

ANALYTIC NUMBER THEORY AND THE MEAN VALUES OF L -FUNCTIONS

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This thesis focuses on moments of the Riemann zeta-function and Dirichlet L -functions. Finding asymptotic formulae for moments of L -functions is an important pursuit, as they can be used to reveal information about the distribution of the functions' zeros and bounds on their order of magnitude, as well as having numerous other applications. Moreover, the moments are interesting in themselves, exhibiting a remarkable structure and often fascinating behaviour.

The first three parts of this thesis concern Dirichlet L -functions, while the final chapter examines a particular moment of the Riemann zeta-function. In Chapter 2, we obtain an asymptotic formula for the twisted second moment of Dirichlet L -functions that breaks the half-barrier with an error term that is uniform in both the q -aspect and the t -aspect. For general coefficients, we demonstrate that it is possible to take the length of the twist up to height $(qT)^{\frac{1}{2}+\frac{1}{66}}$. We then continue on to prove results with longer twists, under certain assumptions. In the next chapter, we demonstrate two applications of the twisted second moment from the first chapter, namely proving a new zero-density result that is strong near the critical line, and a new result on the proportion of critical zeros of Dirichlet L -functions. In Chapter 4 we obtain an asymptotic for the twisted fourth moment of Dirichlet L -functions, again simultaneously in the q and t -aspect. The final chapter investigates the twisted fourth moment of the Riemann zeta-function in vertical arithmetic progressions up the critical line. Li and Radziwiłł have shown that the second moment of $\zeta(s)$ does not behave uniformly over all vertical arithmetic progressions [84]. We derive an asymptotic formula for the fourth moment over a certain set of arithmetic progressions that still exhibits this non-uniform behaviour.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Notation

- $\mathbb{N}$ is the set of natural numbers, $1, 2, 3, \dots$ and $\mathbb{Z}$ is the set of integers. $\mathbb{Q}$ is the set of rational numbers. $\mathbb{R}$ is the set of real numbers. $\mathbb{C}$ is the set of complex numbers.
- Given a complex number z , the real part of z is represented by $\Re(z)$ and the imaginary part by $\Im(z)$. It is common to write $s = \sigma + it$ where $\sigma = \Re(s)$ and $t = \Im(s)$.
- We use standard asymptotic notation. For a positive function $h(x)$, we write $f(x) = O(h(x))$, or equivalently $f(x) \ll h(x)$ or $h(x) \gg f(x)$, to mean that there exists a positive constant K such that $\forall x, |f(x)| \leq Kh(x)$. If this constant depends on some parameter, ϵ say, then that is shown in the subscript $f(x) = O_\epsilon(h(x))$ or $f(x) \ll_\epsilon h(x)$ or $h(x) \gg_\epsilon f(x)$. We write $f(x) \asymp h(x)$ to mean $f(x) \ll h(x) \ll f(x)$, $f(x) \sim g(x)$ to represent $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$ while $f(x) = o(h(x))$ represents $\lim_{x \rightarrow \infty} \frac{f(x)}{h(x)} = 0$.
- $\tau_k(n)$ represents the number of ways of writing an integer n as a product of k factors.

$$\tau_k(n) = \sum_{d_1 d_2 \dots d_k = n} 1.$$

Often we will write $\tau_2(n)$ as $\tau(n)$. This satisfies a useful bound called the divisor bound: $\forall \epsilon > 0, \tau_k(n) \ll_{k, \epsilon} n^\epsilon$.

- Unless stated otherwise, p always represents a prime, and $\prod_p$ denotes the product over all primes.

- The Möbius function $\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$ is a multiplicative function defined by

$$\mu(p^k) = \begin{cases} 1 & \text{if } k = 0, \\ -1 & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

- We define the von Mangoldt function as

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- $e(x) := e^{2\pi i x}$ and $e_q(x) := e^{\frac{2\pi i x}{q}}$.

- For a real number x , $\{x\}$ shall represent its fractional part.

- For $\Re(s) > 1$, we write

$$\zeta_q(s) = \prod_{p \nmid q} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

- Unless stated otherwise or clear in the context, γ shall represent the Euler–Mascheroni constant.

