu-\mu'|} \ll N_m^{\epsilon}, \quad \forall\epsilon>0. \tag{5.24}$$

5.3. **A result of Mordell.** Denote by $\mathcal{H}$ the set of $h \le H$ for which the system

$$|\mu|^2 = m = |\mu'|^2,\ |\mu-\mu'|^2 = 2h \tag{5.25}$$

has integer solutions, and by $A(m,h)$ the number such solutions.

We give an arithmetic characterization of the set $\mathcal{H}$. To do so, we will need a result of Mordell [22] (see also Niven [23]) on the representation of a binary quadratic form as a sum of two squares of integer linear forms.

Theorem 5.4 (Mordell [22])**.** *Let $A, B, C \in \mathbb{Z}$. Assume that the integer binary quadratic form*

$$F(x,y) := Ax^2 + 2Bxy + Cy^2$$

is positive definite, i.e. that $A, C > 0$ and $AC - B^2 > 0$. Then we can represent

$$F(x,y) = (ux+u'y)^2 + (vx+v'y)^2$$

with integer u, v, u', v' if and only if

$$AC - B^2 = \square \text{ is a perfect square,} \tag{5.26}$$

and

$$\gcd(A,B,C) = \square + \square \text{ is a sum of two integer squares.} \tag{5.27}$$

Pall [24] gives the exact number of solutions as $r_2(\gcd(A,B,C))$ if $AC-B^2>0$, and $2r_2(\gcd(A,B,C))$ if $AC-B^2=0$, where $r_2(n)$ is the number of representations of n as a sum of two integer squares.

Writing $\mu=(u,v)$ and $\mu'=(u',v')$ we have

$$(ux+u'y)^2+(vx+v'y)^2=|x\mu+y\mu'|^2$$

so that we can interpret Mordell's theorem as saying that given A,B,C as above, there are integer vectors $\mu,\mu'\in\mathbb{Z}^2$ satisfying

$$|\mu|^2=A,\quad \langle\mu,\mu'\rangle=B,\quad |\mu'|^2=C \tag{5.28}$$

if and only if (5.26) and (5.27) hold.

A consequence is

Corollary 5.5. *Let $m,h\in\mathbb{Z}$, $0<h<m$. There are two integer vectors μ,μ' with $|\mu|^2=m=|\mu'|^2$ and $|\mu-\mu'|^2=2h$ if and only if*

(i) *$h(2m-h)=\Box$ is a perfect square, and*

(ii) *$\gcd(m,h)=\Box+\Box$ is a sum of two squares.*

In this case the number of solutions is $A(m,h)=r_2(\gcd(m,h))\ll h^{o(1)}$.

5.4. **Proof of Proposition 5.3.** Let $H=N_m^4$. We separate the sum into that over "distant" pairs $|\mu-\mu'|^2>H$ and "close" pairs $1\le|\mu-\mu'|^2\le H$. For the sum over distant pairs, we crudely use

$$\sum_{\substack{\mu,\mu'\in\mathcal{E}\\ |\mu-\mu'|^2>H}}\frac{1}{|\mu-\mu'|}\ll\frac{N_m^2}{\sqrt{H}}. \tag{5.29}$$

To handle the sum over "close" pairs, we write

$$\sum_{\substack{(\mu,\mu')\in\mathcal{E}\times\mathcal{E}\\ 0<|\mu-\mu'|^2<H}}\frac{1}{|\mu-\mu'|}=\sum_{\substack{h\in\mathcal{H}\\ 0<h<H}}\frac{A(m,h)}{\sqrt{2h}}\ll H^{\epsilon}\sum_{\substack{h\in\mathcal{H}\\ 0<h<H}}\frac{1}{\sqrt{h}}. \tag{5.30}$$

For $h\in\mathcal{H}$, we write $d=(h,m)$, which is a sum of two squares ($d=\Box+\Box$), $h=dh'$, $m=dm'$ with $(m',h')=1$. Then $h\in\mathcal{H}$ means $h(2m-h)=\Box$ and so $h'(2m'-h')=\Box$. Thus we find

$$\sum_{\substack{h\in\mathcal{H}\\ 0<h<H}}\frac{1}{\sqrt{h}}=\sum_{\substack{d|m\\ d=\Box+\Box\\ d<H}}\frac{1}{\sqrt{d}}\sum_{\substack{h'(2m'-h')=\Box\\ (h',m')=1\\ h'<H/d}}\frac{1}{\sqrt{h'}}. \tag{5.31}$$

We claim that the inner sum over h' is $O(1)$. To see this, use $1/\sqrt{h'}\le 1$ and separate into cases according to h' being odd or even. If h' is odd and $(h',m')=1$, then the condition $h'(2m'-h')=\Box$ implies $h'=\Box$ and $2m'-h'=\Box$, that is $h'=u^2$ and $2m-h'=v^2$ with $v>0$, $0<u<\sqrt{H/d}$. If h' is even, the the condition $h'(2m'-h')=\Box$ and $(h',m')=1$ implies $(h'/2,m'-h'/2)=1$ and $h'/2=\Box$, $m'-h'/2-\Box$ so that $h'/2=u^2$,

$m' - h'/2 = v^2$ with $v > 0$, $0 < u < \sqrt{H/d}$. Summarizing, we get lattice points on the circle $u^2 + v^2 = 2m'$ or $u^2 + v^2 = m'$ depending on the parity of h', with $0 < u < \sqrt{H/d}$, $v > 0$. These conditions puts these lattice points on a "short" arc on the circle, since $H \ll m^{o(1)}$. Recall Jarnik's theorem [15], which states that on an arc of size $< R^{1/3}$ on a circle of radius R there can be at most two lattice points. Hence there are at most two such lattice points in each of the two cases, and thus the number of participating h' is at most 4. This proves that the inner sum in (5.31) is bounded.

