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## The exact Schur index of $\mathcal{N} = 4$ SYM

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**ABSTRACT:** The Witten index counts the difference in the number of bosonic and fermionic states of a quantum mechanical system. The Schur index, which can be defined for theories with at least  $\mathcal{N} = 2$  supersymmetry in four dimensions is a particular refinement of the index, dependent on one parameter  $q$  serving as the fugacity for a particular set of charges which commute with the hamiltonian and some supersymmetry generators. This index has a known expression for all Lagrangian and some non-Lagrangian theories as a finite dimensional integral or a complicated infinite sum. In the case of  $\mathcal{N} = 4$  SYM with gauge group  $U(N)$  we rewrite this as the partition function of a gas of  $N$  non interacting and translationally invariant fermions on a circle. This allows us to perform the integrals and write down explicit expressions for fixed  $N$  as well as the exact all orders large  $N$  expansion.

**KEYWORDS:** Matrix Models, Supersymmetric gauge theory

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The superconformal index is a beautiful generalization of the Witten index [1] to supersymmetric field theories. For four dimensional supersymmetric theories on  $S^3 \times \mathbb{R}$  it is possible to define an index which depends on three fugacities, coupling to conserved charges which commute with the supersymmetry generator used to define the index.<sup>1</sup> Explicit expressions in terms of integrals of elliptic gamma functions exist for all Lagrangian theories [2–4].

The crucial property of an index is that it does not depend on continuous moduli of the theory. In the case of conformal  $\mathcal{N} = 2$  theories there are many such moduli, the gauge couplings. These theories have a rich structure of conjectured S-dualities; seemingly different theories which are in fact the same under the appropriate mapping of couplings. Since the index should be the same for dual theories, it is possible to find expressions for the index of strongly interacting theories without a known Lagrangian (for a review, see [5]). It was also shown that the expressions for the index satisfy the axioms of a topological field theory in 2d living on an auxiliary Riemann surface [6].

A particularly simple version of the index of  $\mathcal{N} = 2$  theories which depends only on a single fugacity is known as the Schur index [7, 8]. The contributions of both the vector and hyper multiplets can be expressed in terms of  $q$ -theta functions, rather than elliptic gamma functions [9]. The topological field theory is then the zero area limit of  $q$ -deformed two dimensional Yang-Mills and the form of the index of the strongly interacting theories are given by explicit infinite sums [7].

The purpose of this note is to write explicit expressions in terms of elementary functions for the Schur index of the simplest interacting field theory, namely  $\mathcal{N} = 4$  SYM.<sup>2</sup>

The integral representation of the Schur index of  $\mathcal{N} = 4$  SYM with gauge group  $U(N)$  is<sup>3</sup>

$$\mathcal{I}(N) = \frac{q^{-N^2/4} \eta^{3N}(\tau)}{N! \pi^N} \int_0^\pi d^N \alpha \frac{\prod_{i < j} \vartheta_1^2(\alpha_i - \alpha_j)}{\prod_{i, j} \vartheta_4(\alpha_i - \alpha_j)} \quad (1)$$

This expression is written in different notations than normally found in the literature; we use standard Jacobi theta functions and the Dedekind eta function rather than  $q$ -theta functions and  $q$ -Pochhammer symbols. Due to that, we also choose our  $q$  to be the square root of that used more often. We also rescaled  $\alpha$ . Some simple definitions and relations of theta functions are given in appendix A. We use the shorthand  $\vartheta_i(z) = \vartheta_i(z, q)$  and  $q \equiv e^{i\pi\tau}$ .

The product in (1) can be rewritten using an elliptic determinant identity. For arbitrary  $\alpha_i$  and  $\alpha'_i$ , one has [11–14]

$$\frac{\prod_{i < j} \vartheta_1(\alpha_i - \alpha_j) \vartheta_1(\alpha'_i - \alpha'_j)}{\prod_{i, j} \vartheta_4(\alpha_i - \alpha'_j)} = \frac{q^{-N^2/4} e^{-iN \sum_{i=1}^N (\alpha_i - \alpha'_i)}}{\vartheta_3(\sum_{i=1}^N (\alpha_i - \alpha'_i) + \pi\tau \frac{N}{2}) \vartheta_3^{N-1}} \det_{i, j} \left( \frac{\vartheta_2(\alpha_i - \alpha'_j)}{\vartheta_4(\alpha_i - \alpha'_j)} \right). \quad (2)$$

<sup>1</sup>In addition one can define fugacities for flavor symmetries.