- The Barnes G -function is defined via its Weierstrass product formula as

$$G(1+z) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z + z^2(1+\gamma)}{2}\right) \prod_{k=1}^{\infty} \left(\left(1 + \frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} - z\right) \right),$$

where γ is the Euler–Mascheroni constant.

- We use ϵ to represent an arbitrarily small positive constant that may change line by line.

Chapter 1

Introduction

1.1 The Riemann zeta-function

In this chapter we will discuss some of the basic theory of the Riemann zeta-function and Dirichlet L -functions, as well as their history and mathematical significance. This area is rich with fascinating detail, so much so that if everything were included, then this thesis would have to be spread across several books. Luckily for the interested reader, these books have already been written (see for example [11, 27, 72, 91, 92, 115]), and cover a great deal of surrounding theory as well.

The **Riemann zeta-function** is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.1}$$

for a complex variable s such that $\Re(s) > 1$. In one of those peculiar twists of mathematical history, the Riemann zeta-function was not entirely invented by Riemann, and was in fact known to Euler over a hundred years before Riemann became involved with it. The first record of $\zeta(s)$ (albeit in a very early form) is in a letter from Euler to Bernoulli in 1736 [40] and it was Euler who first proved the **Euler product formula** in 1748, that for $\Re(s) > 1$

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1.2}$$

which follows from the Fundamental Theorem of Arithmetic. This simple formula shows clearly the inherent link between $\zeta(s)$ and the primes. Euler used this formula to show that the series $\sum_p \frac{1}{p}$ diverges and hence there is an infinite number of primes [35].

If the zeta-function was born with Euler then it grew up with Riemann. Euler only considered the function over the reals while Riemann extended its definition to the complex numbers and revealed its importance in his seminal paper [106], sadly his only paper on number theory. This paper has influenced thousands of researchers and continues to do so even up to the present day.

As the series in (1.1) is absolutely convergent for $\Re(s) > 1$, and uniformly convergent for $\Re(s) > 1 + \delta$ for any fixed $\delta > 0$, $\zeta(s)$ is holomorphic for $\Re(s) > 1$. Riemann showed that the definition of $\zeta(s)$ can be meromorphically continued to the entire complex plane (its only pole is a simple pole at $s = 1$) and satisfies the **functional equation**¹

$$\begin{aligned}\xi(s) &:= \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \\ &= \xi(1-s).\end{aligned}\tag{1.3}$$

He proved this by showing that when $\Re(s) > 1$,

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty x^{\frac{s}{2}-1} \psi(x) dx,\tag{1.4}$$

where

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}.$$

The Jacobi theta-function $\theta(x) = \sum_{n \in \mathbb{Z}} e^{-n^2 \pi x}$ is related to $\psi(x)$ as

$$\psi(x) = \frac{\theta(x) - 1}{2}.$$

It also satisfies its own useful functional equation, as the Poisson summation formula reveals that for $x > 0$

$$\theta\left(\frac{1}{x}\right) = x^{\frac{1}{2}} \theta(x).\tag{1.5}$$

Hence, by (1.4) and (1.5),

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{1-s}{2}-1} + x^{\frac{s}{2}-1}) \psi(x) dx.\tag{1.6}$$

The integral in (1.6) converges for all values of s , showing that $\zeta(s)$ is analytic everywhere except for at $s = 1$. The functional equation follows immediately. Another useful consequence from (1.6) is that for s in a neighbourhood of 1,

$$\zeta(s) = \frac{1}{s-1} + O(1).\tag{1.7}$$

¹A much more basic form of the functional equation was conjectured by Euler, although he could not prove it [36].

Another way of seeing this is by applying partial summation to the series representation of $\zeta(s)$ (1.1) to find

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + s \int_1^\infty \frac{\{x\} + \frac{1}{2}}{x^{s+1}} dx. \quad (1.8)$$

As $\{x\}$ is bounded by 1, the integral converges for $\sigma > 0$.