We conclude that

$$\sum_{\substack{h \in \mathcal{H} \\ 0<h<H}} \frac{1}{\sqrt{h}} \ll \sum_{\substack{d|m \\ d=\square+\square \\ d<H}} \frac{1}{\sqrt{d}}. \tag{5.32}$$

Below in Lemma 5.6 we will show that this sum is bounded by $O(N_m^\epsilon)$. This will show that the contribution of close pairs is $O(N_m^\epsilon)$. Combining with the bound (5.29) on distant pairs we get

$$\sum_{\substack{\mu,\mu' \in \mathcal{E} \\ \mu \neq \mu'}} \frac{1}{|\mu - \mu'|} \ll \frac{N_m^2}{\sqrt{H}} + N_m^\epsilon \ll N_m^\epsilon \tag{5.33}$$

on recalling that $H = N_m^4$. This will conclude the proof of Proposition 5.3, once we prove:

Lemma 5.6. *Suppose that $H = N_m^\alpha$ for some $\alpha > 0$. Then*

$$\sum_{\substack{d|m \\ d=\square+\square \\ d<H}} \frac{1}{\sqrt{d}} \ll_\epsilon N_m^\epsilon.$$

Proof. Write $m = m_1^2 m_2$ where $m_1 = 2^r \prod_{q_k = 3 \bmod 4} q_k^{b_k}$ is a product of powers of primes $q_k = 3 \bmod 4$ and possibly a power of 2, and $m_2 = 2^c \prod_j p_j^{a_j}$ is a product of powers of primes $p_j = 1 \bmod 4$, possibly times 2 ($c = 0, 1$). Then

$$N_m = \prod_j (a_j + 1).$$

Likewise we write $d = d_1^2 d_2$ in the same fashion, so that $d \mid m$ is equivalent to $d_1 \mid m_1$ and $d_2 \mid m_2$.

The sum over d's is bounded by

$$\sum_{\substack{d|m \\ d=\square+\square \\ d<H}} \frac{1}{\sqrt{d}} \ll \sum_{\substack{d_1|m_1 \\ d_1<\sqrt{H}}} \frac{1}{d_1} \sum_{d_2|m_2} \frac{1}{\sqrt{d_2}} \ll \log H \sum_{d_2|m_2} \frac{1}{\sqrt{d_2}} \ll \log N_m \sum_{d_2|m_2} \frac{1}{\sqrt{d_2}},$$

where in the sum over d_2 we have dropped the condition $d < H$.

It now suffices to show that for all $\epsilon > 0$, there is some $C(\epsilon) > 0$ so that

$$\sum_{d_2|m_2} \frac{1}{\sqrt{d_2}} \leq C(\epsilon) N_m^{\epsilon}.$$

Ignoring the possible factor of 2,

$$\sum_{d_2|m_2} \frac{1}{\sqrt{d_2}} \ll \prod_j \left(1 + \frac{1}{\sqrt{p_j}} + \cdots + \frac{1}{p_j^{(a_j+1)/2}}\right) \leq \prod_j \frac{1}{1 - \frac{1}{\sqrt{p_j}}}.$$

Recalling that $N_m = \prod_j (a_j + 1) \geq \prod_j 2$ we find

$$\begin{aligned} \frac{1}{N_m^{\epsilon}} \sum_{d_2|m_2} \frac{1}{\sqrt{d_2}} &\ll \prod_{\substack{p_j|m \\ p_j=1 \bmod 4}} \frac{1}{(1 - \frac{1}{\sqrt{p_j}})2^{\epsilon}} \\ &\leq \prod_{*} \frac{1}{(1 - \frac{1}{\sqrt{p}})2^{\epsilon}} =: C(\epsilon), \end{aligned} \tag{5.34}$$

where in the last line, the product is over all primes satisfying $(1 - \frac{1}{\sqrt{p}})2^{\epsilon} < 1$. This gives $\sum_{d_2|m_2} 1/\sqrt{d_2} \leq C(\epsilon) N_m^{\epsilon}$ as claimed. □

6. Bounds for the higher moments of r and its derivatives

The estimates of the following lemma were used in section 4 in the proof of Proposition 1.3.

Lemma 6.1. *We have the following estimates on the 4th moments of the covariance function and its various derivatives along a (smooth) reference curve γ with nowhere vanishing curvature:*

$$\iint_{[0,L]^2} r(t_1, t_2)^4 dt_1 dt_2 = O\left(\frac{1}{N_m^{3/2}}\right), \tag{6.1}$$

$$\frac{1}{m^2} \iint_{[0,L]^2} r_1(t_1, t_2)^4 dt_1 dt_2 = O\left(\frac{1}{N_m^{3/2}}\right),$$

$$\frac{1}{m^2} \iint_{[0,L]^2} r_2(t_1, t_2)^4 dt_1 dt_2 = O\left(\frac{1}{N_m^{3/2}}\right),$$

$$\frac{1}{m^4} \iint_{[0,L]^2} r_{12}(t_1, t_2)^4 dt_1 dt_2 = O\left(\frac{1}{N_m^{3/2}}\right).$$

Proof. Abbreviating $e(z) := e^{2\pi i z}$, we have

$$\iint\limits_{[0,L]^2} r(t_1,t_2)^4 dt_1 dt_2$$
$$= \frac{1}{N_m^4} \sum_{\mu_1,\dots,\mu_4\in\mathcal{E}} \iint_{[0,L]^2} e(\langle \mu_1+\mu_2+\mu_3+\mu_4, \gamma(t_1)-\gamma(t_2)\rangle) dt_1 dt_2$$
$$= \frac{1}{N_m^4} \sum_{\mu_1,\dots,\mu_4\in\mathcal{E}} |I_1(\mu_1,\mu_2,\mu_3,\mu_4)|^2$$

with

$$I_1(\mu_1,\mu_2,\mu_3,\mu_4) = \int\limits_{[0,L]} e\left(\langle \mu_1+\mu_2+\mu_3+\mu_4, \gamma(t)\rangle\right) dt. \tag{6.2}$$