<sup>2</sup>Other limits of fugacities of the index of  $\mathcal{N} = 4$  SYM were studied in [10].

<sup>3</sup>For  $SU(N)$  gauge group the prefactor should be rescaled by  $q^{1/4} \eta^2(\tau/2) \eta^{-4}(\tau)$ . In addition a delta function should be introduced so that sum  $\alpha_i$  vanishes modulo  $\pi$ . The final answer is the same as for  $U(N)$ , up to the aforementioned prefactor. We thank Arash Arabi Ardehali for pointing out an omission in this factor.

Here we used the notation  $\vartheta_i = \vartheta_i(0, q)$ . Setting  $\alpha'_i = \alpha_i$  simplifies the prefactor and we can also use the quasi-periodicity of  $\vartheta_3$  to write

$$\vartheta_3(\pi\tau N/2) = q^{-N^2/4} \vartheta_3 \Delta_N, \quad \Delta_N = \begin{cases} 1, & N \text{ even,} \\ \frac{\vartheta_2}{\vartheta_3}, & N \text{ odd.} \end{cases} \quad (3)$$

Now we note that the ratio of Jacobi theta functions appearing in the determinant is in fact closely related to the elliptic function  $\text{cn}$  with modulus  $k = \vartheta_2^2/\vartheta_3^2$

$$\frac{\vartheta_2(z)}{\vartheta_4(z)} = \frac{\vartheta_2}{\vartheta_4} \text{cn}(z\vartheta_3^2). \quad (4)$$

Finally using the relation  $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3(\tau)$  we can write the index (1) as<sup>4</sup>

$$\mathcal{I}(N) = \frac{q^{-N^2/4}}{\Delta_N} Z(N), \quad Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int_0^\pi d^N \alpha \prod_{i=1}^N \frac{\vartheta_2^2}{2\pi} \text{cn}((\alpha_i - \alpha_{\sigma(i)})\vartheta_3^2), \quad (5)$$

Equation (5) has the form of the partition function of  $N$  free fermions on a circle (see e.g. [15]). The Fermi gas partition function is completely determined by the spectral traces

$$\begin{aligned} Z_\ell &= \text{Tr}(\rho^\ell) = \int_0^\pi d\alpha_1 \dots d\alpha_\ell \rho(\alpha_1, \alpha_2) \dots \rho(\alpha_\ell, \alpha_1), \\ \rho(\alpha, \alpha') &= \frac{\vartheta_2^2}{2\pi} \text{cn}((\alpha - \alpha')\vartheta_3^2) = \frac{1}{\pi} \sum_{p \in \mathbb{Z}} \frac{e^{i(2p-1)(\alpha-\alpha')}}{q^{p-\frac{1}{2}} + q^{-(p-\frac{1}{2})}}, \end{aligned} \quad (6)$$

where we used the Fourier expansion of the  $\text{cn}$  function.

Performing the integrals over  $\alpha$  identifies the Fourier coefficients in the expansion of the different  $\rho$ 's giving an exceedingly simple result

$$Z_\ell = \sum_{p \in \mathbb{Z}} \left( \frac{1}{q^{p-\frac{1}{2}} + q^{-(p-\frac{1}{2})}} \right)^\ell. \quad (7)$$

As we show in appendix B, these infinite sums can be expressed as polynomials of complete elliptic integrals  $K$  and  $E$ . One can then use the combinatorics of the conjugacy classes of the symmetric group  $S_N$  with  $m_\ell$  cycles of length  $\ell$  to write  $Z(N)$  (5) as

$$Z(N) = \sum'_{\{m_\ell\}} \prod_{\ell} \frac{Z_\ell^{m_\ell} (-1)^{(\ell-1)m_\ell}}{m_\ell! \ell^{m_\ell}}, \quad (8)$$

where the prime denotes a sum over sets that satisfy  $\sum_{\ell} \ell m_\ell = N$ . Plugging (B.5) into (8) and including the normalization in (5), we find for  $N = 1, 2, 3, 4$