The importance of the functional equation (1.3) cannot be overstated. This seemingly simple formula exhibiting a symmetry condition on $\zeta(s)$ reveals a great deal about the function. By defining

$$\chi(s) := 2^{s-1} \pi^s \frac{\sec\left(\frac{s\pi}{2}\right)}{\Gamma(s)},$$

(1.3) can be re-written as

$$\zeta(s) = \chi(s) \zeta(1-s)$$

to more clearly show the symmetry of $\zeta(s)$. A useful consequence of the functional equation is the **approximate functional equation** of the Riemann zeta-function, which can take many slightly varying forms but are all roughly of the shape of Theorem 4.13 of [115], that for a positive constant h , $0 < \sigma < 1$, $2\pi xy = t$, $x > h > 0$, $y > h > 0$,

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + O\left(x^{-\sigma} \log |t| + |t|^{\frac{1}{2}-\sigma} y^{\sigma-1}\right). \quad (1.9)$$

This is extremely useful in applications, allowing us to approximate $\zeta(s)$ by two Dirichlet polynomials which are individually easier to manipulate.

As well as $\zeta(s)$, it is useful to study $\frac{1}{\zeta(s)}$. By considering the properties of the Möbius function μ , specifically Möbius inversion, it is easy to see that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $\Re(s) > 1$. This relationship is useful when we wish to dampen the extreme values of $\zeta(s)$, called ‘mollifying’. To do this we multiply $\zeta(s)$ by the truncated Dirichlet series of its inverse to find that

$$\zeta(s) \sum_{n=1}^N \frac{\mu(n)}{n^s} = 1 + \sum_{\substack{ab > N \\ b \leq N}} \frac{\mu(b)}{(ab)^s} = 1 + O_\epsilon(N^{1-\sigma+\epsilon}) \quad (1.10)$$

for $\Re(s) > 1$. The mollifier has a smoothing effect on $\zeta(s)$ that still exists when $\Re(s) \leq 1$, and becomes extremely useful in a number of techniques that we shall use. Strictly speaking (1.10) is a very simple form of mollifying and most mollifiers that we use in the following chapters will be more complex, but the essence of the idea remains the same.

1.2 Zeros of the Riemann zeta-function

Of particular interest are the zeros of $\zeta(s)$ as they are linked deeply with the primes in a way that we shall demonstrate later. The Euler product (1.2) shows that $\zeta(s)$ has no zeros when $\Re(s) > 1$, and the convergence of the right hand side of (1.6) implies that $\zeta(s)$ has zeros precisely at the poles of $\Gamma\left(\frac{s}{2}\right)$, which can be found at the negative even integers. These zeros are called the **trivial zeros**. Note that this implies that $\xi(s)$ in (1.3) is an integral function. By the symmetry of $\xi(s)$ about the point $s = \frac{1}{2}$ apparent in (1.3), there can be no more zeros except in the strip $0 \leq \Re(s) \leq 1$, called the **critical strip**. By the same symmetry and as $\overline{\xi(s)} = \xi(\bar{s})$, if ρ is a (non-trivial) zero, then so is $1 - \rho$, $\bar{\rho}$, and $1 - \bar{\rho}$. Riemann sketched a proof that no zeros lie on the line $\Re(s) = 1$, which was confirmed by Hadamard [44] and de la Vallée Poussin [28] independently in 1896, who both gave proofs of the famous **Prime Number Theorem** (PNT)². The PNT states that if $\pi(x)$ is defined to be the number of primes less than or equal to x , then

$$\pi(x) \sim \text{li}(x) := \int_0^x \frac{1}{\log t} dt. \quad (1.11)$$

It can be shown without too much difficulty that $\text{li}(x) \sim \frac{x}{\log x}$, and hence the PNT is often stated in the equivalent form that

$$\pi(x) \sim \frac{x}{\log x}.$$

There are a number of ways of proving the PNT but one of the most well known methods is to consider the von Mangoldt function and then let

$$\psi(x) := \sum_{n \leq x} \Lambda(n).$$

The PNT is equivalent to the statement that $\psi(x) \sim x$ as

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p \leq x} \sum_{1 \leq k \leq \frac{\log x}{\log p}} \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x,$$

and for any $\epsilon > 0$

$$\psi(x) \geq \sum_{x^{1-\epsilon} \leq p \leq x} \log p \geq \sum_{x^{1-\epsilon} \leq p \leq x} (1 - \epsilon) \log x = (1 - \epsilon)(\pi(x) + O(x^{1-\epsilon})) \log x.$$

²The PNT had a rich history even before it was proved, starting as an assertion by Euler before being more clearly stated by a teenage Gauss after many arduous calculations. See [11] for more of the history.