Now by Lemma 5.2, for $\mu_1+\mu_2+\mu_3+\mu_4 \neq 0$ we have the estimate

$$|I_1(\mu_1,\mu_2,\mu_3,\mu_4)| \ll \frac{1}{|\mu_1+\mu_2+\mu_3+\mu_4|^{1/2}}. \tag{6.3}$$

Hence

$$\iint_{[0,L]^2} r(t_1,t_2)^4 dt_1 dt_2 \ll \frac{1}{N_m^2} + \frac{1}{N_m^4} \sum_{\substack{\mu_1,\dots,\mu_4\in\mathcal{E} \\ \mu_1+\mu_2+\mu_3+\mu_4\neq 0}} \frac{1}{\|\mu_1+\mu_2+\mu_3+\mu_4\|},$$

since for given $\mu_1,\mu_2 \in \mathcal{E}$ with $\mu_1 \neq -\mu_2$ there exist (precisely) 2 choices for $\mu_3,\mu_4 \in \mathcal{E}$ so that

$$\mu_1+\mu_2+\mu_3+\mu_4 = 0,$$

by an elementary argument due to Zygmund [27]. The estimate (6.1) now follows from Lemma 6.2.

For the derivative r_1 we have:

$$\iint_{[0,L]^2} r_1(t_1,t_2)^4 dt_1 dt_2 = \frac{(2\pi)^4}{N_m^4} \sum_{\mu_1,\dots,\mu_4\in\mathcal{E}} I_2(\mu_1,\mu_2,\mu_3,\mu_4)\cdot \overline{I_1(\mu_1,\mu_2,\mu_3,\mu_4)}, \tag{6.4}$$

where I_1 was defined in (6.2), and

$$I_2(\mu_1,\mu_2,\mu_3,\mu_4) = \int_0^L e(\langle \mu_1+\mu_2+\mu_3+\mu_4, \gamma(t)\rangle) \mu_1^t\dot\gamma(t)\mu_2^t\dot\gamma(t)\mu_3^t\dot\gamma(t)\mu_4^t\dot\gamma(t) dt. \tag{6.5}$$

We invoke Lemma 5.2 again to yield the bound

$$|I_2| \ll m^2 \cdot \frac{1}{|\mu_1+\mu_2+\mu_3+\mu_4|^{1/2}}, \tag{6.6}$$

so that combined with the estimate (6.3) and (6.4) it implies

$$\frac{1}{m^2}\iint\limits_{[0,L]^2} r_1(t_1,t_2)^4 dt_1 dt_2 \ll \frac{1}{N_m^2} + \frac{1}{N_m^4} \sum_{\substack{\mu_1,\dots,\mu_4\in\mathcal{E} \\ \mu_1+\mu_2+\mu_3+\mu_4\neq 0}} \frac{1}{\|\mu_1+\mu_2+\mu_3+\mu_4\|},$$

yielding the statement of the present lemma in this case as before, via Lemma 6.2. The argument for r_2 is identical.

For the second mixed derivative r_{12} we have:

$$r_{12}(t_1,t_2) = -\frac{(2\pi)^2}{N_m}\sum_{\mu\in\mathcal{E}}\mu^t\dot{\gamma}(t_1)\mu^t\dot{\gamma}(t_2)e\left(\langle\mu_1+\mu_2+\mu_3+\mu_4,\gamma(t_1)-\gamma(t_2)\rangle\right),$$

and

$$r_{12}(t_1,t_2)^4 =$$

$$\frac{(2\pi)^8}{N_m^4}\sum_{\mu_1,\dots,\mu_4\in\mathcal{E}}\prod_{j=1}^{4}\langle\mu_j,\dot{\gamma}(t_1)\rangle\cdot\langle\mu_j,\dot{\gamma}(t_2)\rangle\times e\left(\langle\mu_1+\mu_2+\mu_3+\mu_4,\gamma(t_1)-\gamma(t_2)\rangle\right)$$

so that by separation of variables and upon recalling (6.5), we have

$$\iint\limits_{[0,L]^2} r_{12}(t_1,t_2)^4dt_1dt_2 = \frac{(2\pi)^8}{N_m^4}\sum_{\mu_1,\dots,\mu_4\in\mathcal{E}}\left|I_2(\mu_1,\mu_2,\mu_3,\mu_4)\right|^2,$$

and invoking (6.6) (valid for $\mu_1+\mu_2+\mu_3+\mu_4\neq 0$), we finally have

$$\begin{aligned}\frac{1}{m^4}\iint\limits_{[0,L]^2} r_{12}(t_1,t_2)^4dt_1dt_2 &\ll \frac{1}{N_m^2}+\frac{1}{N_m^4}\sum_{\substack{\mu_1,\dots,\mu_4\in\mathcal{E}\\ \mu_1+\mu_2+\mu_3+\mu_4\neq 0}}\frac{1}{\|\mu_1+\mu_2+\mu_3+\mu_4\|}\\ &= O\left(\frac{1}{N_m^{3/2}}\right),\end{aligned}$$

by Lemma 6.2. □

Lemma 6.2. *We have the following bound*

$$\sum_{\substack{\mu_1,\dots,\mu_4\in\mathcal{E}\\ \mu_1+\mu_2+\mu_3+\mu_4\neq 0}}\frac{1}{\|\mu_1+\mu_2+\mu_3+\mu_4\|} = O\left(N_m^{5/2}\right). \tag{6.7}$$

Proof. Let us denote $v=\mu_1+\mu_2+\mu_3+\mu_4$. We choose a big parameter $A>0$ and split the summation into 3 ranges:

(i) $\|v\|\le A$.