$$\begin{aligned} \mathcal{I}(1) &= \frac{q^{-1/4}}{\pi} \sqrt{k} K, & \mathcal{I}(2) &= \frac{q^{-1}}{2\pi^2} K(K - E), \\ \mathcal{I}(3) &= \frac{q^{-9/4}}{24\pi^3} \sqrt{k} K(12K(K - E) - 4(1 + k^2)K^2 + \pi^2), \\ \mathcal{I}(4) &= \frac{q^{-4}}{24\pi^4} K(3K(K - E)^2 - 2k^2K^3 + \pi^2(K - E)). \end{aligned} \quad (9)$$

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<sup>4</sup>The factor of  $\vartheta_2^2/2\pi$  is included with the  $\text{cn}$  functions to simplify their Fourier expansion below.

It is easy to generate explicit expressions for larger  $N$ . The powers of  $q$  in these expressions as well as  $\sqrt{k}$  for odd  $N$  come from the normalization prefactor in (5). We see that by removing this power of  $q$ , the resulting expressions are polynomials of complete elliptic integrals and the elliptic modulus and have nice modular properties. We comment on that further below.

An alternative approach is to consider the grand canonical partition function with fugacity  $\kappa$

$$\Xi(\kappa) = 1 + \sum_{N=1}^{\infty} Z(N)\kappa^N. \tag{10}$$

This is a Fredholm determinant of a very simple form

$$\Xi(\kappa) = \exp\left(-\sum_{\ell=1}^{\infty} \frac{(-\kappa)^\ell}{\ell} Z_\ell\right) = \prod_{p \in \mathbb{Z}} \left(1 + \frac{\kappa}{q^{p-\frac{1}{2}} + q^{-(p-\frac{1}{2})}}\right) \tag{11}$$

This product turns out to be expressible in terms of theta functions (see appendix A)

$$\begin{aligned} \Xi(\kappa) &= \prod_{p=1}^{\infty} \left(\frac{1 + q^{2p-1} + \kappa q^{p-\frac{1}{2}}}{1 + q^{2p-1}}\right)^2 = \frac{(\prod_{p=1}^{\infty} (1 - q^p)(1 + q^{2p-1} + \kappa q^{p-\frac{1}{2}}))^2}{\prod_{p=1}^{\infty} (1 - q^p)^2 (1 + q^{2p-1})^2} \\ &= \frac{1}{\vartheta_3 \vartheta_4} \vartheta_3^2\left(\frac{1}{2} \arccos \frac{\kappa}{2}, q^{1/2}\right) = \frac{1}{\vartheta_4} \left(\vartheta_3\left(\arccos \frac{\kappa}{2}\right) + \frac{\vartheta_2}{\vartheta_3} \vartheta_2\left(\arccos \frac{\kappa}{2}\right)\right). \end{aligned} \tag{12}$$

Recall that the index has the extra factor of  $1/\Delta_N$  (3) compared with the free fermion partition function (5). This factor distinguishes even and odd  $N$ , so it is natural to split  $\Xi$  into its even and odd parts. In fact, since  $\vartheta_3(z)$  is even and  $\vartheta_2(z)$  is odd under  $z \rightarrow \pi - z$ , the two terms in the last line of (12) give the decomposition into

$$\Xi_{\pm} \equiv \frac{1}{2}(\Xi(\kappa) \pm \Xi(-\kappa)) \tag{13}$$

It is then possible to define the *grand index*  $\hat{\Xi}$ , where the  $\Xi_-$  part is divided by the value of  $\Delta_N$  for odd  $N$

$$\hat{\Xi}(\kappa) \equiv 1 + \sum_{N=1}^{\infty} \mathcal{I}(N) q^{N^2/4} \kappa^N = \frac{1}{\theta_4} \left[\vartheta_3\left(\arccos \frac{\kappa}{2}\right) + \vartheta_2\left(\arccos \frac{\kappa}{2}\right)\right]. \tag{14}$$

By expanding the last expression in powers of  $\kappa$ , it is very easy to obtain explicit formulae for  $\mathcal{I}(N)$  for finite  $N$ . Using relations between derivatives of theta functions and elliptic integrals, one recovers (9).