Therefore

$$\psi(x) \sim \pi(x) \log x$$

and the equivalence follows. The reason we consider $\psi(x)$ is because by applying Perron's formula to $\psi(x)$,

$$\psi(x) = - \int_{(2)} \frac{\zeta'}{\zeta}(z) x^z \frac{dz}{z}$$

for any non-integer x . By shifting the contour of integration and using Cauchy's residue formula, we arrive at the **explicit formula**:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2} \log(1 - x^{-2}) \quad (1.12)$$

for any non-integer $x > 1$, where the sum runs over all non-trivial zeros ρ of $\zeta(s)$. As the series does not converge absolutely, the sum should be taken in increasing order of imaginary part. This allows us to connect the distribution of the zeros with the distribution of the primes. From here, it is enough to show that no zeros have real part equal to 1 as then $\psi(x) \sim x$ (brushing over many details) and the PNT follows.

Riemann stated in his paper that there are infinitely many zeros of $\zeta(s)$ in the critical strip and conjectured that they all lie on the **critical line** $\Re(s) = \frac{1}{2}$, which has come to be known as the **Riemann Hypothesis** (RH). There are many powerful consequences of RH (see [12] for lists of equivalences and repercussions). To give one example, if the hypothesis is true, then from (1.12) we can see that (glossing over issues of convergence) RH would give a much more precise version of the PNT: that $|\pi(x) - \text{li}(x)| \ll_{\epsilon} x^{\frac{1}{2}+\epsilon}$. However, despite many attempts throughout history, RH is still open and has since been included on Hilbert's list of problems as well as becoming one of the Millennium Problems. In fact, even showing that $\zeta(\rho) = 0 \Rightarrow \Re(\rho) < \delta$ for any $\frac{1}{2} < \delta < 1$ would imply the significant fact that $|\pi(x) - \text{li}(x)| \ll_{\epsilon} x^{\delta+\epsilon}$, but this has also avoided proof. Due to this important link between zeta zeros and primes, a great deal of work has been done on finding zero-free regions in the critical strip. The classical result in this direction is that there are no zeros with

$$\sigma \geq 1 - \frac{c}{\log |t|}$$

for some absolute constant c and $|t| \geq 3$. See Chapter 13 of [27] for details. In modern times, there is now a vast array of results. For example, Korobov [80] and Vinogradov

[117] showed that there are no zeros of $\zeta(\sigma + it)$ for $t \geq 3$ and

$$\sigma \geq 1 - \frac{1}{R(\log |t|)^{\frac{2}{3}}(\log \log |t|)^{\frac{1}{3}}} \quad (1.13)$$

for some positive constant R . This is asymptotically the largest (inexplicit) region, and much effort has been expended to make this region as explicitly large as possible by reducing the size of the constant R . The smallest constant to date was found recently by Mosinghoff, Trudgian, and Yang [93], who showed that one may take $R = 55.241$ and that this may be reduced to 48.1588 for sufficiently large t .

There is strong evidence to suggest that RH is true. For example, Platt and Trudgian [99] have shown that the first 12,363,153,437,138 non-trivial zeros of $\zeta(s)$ all lie on the critical line. This is an impressively large number of zeros to check, but compared to infinity it is still just a drop in a never-ending ocean. While most mathematicians expect that RH is true, there are some exceptions, such as Littlewood and Turán who firmly believed that there was no reason for RH to hold [71]. Counter arguments to RH can be made by demonstrating similar functions to $\zeta(s)$ that we know have badly behaved zeros, such as the existence of the zeta-function of the class of principal ideals in $\mathbb{Q}(\sqrt{-5})$. This function acts similarly to $\zeta(s)$, satisfies a similar functional equation, and even has a positive proportion of zeros on the critical line, but still has an infinite number of non-trivial zeros with real part greater than 1 [11]. For further discussion on the plausibility of RH, see [71].