We invoke Zygmund's elementary observation [27] again to deduce that, given μ_1 and μ_2 and v such that

$$\mu_1+\mu_2\neq v,$$

there are (at most) two choices for $\mu_3,\mu_4\in\mathcal{E}$ that solve

$$\mu_1+\mu_2+\mu_3+\mu_4=v.$$

Therefore we may bound the contribution to the sum (6.7) of this range as

$$\sum_{\substack{\mu_1,\ldots,\mu_4\in\mathcal{E}\\ \|\mu_1+\mu_2+\mu_3+\mu_4\|\le A}} \frac{1}{\|\mu_1+\mu_2+\mu_3+\mu_4\|} \tag{6.8}$$
$$\le N_m^2\cdot \sum_{\|v\|\le A}\frac{1}{\|v\|} \ll N_m^2 \int\limits_{1\le|x\|\le A} \frac{dx}{\|x\|} = N_m^2\int\limits_1^A dt = A\cdot N_m^2,$$

by comparing the sum $\sum\limits_{\|v\|\le A}\frac{1}{\|v\|}$ to the integral $\int\limits_{1\le|x\|\le A}\frac{dx}{\|x\|}$.

(ii) $A\le\|v\|\le N_m^{3/2}$.

We claim that given μ_1,μ_2,μ_3 there exist at most 2 lattice points μ_4 that lie in the relevant range so that $\|\mu_1+\mu_2+\mu_3+\mu_4\|\le N_m^{3/2}$. Once established the above, the contribution of this range is, bounding the summands point-wise,

$$\sum_{\substack{\mu_1,\ldots,\mu_4\in\mathcal{E}\\ A\le\|\mu_1+\mu_2+\mu_3+\mu_4\|\le N_m^{3/2}}} \frac{1}{\|\mu_1+\mu_2+\mu_3+\mu_4\|} \le \frac{1}{A}\cdot N_m^3. \tag{6.9}$$

To see that indeed, given $\mu_1,\ldots\mu_3$ there are at most two vectors μ_4 that return us to the relevant range, we consider the geometric picture. Let μ_1,μ_2,μ_3 be fixed, define $w=\mu_1+\mu_2+\mu_3$ and suppose that there exists a vector μ_4 so that $v=\mu_1+\mu_2+\mu_3+\mu_4$ satisfies $N_m/\log N_m\le\|v\|\le N_m^{3/2}$ indeed. By the triangle inequality, the vector w satisfies

$$\sqrt{m}-N_m^{3/2}\le\|w\|\le N_m^{3/2}+\sqrt{m};$$

adding the vector μ_4 translates it to a circle of a small radius $N_m\log N_m$ around the origin, which means that μ_4 has to be on a circular arc of angle α of the order at most

$$\alpha\sim\sin\alpha\le\frac{N_m^{3/2}}{\sqrt{m}-N_m^{3/2}},$$

with arc length $\le\sqrt{m}\frac{N_m^{3/2}}{\sqrt{m}-N_m^{3/2}}$, which is much smaller than $m^{1/3}$, so by Jarnik there exists at most two such lattice points, as claimed.

(iii) $\|v\|\ge N_m^{3/2}$.

Here it is sufficient to bound the summands in (6.7) pointwise; since the total number of summands is N_m^4 the sum is bounded as

$$\sum_{\substack{\mu_1,\ldots,\mu_4\in\mathcal{E}\\ \|\mu_1+\mu_2+\mu_3+\mu_4\|\ge N_m^{3/2}}} \frac{1}{\|\mu_1+\mu_2+\mu_3+\mu_4\|} \le \frac{1}{N_m^{3/2}}\cdot N_m^4 = N_m^{5/2}. \tag{6.10}$$

Consolidating (6.8), (6.9) and (6.10) we find that the sum (6.7) is bounded by

$$A \cdot N_m^2 + \frac{1}{A} \cdot N_m^3 + N_m^{5/2},$$

and the lemma follows by taking $A = N_m^{1/2}$. □

7. Fluctuations of the leading constant

7.1. **Some basic observations.** Recall that given m we denoted $\mathcal{E}$ to be the set of lattice points on the circle of radius $\sqrt{m}$, and that we defined the probability measures τ_m on $\mathcal{S}^1$ as in (1.6). We may then rewrite $B_{\mathcal{C}}(\mathcal{E})$ (1.11) as

$$B_{\mathcal{C}}(\mathcal{E}) := \int_{\mathcal{C}}\int_{\mathcal{C}}\int_{\mathcal{S}^1} \langle \theta, \dot{\gamma}(t_1)\rangle^2 \langle \theta, \dot{\gamma}(t_2)\rangle^2 d\tau_m(\vartheta) dt_1 dt_2.$$

More generally, for any probability measure τ on $\mathcal{S}^1$, invariant w.r.t. $\frac{\pi}{2}-$rotations and the reflection $(x,y)\mapsto(x,-y)$ we define the number

(7.1)

$$c(\tau,\gamma) = \int_0^L\int_0^L\int_{\mathcal{S}^1} \langle \theta, \dot{\gamma}(t_1)\rangle^2 \langle \theta, \dot{\gamma}(t_2)\rangle^2 d\tau(\vartheta) dt_1 dt_2 = \int_{\mathcal{S}^1} d\tau(\theta) \left[\int_0^L \langle \theta, \dot{\gamma}(t)\rangle^2 dt\right]^2,$$

so that, in particular,

$$B_{\mathcal{C}}(\mathcal{E}) = c(\tau_m,\gamma). \tag{7.2}$$

The leading constant (7.1) is intimately related with the (weak) limiting angular distribution of lattice points in $\mathcal{E}$. As usual when we deal with convergence of measures, weak convergence is denoted by "$\Rightarrow$". Thus if $\{m_j\}$ is a subsequence of energy levels such that $\tau_{m_j} \Rightarrow \tau$ for some probability measure τ on $\mathcal{S}^1$ then

$$c(\tau_{m_j},\gamma) \rightarrow c(\tau,\gamma).$$

Therefore the variety of limiting values of B is related to the weak partial limits of $\{\tau_m\}$, i.e. probability measures τ on $\mathcal{S}^1$ such that for some subsequence m_j of energy levels, such that $\tau_{m_j}\Rightarrow\tau$. The classification of *all* such measures τ, called *attainable*, was first addressed in [18], and was subsequently studied in more detail in [20]. It is well known that the lattice points $\mathcal{E}$ are equidistributed on $\mathcal{S}^1$ along generic subsequences of energy levels (see e.g. [13], Proposition 6) in the sense that $\tau_{m_j}\Rightarrow \frac{1}{2\pi}d\theta$ along some density 1 sequence $\{m_j\}$, and thus, in particular, the normalized arc-length measure $\frac{1}{2\pi}d\theta$ on $\mathcal{S}^1$ is attainable. Among other things it was shown in [20] that for τ attainable the value of the Fourier transform $\widehat{\tau}(4)$ attains the whole interval $[-1,1]$, a fact that is going to be important in the example considered in section 7.2 below.