We can also obtain a closed formula for the coefficients of the large  $N$  expansion of  $\mathcal{I}(N)$  by expanding (14) at large  $\kappa = e^\mu$  and using the integral transform

$$\mathcal{I}(N) = \frac{q^{-N^2/4}}{2\pi i} \int_{-i\pi}^{i\pi} d\mu \hat{\Xi}(e^\mu) e^{-\mu N}, \tag{15}$$

Given the large  $\kappa$  expansion of  $\arccos$

$$\arccos \frac{e^\mu}{2} = i \operatorname{arccosh} \frac{e^\mu}{2} = i\mu + i \log \frac{1 + \sqrt{1 - 4e^{-2\mu}}}{2}, \tag{16}$$

we find

$$e^{-\mu N} \hat{\Xi}(e^\mu) = \frac{e^{-\mu N}}{\vartheta_4} : e^{\log \frac{1+\sqrt{1-4e^{-2\mu}}}{2}} \partial_\mu : (\vartheta_3(i\mu) + \vartheta_2(i\mu)), \quad (17)$$

where the exponent is normal ordered. Expanding in powers of  $e^{-2\mu}$  gives

$$e^{-\mu N} : e^{\log \frac{1+\sqrt{1-4e^{-2\mu}}}{2}} \partial_\mu : = e^{-\mu N} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} e^{-(N+2n)\mu} \partial_\mu \prod_{k=1}^{n-1} (\partial_\mu - n - k). \quad (18)$$

The index (15) can now be easily evaluate through integration by parts. For any integer  $l \geq 0$ , we have

$$\frac{1}{2\pi i} \int_{-i\pi}^{i\pi} d\mu e^{-\mu(N+2n)} \partial_\mu^l \vartheta_3(i\mu) = \frac{(N+2n)^l}{2\pi i} \int_{-i\pi}^{i\pi} d\mu e^{-\mu(N+2n)} \vartheta_3(i\mu) \quad (19)$$

and similarly for  $\vartheta_2(i\mu)$ . Using the Fourier expansion

$$\frac{1}{2\pi i} \int_{-i\pi}^{i\pi} d\mu e^{-\mu(N+2n)} (\vartheta_3(i\mu) + \vartheta_2(i\mu)) = q^{\frac{(N+2n)^2}{4}}, \quad (20)$$

we finally obtain

$$\mathcal{I}(N) = \frac{1}{\vartheta_4} \sum_{n=0}^{\infty} (-1)^n \left[ \binom{N+n}{N} + \binom{N+n-1}{N} \right] q^{nN+n^2}. \quad (21)$$

This sum is convergent for  $|q| < 1$  and it is easy to check that for  $N = 1, 2, 3, 4$  it is exactly the  $q$ -expansion of the finite  $N$  expressions in (9).

Equations (21), (14), (12) and (9) (with the algorithm in appendix B) are the main results of this paper.

The large  $N$  expansion (21) should have an interpretation in string theory, but it is a long enduring puzzle to get a match. The leading contribution to the Schur index with  $n = 0$  is just  $1/\vartheta_4$ , which is  $N$  independent, unlike the supergravity action, which should scale with  $N^2$ . This was already observed for the large  $N$  limit of the most general index of  $\mathcal{N} = 4$  SYM in [3]. Indeed, the correct quantity to match should be the partition function on  $S^3 \times S^1$ , which is related to our definition of the index by the Casimir factor, which does scale as  $N^2$  [16, 17]. It still does not match the gravity calculation, possibly due to missing counterterms of supersymmetric holographic renormalization.

Still one could hope that the terms with  $n > 1$  correspond to some instantons in string theory. Identifying  $\log 1/q$  with the radius of the compact  $S^1$  (relative to the radius of  $S^3$ ), it seems to match the action of  $n$  D3-branes plus  $n^2$  excitations between the branes. Natural candidates would be  $1/8$  BPS giant gravitons [18–20], which should play a similar role to the string and membrane instantons in the case of ABJM [21, 22].