7.2. **An example: explicit computation of $c(\tau,\gamma)$ for circular arcs.** Let $\mathcal{C}$ be the circular arc

$$\gamma(t) = (r\cos(t/r), r\sin(t/r)),$$

$t \in [0,L]$. Here we obtain after some elementary manipulations

$$c(\tau,\gamma) = \frac{1}{4}L^2 + \frac{1}{8}r^2\sin^2(L/r) + \frac{1}{8}r^2\sin^2(L/r)\cos(2L/r)\cdot\widehat{\tau}(4), \tag{7.3}$$

where we exploited the $\pi/2$–invariance of τ to write $\widehat{\tau}(2) = 0$. Since, as it was mentioned in section 7.1, all the values of $\widehat{\tau}(4) \in [-1,1]$ are hit by attainable measures, the leading constant $4c(\tau,\gamma) - L^2$ in (1.10) takes all values between

$$r^2\sin^4(L/r) \text{ and } r^2\sin^2(L/r)\cos^2(L/r).$$

We may also infer from (7.3) that if γ is a $\frac{1}{8}$-circle plus a multiple of a quarter-circle ($L/r = \frac{\pi}{4} + k\pi/2$, $k = 0,1,2,3$), or a multiple of a semi-circle ($L/r = \pi, 2\pi$), then the leading constant is independent of τ. For the latter case the constant *vanishes universally*; here the nodal length fluctuations are of lower order of magnitude than prescribed by Theorem 1.2. The only other case when the leading constant vanishes occurs for quarter circles plus multiples of semi-circles and

$$\tau = \frac{1}{4}(\delta_{\pm\pi/4} + \delta_{\pm3\pi/4}) \tag{7.4}$$

the "tilted Cilleruelo measure" (attainable), name inspired from the "Cilleruelo measure" [18, 9]

$$\tau = \frac{1}{4}(\delta_{\pm1} + \delta_{\pm i}) \tag{7.5}$$

(when thinking $\mathcal{S}^1 \subseteq \mathbb{C}$); these are excluded from our discussion by bounding $|\widehat{\tau}(4)|$ away from ±1 (see e.g. the formulation of Theorem 1.1).

7.3. **Classification of the leading constants.** By applying the Cauchy-Schwartz inequality on (7.1) it is obvious that for all τ, γ, one has $c(\tau,\gamma) \le L^2$. A stronger bound is possible, thanks to the $\pi/2$–rotation invariance of τ.

We will employ an auxiliary notation in order to rewrite the definition (7.1) of $c(\tau,\gamma)$ in a more useful way for our purposes. Given a direction

$$\theta = e^{i\vartheta} \in \mathcal{S}^1$$

we denote the L^2-squared energy of the projection of the tangent directions of γ in the direction θ:

$$A(\gamma,\theta) := \int_0^L \langle\theta, \dot{\gamma}(t)\rangle^2 dt, \tag{7.6}$$

so that we may rewrite (7.1) as

$$c(\tau,\gamma)=\int\limits_{\mathcal{S}^1} A(\gamma,\theta)^2 d\tau(\theta). \tag{7.7}$$

Proposition 7.1. (i) *For all τ measures on $\mathcal{S}^1$, and smooth toral curves γ one has*

$$\frac{L^2}{4}\le c(\tau,\gamma)\le L^2/2. \tag{7.8}$$

(ii) *The minimum value*

$$c(\tau,\gamma)=\frac{L^2}{4}$$

is attained for a given measure τ if and only if for all θ in the support of τ, $A(\gamma,\theta)=\frac{L}{2}$.

Proof. We observe that for $\theta^{\perp}$ a perpendicular direction to θ (any of the two),

$$A(\gamma,\theta)+A(\gamma,\theta^{\perp})=L,$$

from which it is easy to show that

$$\frac{L^2}{2}\le A(\gamma,\theta)^2+A(\gamma,\theta^{\perp})^2\le L^2. \tag{7.9}$$

We then use the invariance properties of τ to write (7.7) as

$$c(\tau,\gamma)=\int\limits_{\mathcal{S}^1/i} 2(A(\gamma,\theta)^2+A(\gamma,\theta^{\perp})^2)d\tau(\theta),$$

where $\mathcal{S}^1/i$ is a quarter of the circle identifying ϑ and $\vartheta+\pi/2$ of measure

$$\tau(\mathcal{S}^1/i)=\frac{1}{4}$$

by the invariance. It then readily yields via (7.9) that

$$c(\tau,\gamma)=\int\limits_{\mathcal{S}^1/i} 2(A(\gamma,\theta)^2+A(\gamma,\theta^{\perp})^2)d\tau(\theta)\ge\frac{L^2}{4}, \tag{7.10}$$

and also (7.8). This concludes the proof of the first statement of the present proposition, and, in fact, this proof also yields the second one. □

The following corollary from Proposition 7.1 gives the necessary and sufficient conditions for the leading constant to vanish (equivalently, for $c(\tau,\gamma)$ to attain its theoretical minimum $c(\tau,\gamma)=\frac{L^2}{4}$). Define the complex number $\mathcal{I}(\gamma)$ as

$$\mathcal{I}(\gamma)=\int\limits_0^L e^{2i\varphi(t)}dt=0,$$

where $\dot{\gamma}(t) = e^{i\varphi(t)}$, i.e. $\varphi(t)$ is the angle of $\dot{\gamma}(t)$ w.r.t. the coordinate axes.

Corollary 7.2. (i) *The minimum value*

$$c(\tau, \gamma) = \frac{L^2}{4}$$

is attained universally (i.e. for all τ), if and only if

$$\mathcal{I}(\gamma) = 0, \tag{7.11}$$

(ii) *If* (7.11) *is not satisfied, then the the only measures τ where $c(\tau, \gamma)$ may equal $\frac{L^2}{4}$ are the Cilleruelo measure* (7.5) *and the tilted Cilleruelo* (7.4)*; it will occur if and only if*

$$\mathrm{Re}\,\mathcal{I}(\gamma) = \int_0^L \cos(2\varphi(t))dt = 0 \text{ or } \mathrm{Im}\,\mathcal{I}(\gamma) = \int_0^L \sin(2\varphi(t))dt = 0$$

respectively.

Proof. Under the notation $\dot{\gamma}(t) = e^{i\varphi(t)}$ as above,

$$A(\gamma, \theta) = \int_0^L \cos(\vartheta - \varphi(t))^2 dt = \frac{L}{2} + \frac{1}{2}\int_0^L \cos(2(\vartheta - \varphi(t)))dt,$$

and therefore $A(\gamma, \theta) = \frac{L}{2}$ if and only if

$$\int_0^L \cos(2(\vartheta - \varphi(t)))dt = 0.$$

Now the latter integral is

$$\int_0^L \cos(2(\vartheta - \varphi(t)))dt = \cos(2\vartheta) \cdot \int_0^L \cos(2\varphi(t))dt + \sin(2\vartheta)\int_0^L \sin(2\varphi(t))dt.$$

Thus, if the tuple $(\cos(2\vartheta), \sin(2\vartheta))$ attains at least two not co-linear values with $\vartheta \in \mathrm{supp}(\tau)$, it implies that

$$\int_0^L \cos(2\varphi(t))dt = \int_0^L \sin(2\varphi(t))dt = 0,$$

which is equivalent to (7.11); in this case the constant $c(\tau, \gamma)$ vanishes *universally*, i.e. for all measures τ.

The only two attainable measures that violate the condition of

$$(\cos(2\vartheta), \sin(2\vartheta))$$

attaining at least two not co-linear values with $\vartheta \in \mathrm{supp}(\tau)$ as above are Cilleruelo (7.5) and tilted Cilleruelo (7.4). Here the condition for vanishing of the leading constant is $\int_0^L \cos(2\varphi(t))dt = 0$ or $\int_0^L \sin(2\varphi(t))dt = 0$ respectively, as prescribed. □

The next proposition studies when $c(\tau,\gamma)$ attains the "theoretical maximum" $\frac{L^2}{2}$.

Proposition 7.3. *The maximum value $c(\tau,\gamma) = \frac{L^2}{2}$ is attained for τ the Cilleruelo measure* (7.5) *and $\mathcal{C}$ a straight line parallel to either of the axes, or τ the tilted Cilleruelo measure* (7.4) *and $\mathcal{C}$ parallel to $y = \pm x$. Though excluded by Theorem 1.2, this could be approximated arbitrarily well by $c(\tau,\gamma)$ for length-L smooth curves with non-vanishing curvature.*

Proof. By the proof of Proposition 7.1 above the upper bound in (7.8) is attained if and only if for all $\theta \in \operatorname{supp}(\tau)$,

$$A(\gamma,\theta)^2 + A(\gamma,\theta^{\perp})^2 = L^2,$$

which happens if and only if for all $\theta \in \operatorname{supp}(\tau)$ one has $A(\gamma,\theta) = 0$ or $A(\gamma,\theta^{\perp}) = 0$. Equivalently, for all $\theta \in \operatorname{supp}(\tau)$ and all $t \in [0,L]$, either $\theta \perp \dot{\gamma}(t)$ or $\theta^{\perp} \perp \dot{\gamma}(t)$. Thus there is a "unique" maximizer for $c(\tau,\gamma)$, where τ is an attainable measure and γ is a curve, namely the only cases prescribed in the statement of the present proposition. Since we exclude the straight lines from our discussion, this is the supremum rather than maximum. □

Appendix A. Non-degeneracy of the covariance matrix

In this section we prove Lemma 4.3: given a fixed $0 < \epsilon_0 < 1$ we are to find a constant $c_0 = c_0(\epsilon_0)$, so that for all m satisfying $|\tau_m| < 1 - \epsilon_0$ (with τ_m defined in (1.6)), we have $\det \Sigma(t_1,t_2) > 0$ (with $\Sigma(t_1,t_2)$ given by (3.6)) is strictly positive for $|t_2 - t_1| \le \frac{c_0}{\sqrt{m}}$. Recall that μ and ρ are given by (3.4) and (3.5) respectively (with $\alpha = 2\pi^2 m$); we have explicitly

(A.1)
$$\det \Sigma(t_1,t_2) = \det A \cdot \det \Omega = (1-r^2)\cdot(1-r^2)^{-2}\mu^2(1-\rho^2) = (1-r^2)^{-1}\mu^2(1-\rho^2).$$

As above, $\dot{\gamma}(t) = e^{i\varphi(t)}$, i.e. the vector $\dot{\gamma}(t)$ is a unit vector in the direction $\varphi(t)$, and

$$A(t) := \widehat{\tau_m}(4)\cdot\cos(4\varphi(t)).$$

In order to establish the positivity of $\det \Sigma(t_1,t_2)$ we Taylor expand the expression $\mu^2 \cdot (1-\rho)^2$, considered as a function of t_2 and t_1 constant, around $t_2 = t_2$, as in the following lemma, with the other term $(1-r^2)^{-1}$ having been readily expanded (4.9).