The grand canonical partition function  $\Xi$  (12) and the grand index  $\hat{\Xi}$  (14) bear a very strong resemblance to the grand canonical partition function of  $\mathcal{N} = 8$  superconformal Chern-Simons-matter theories, which was expressed in [23] also in terms of a Jacobi theta function. In the 3-dimensional case, the classical limit of the grand potential for a wide class of theories contains a cubic term in  $\mu$  which leads to the universal Airy function

behavior for the partition function [24]. The analog in 4 dimensions is a quadratic term in  $\mu$  (related to the structure of the Fermi surface on the circle). The analog of the Airy function is then a  $q^{kN^2}$  scaling of the partition function of the fermions, here with  $k = 1/4$ . But this behavior is not evident in the final answer, due to the overall prefactor in the definition (1), (5). If the  $\mu^2$  scaling is universal, it would be natural to try to reproduce it from a quantum supergravity calculation, along the lines of [25], though an understanding of classical supergravity is required first.

Comparing to the Fermi-gas formulation of 3d theories [24], the expressions we find are even simpler, since the density  $\rho$  (6) is a function of difference form, which means that the Hamiltonian  $H = -\log \rho$  is a function only of the momentum. So the free fermions have a complicated kinetic term, but no potential and no interactions. Indeed the exact spectrum is pairwise degenerate with

$$E_p = \log \left( q^{p-\frac{1}{2}} + q^{-p+\frac{1}{2}} \right) = - \left( p - \frac{1}{2} \right) \log q - \sum_{l=1}^{\infty} \frac{(-1)^l}{l} q^{l(2p-1)}, \quad p \geq 1. \quad (22)$$

Identifying  $\hbar = -i\pi\tau$ , this is the spectrum of the harmonic oscillator with exponentially small corrections at large  $\hbar$ . In the 3d case there is a connection between the grand canonical partition functions and the spectrum of Schrödinger equations on the line related to topological string theory [26, 27]. One can easily enrich our model such that it would calculate non-trivial spectra on the circle. A natural modification is to include extra fundamental hypermultiplets. While the resulting matrix model looks interesting, the 4d field theory is not asymptotically free, so it is not obvious at all what this would be capturing.

It was advocated in [9] that the index would have nice modular properties once rescaled by  $q^c$ , where  $c$  is the central charge. This rescaling is exactly that mentioned after (9) and is also the one appearing in our definition of the grand index  $\hat{\Xi}$  (14). Indeed for finite  $N$  we find elliptic integrals and at fixed  $\mu$  we get theta functions of nome  $q$ , all of which have nice modular properties. It was also proposed in [9] to modify the index by changing some charge assignments, which amounts in our case to introducing a mass  $m = \pi/2$  for the adjoint hypermultiplet. It is easy to repeat our calculation for this particular mass and the result for the grand index is equal to our  $\Xi_+(ie^\mu, -q)$  (13). The modified index therefore vanishes for all odd  $N$ , and is essentially the same as the usual index for even  $N$ .

It would also be interesting to reproduce our expression from the topological field theory formulation as a sum over  $U(N)$  representations [6, 8]. Given the explicit relation in this case to  $q$ -deformed 2d Yang-Mills [7], it should also be possible to reproduce our results from an analysis similar to [28]. The Schur index is also conjectured to be equal to the torus partition function of certain 2d chiral algebras [29]. For  $\mathcal{N} = 4$  SYM the conjectured chiral theory has super  $\mathcal{W}$ -algebra symmetry and our expressions should give the full partition function of this theory. In addition there is a relation between the index and integrable models [30], which should also yield the same result.

One can also apply such ideas to other theories: does the grand index take a simple form for all class- $\mathcal{S}$  theories? The most obvious generalization is necklace quivers, where a lot of the structure persists. It is also possible to include fugacities for the flavor charges, which in the case of  $\mathcal{N} = 4$  is analogous to considering  $\mathcal{N} = 2^*$ . Other generalizations are

the inclusion of line operators, for which the matrix models were computed in the Schur limit in [31]. It would also be interesting to see whether it is possible to generalize our analysis beyond the Schur limit and/or to the lens space index [32]. We hope to report on some of these issues in the near future.

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## A Theta functions

In the literature on the superconformal index, the expressions are mostly written in terms of  $q$ -theta functions and  $q$ -Pochhammer symbols. Those are related to the Jacobi theta functions and Dedekind eta function by

$$\theta(e^{2iz}, q^2) = \frac{-ie^{iz} \vartheta_1(z, q)}{q^{1/6} \eta(\tau)}, \quad (q^2; q^2)_\infty = \prod_{k=1}^{\infty} (1 - q^{2k}) = q^{-1/12} \eta(\tau). \quad (\text{A.1})$$

where the (quasi) period  $\tau$  is related to the nome  $q$  by  $q = e^{i\pi\tau}$ .

The Jacobi theta function  $\vartheta_3(z, q)$  is given by the series and product representations

$$\vartheta_3(z, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz} = \prod_{k=1}^{\infty} (1 - q^{2k})(1 + 2q^{2k-1} \cos(2z) + q^{4k-2}), \quad (\text{A.2})$$

in terms of which the auxiliary theta functions are

$$\begin{aligned} \vartheta_1(z, q) &= iq^{1/4} e^{-iz} \vartheta_3\left(z - \frac{1}{2}\pi\tau - \frac{1}{2}\pi, q\right) \\ \vartheta_2(z, q) &= q^{1/4} e^{-iz} \vartheta_3\left(z - \frac{1}{2}\pi\tau, q\right) \\ \vartheta_4(z, q) &= \vartheta_3\left(z - \frac{1}{2}\pi, q\right). \end{aligned} \quad (\text{A.3})$$

We also need Watson’s identity (eq. (20.7v) in [33])

$$\vartheta_3(z, q^{1/2}) \vartheta_3(w, q^{1/2}) = \vartheta_3(z + w, q) \vartheta_3(z - w, q) + \vartheta_2(z + w, q) \vartheta_2(z - w, q) \quad (\text{A.4})$$

## B Finite $N$ expressions

The quantities  $Z_\ell$  (7) can be evaluate by the procedure given in [34] which we briefly summarise (correcting some small typos). For even  $\ell = 2s$  one has

$$Z_{2s} = 2 \frac{(-1)^{s+1}}{(2s-1)!} \sum_{m=1}^{s-1} \alpha_m(s) D_{2s-2m-1}, \quad (\text{B.1})$$

where  $\alpha_m(s)$  and  $D_s$  are generated by

$$\sum_{m=0}^{s-1} \alpha_m(s) t^m = \prod_{j=1}^{s-1} (1 - j^2 t), \quad \sum_{s=0}^{\infty} D_{2s+1} \frac{(-4t^2)^s}{(2s)!} = \frac{EK}{2\pi^2} - \frac{K^2}{2\pi^2} (1 - k^2) \text{nd}^2 \left( \frac{2Kt}{\pi} \right), \quad (\text{B.2})$$

where  $K \equiv K(k^2)$  and  $E \equiv E(k^2)$  are complete elliptic integrals of the first and second kind respectively, nd is an elliptic function (likewise cd below) and the elliptic modulus is  $k = \vartheta_2^2 / \vartheta_3^2$ .

Similarly for odd  $\ell = 2s + 1$

$$Z_{2s+1} = \frac{(-1)^s}{2^{2s-1} (2s)!} \sum_{m=0}^s \tilde{\alpha}_m(s) J_{2s-2m}, \quad (\text{B.3})$$

where

$$\sum_{m=0}^s \tilde{\alpha}_m(s) t^m = \prod_{j=1}^s (1 - (2j-1)^2 t), \quad \sum_{s=0}^{\infty} J_{2s} \frac{(-t^2)^s}{(2s)!} = \frac{Kk}{2\pi} \text{cd} \left( \frac{2Kt}{\pi} \right). \quad (\text{B.4})$$

We find for  $N = 1, 2, 3, 4$

$$\begin{aligned} Z_1 &= \frac{kK}{\pi}, & Z_2 &= \frac{KE - (1 - k^2)K^2}{\pi^2}, \\ Z_3 &= \frac{Z_1}{8} - \frac{(1 - k^2)kK^3}{2\pi^3}, & Z_4 &= \frac{Z_2}{6} - \frac{(1 - k^2)k^2K^4}{3\pi^4}. \end{aligned} \quad (\text{B.5})$$

Plugging these expressions into (8) and including the normalization in (5) gives (9). This algorithm can be easily pushed to larger values of  $N$  and is easy to implement in Mathematica.

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