Lemma A.1. *We have*

$$\mu^2(1-\rho^2) = \frac{2}{9}\pi^{14}m^7(A(t_1)-1)(A(t_1)^2-1)(t_2-t_1)^{10} + O(m^{13/2}(t_2-t_1)^{10} + m^{15/2}(t_2-t_1)^{11}),$$

valid for $|t_2 - t_1| \ll \frac{1}{\sqrt{m}}$.

Proof of Lemma 4.3 assuming Lemma A.1. Recall that by (A.1) we have

$$\det \Sigma = (1-r^2)^{-1} \cdot \mu^2(1-\rho^2).$$

It is obvious from (4.9) that $(1-r^2)$ (and hence its reciprocal) is strictly positive for $|t_2 - t_1| < \frac{c_0}{\sqrt{m}}$ with c_0 depending on γ only. Concerning the other factor, we use Lemma A.1 to expand

$$\begin{aligned} \mu^2(1-\rho^2) &= \frac{2}{9}\pi^{14} m^7 (A(t_1)-1)(A(t_1)^2-1)(t_2-t_1)^{10} \\ &+ O(m^{13/2}(t_2-t_1)^{10} + m^{15/2}(t_2-t_1)^{11}). \end{aligned} \tag{A.2}$$

Note that

$$|A(t_1)| \le |\widehat{\tau_m}(4)| < 1 - \epsilon_0$$

is bounded away from 1. That implies that the leading term in (A.2),

$$\frac{2}{9}\pi^{14} m^7 (1-A(t_1))(1-A(t_1)^2)(t_2-t_1)^{10} \ge \frac{2}{9}\pi^{14}\epsilon_0^3 \cdot m^7 (t_2-t_1)^{10} \gg m^7(t_2-t_1)^{10},$$

is bigger than the remainder term in (A.2) for $|t_2 - t_1| < \frac{c_0}{\sqrt{m}}$ for c_0 chosen sufficiently small. □

Proof of Lemma A.1. We have

$$\mu^2(1-\rho^2) = \left(\alpha(1-r^2) - r_1^2\right)\left(\alpha(1-r^2) - r_2^2\right) - \left(r_{12}(1-r^2) + r r_1 r_2\right)^2. \tag{A.3}$$

Let $c_m = c_m(t_1) := \frac{\partial^4 r}{\partial t_2^4}(t_1, t_1)$, $e_m = e_m(t_1) := \frac{\partial^6 r}{\partial t_2^6}(t_1, t_1)$. Using the identities

$$\cos^4\theta = \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta),$$

and

$$\cos^6(\theta) = \frac{5}{16} + \frac{15}{32}\cos(2\theta) + \frac{3}{16}\cos(4\theta) + \frac{1}{32}\cos(6\theta),$$

and $\widehat{\tau_m}(k) = 0$ unless $4|k$, by the $\pi/2$ rotation invariance, we may compute

$$\begin{aligned} c_m &= \frac{(2\pi)^4}{N}\sum_{\mu\in\mathcal{E}} (\mu^t \cdot \dot\gamma(t))^4 + O(m^{3/2}) \\ &= (2\pi)^4 m^2 \left(\frac{3}{8} + \frac{1}{8}\widehat{\tau_m}(4)\cos(4\phi)\right) + O(m^{3/2}), \end{aligned} \tag{A.4}$$

$$c_m' = O(m^2),$$

$$\begin{aligned} e_m &:= -\frac{(2\pi)^6}{N}\sum_{\mu\in\mathcal{E}} (\mu^t \cdot \dot\gamma(t))^6 + O(m^{5/2}) \\ &= -(2\pi)^6 m^3 \left(\frac{5}{16} + \frac{3}{16}\widehat{\tau_m}(4)\cos(4\phi)\right) + O(m^{5/2}). \end{aligned} \tag{A.5}$$

Let $z := t_2 - t_1$. Bearing in mind that (4.8),

$$\frac{\partial r}{\partial t_2}(t_1,t_1) = \frac{\partial^3 r}{\partial t_2^3}(t_1,t_1) = 0$$

(cf. (4.6) and (4.8)), and

$$\left|\frac{\partial^5 r}{\partial t_2^5}(t_1,t_1)\right| = O(m^2),$$

we may Taylor expand $r = r(t_1,t_2)$ for t_1 fixed as:

$$r = 1 - \frac{\alpha}{2}z^2 + \frac{1}{24}c_m(t_1)z^4 + \frac{1}{720}e_m(t_1)z^6 + O(m^2z^5 + m^{7/2}z^7),$$

where the constant involved in the "O"-notation depends on γ only. We may differentiate term-wise to obtain (the terms involving c_m', e_m' are of smaller order and are absorbed in the various error terms)

$$r_2 = -\alpha z + \frac{1}{6}c_m z^3 + \frac{1}{120}e_m z^5 + O\left(m^2z^4 + m^{7/2}z^6\right),$$

$$\begin{aligned} r_1 &= \alpha z - \frac{1}{6}c_m z^3 - \frac{1}{120}e_m z^5 + O(m^2z^4 + m^{7/2}z^6) \\ &= z\left(\alpha - \frac{1}{6}c_m z^2 - \frac{1}{120}e_m z^4\right) + O(m^2z^4 + m^{7/2}z^6), \end{aligned}$$

$$r_{12} = \alpha - \frac{1}{2}c_m z^2 - \frac{1}{24}e_m z^4 + O(m^2z^3 + m^{7/2}z^5).$$

Incorporating the above, we have (using $|z| \ll \frac{1}{\sqrt{m}}$ to consolidate the various error terms throughout)

$$\begin{aligned} 1 - r^2 &= (1-r)(1+r) \\ &= z^2\left(\frac{\alpha_2}{2} - \frac{1}{24}c_m z^2 - \frac{1}{720}e_m z^4\right)\cdot\left(2 - \frac{\alpha}{2}z^2 + \frac{1}{24}c_m z^4\right) + O(m^2z^5 + m^{7/2}z^7) \\ &= z^2\left(\alpha - \left(\frac{c_m}{12} + \frac{\alpha^2}{4}\right)z^2 + \left(-\frac{e_m}{360} + \frac{\alpha}{24}c_m\right)z^4\right) + O(m^2z^5 + m^{7/2}z^7), \end{aligned}$$

$$r_1^2 = z^2\left(\alpha^2 - \frac{\alpha}{3}c_m z^2 + \left(\frac{c_m^2}{36} - \frac{\alpha}{60}e_m\right)z^4\right) + O(m^3z^5 + m^{9/2}z^7),$$

and the same estimate holds for r_2^2;

$$\begin{aligned} \alpha(1-r^2) - r_1^2 = z^4&\left(\frac{\alpha}{4}\left(c_m - \alpha^2\right) + \frac{1}{72}\left(e_m\alpha + 3\alpha^2 c_m - 2c_m^2\right)z^2\right) \\ &+ O(m^3z^5 + m^{9/2}z^7), \end{aligned}$$

and the same estimate holds for $\alpha(1-r^2)-r_2^2$;

$$\begin{aligned}&(\alpha(1-r^2)-r_1^2)\cdot(\alpha(1-r^2)-r_2^2)\\ &= z^8\left(\frac{\alpha^2}{16}\left(c_m-\alpha^2\right)^2+\frac{\alpha_2}{144}\left(c_m-\alpha^2\right)\left(e_m\alpha+3\alpha^2c_m-2c_m^2\right)z^2\right)\\ &+O(m^6z^9+m^{15/2}z^{11}).\end{aligned} \tag{A.6}$$

Continuing,

$$\begin{aligned}r_{12}(1-r^2) &= z^2\left(\alpha-\frac{1}{2}c_mz^2-\frac{1}{24}e_mz^4\right)\times\\ &\times\left(\alpha-\left(\frac{c_m}{12}+\frac{\alpha^2}{4}\right)z^2+\left(-\frac{e_m}{360}+\frac{\alpha}{24}c_m\right)z^4\right)+O(m^3z^5+m^{9/2}z^7)\\ &= z^2\left(\alpha^2-\alpha\left(\frac{7}{12}c_m+\frac{\alpha^2}{4}\right)z^2+\left(-\frac{2}{45}\alpha e_m+\frac{1}{6}\alpha^2c_m+\frac{c_m(t_1)^2}{24}\right)z^4\right)\\ &+O(m^3z^5+m^{9/2}z^7),\end{aligned}$$

$$\begin{aligned}rr_1r_2 &= -z^2\left(1-\frac{\alpha}{2}z^2+\frac{1}{24}c_mz^4\right)\cdot\left(\alpha-\frac{1}{6}c_mz^2-\frac{1}{120}e_mz^4\right)^2+O(m^3z^5+m^{9/2}z^7)\\ &= -z^2\left(\alpha^2-\frac{\alpha}{6}(3\alpha^2+2c_m)z^2+\left(\frac{5}{24}\alpha^2c_m-\frac{\alpha}{60}e_m+\frac{1}{36}c_m^2\right)z^4\right)\\ &+O(m^3z^5+m^{9/2}z^7)\end{aligned}$$

Combining the last couple of estimates we obtain:

$$\begin{aligned}r_{12}(1-r^2)+rr_1r_2 &= z^4\left(\frac{\alpha}{4}(\alpha^2-c_m+\left(-\frac{1}{36}\alpha e_m-\frac{1}{24}\alpha^2c_m+\frac{1}{72}c_m^2\right)z^2\right)\\ &+O(m^3z^5+m^{9/2}z^7),\end{aligned}$$

and

$$\begin{aligned}&(r_{12}(1-r^2)+rr_1r_2)^2\\ &= \alpha(\alpha^2-c_m)z^8\left(\frac{\alpha}{16}(\alpha^2-c_m)+\frac{1}{144}\left(-2\alpha e_m-3\alpha^2c_m+c_m^2\right)\cdot z^2\right)\\ &+O(m^6z^9+m^{15/2}z^{11}).\end{aligned}$$

Finally using the latter estimate with (A.6) we obtain (the term corresponding to z^8 cancels out precisely, and by the non-negativity the Taylor series necessarily starts from an even power)

$$\begin{aligned}&\left(\alpha(1-r^2)-r_1^2\right)\left(\alpha(1-r^2)-r_2^2\right)-\left(r_{12}(1-r^2)+rr_1r_2\right)^2\\ &= \frac{\alpha}{144}(\alpha^2-c_m)(c_m^2+\alpha e_m)z^{10}+O(m^6z^9+m^{15/2}z^{11}).\end{aligned} \tag{A.7}$$

Note that by (2.6), (A.4) and (A.5) we have

$$\alpha^2-c_m = 2\pi^4m^2\left(\widehat{\tau_m}(4)\cos(4\varphi)-1\right)+O(m^{3/2}),$$

and

$$c_m^2 + \alpha e_m = 4\pi^8 m^4 (\widehat{\tau_m}(4)^2 \cos^2(4\varphi) - 1) + O(m^{7/2}),$$

so that, bearing in mind (A.3), (A.7) is

(A.8)

$$\mu^2(1-\rho^2) = \frac{2}{9}\pi^{14} m^7 (A(t_1)-1)(A(t_1)^2-1)(t_2-t_1)^{10} + O(m^6 z^9 + m^{15/2} z^{11});$$

this is almost identical to the statement of the present lemma, except that we have to improve the error term. To this end we observe that since, in light of (A.1), the expression on the l.h.s. of (A.8) is non-negative, the Taylor expansion on the r.h.s. of (A.8) is guaranteed to begin with an even power of z. Hence the first error term $O(m^6 z^9)$ is $O(m^{13/2} z^{10})$ (recall that this expansion is valid for $|t_2 - t_1| \ll \frac{1}{\sqrt{m}}$). □

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