Another statement of Riemann's concerns the number of zeros of $\zeta(s)$ in the critical strip. Let $N(T)$ denote the number (with multiplicity) of non-trivial zeros of $\zeta(\sigma + it)$ with imaginary part between 0 and T

$$N(T) := \{\rho \in \mathbb{C} : \zeta(\rho) = 0, 0 < \Im(\rho) \leq T\},$$

then

$$N(T) = \frac{T}{2\pi} \log \left(\frac{T}{2\pi} \right) - \frac{T}{2\pi} + O(\log T). \quad (1.14)$$

This was first rigorously proved by von Mangoldt in 1895 but with the larger error term of $O((\log T)^2)$ [118]. To prove (1.14), we note that by the argument principle,

$$N(T) = \frac{\Delta_L \arg \xi(s)}{\pi}$$

where L is the contour from 2 to $2 + iT$ and then to $\frac{1}{2} + iT$. We do not need to specify the value of $\arg \xi(2)$, as we are calculating the change in argument across the contour.

The properties of the gamma function allow us to write

$$\xi(s) = (s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)\zeta(s).$$

To find the change in argument of the gamma function, we use Stirling's asymptotic formula:

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log(s) - s + \frac{1}{2} \log 2\pi + O(|s|^{-1}) \quad (1.15)$$

as $|s| \rightarrow \infty$ with $\arg s \in (-\pi + \delta, \pi - \delta)$ for any fixed $\delta > 0$. Alternatively, in the same region,

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(|s|^{-1}). \quad (1.16)$$

The results in (1.15) and (1.16) are both standard. See for example Chapter 10 of [27]. This with the change in argument of $(s-1)\pi^{-\frac{s}{2}}$ gives

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1}),$$

where

$$S(T) = \frac{\Delta_L \arg \zeta(s)}{\pi}.$$

Hence, (1.14) follows by showing that $S(T) \ll \log T$. This is a result of the following, Lemma 9.4 of [115].

Lemma 1.1 (Titchmarsh). *Let $0 \leq \alpha < \beta < 2$. Let $f(s)$ be an analytic function real for real s , regular for $\sigma \geq \alpha$ except possibly at $s = 1$, with $|\Re(f(2+it))| \geq m > 0$ and $|f(\sigma' + it)| \leq M$ for $\sigma \geq \alpha, 1 \leq t \leq T$. If T is not an ordinate of a zero of $f(s)$, then*

$$|\arg(f(\sigma + iT))| \ll_{\alpha, \beta, m} 1 + \log M$$

for $\sigma \geq \beta$.

As $\zeta(\sigma + it) \ll |t|$ by (1.8) then the bound $S(T) \ll \log T$ follows immediately.

While proving that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line has so far been impossible, progress has been made on showing that at least not too many zeros lie off the line. Bohr and Landau [10] were the first to show a result of this type. Specifically they showed that there exists a positive decreasing function $f(t)$, tending to zero as t tends to infinity such that

$$\left| \beta - \frac{1}{2} \right| < f(\gamma)$$

for almost all zeros $\beta + i\gamma$ of $\zeta(s)$ in the upper half plane. Littlewood [86] improved this to

$$f(t) = \phi(t) \frac{\log \log t}{\log t}$$

for $\phi(t)$ any unbounded positive increasing function. Selberg [109] was then able to improve this by removing the $\log \log t$ in the numerator.

To study the density of zeros away from the critical line, we define

$$N(\sigma, T) := \{\rho \in \mathbb{C} : \zeta(\rho) = 0, \sigma \leq \Re(\rho), 0 < \Im(\rho) \leq T\}. \quad (1.17)$$

Of course, RH implies that $N(\sigma, T) = 0$ for all $\sigma > \frac{1}{2}$, but without RH one can still expect that for $\sigma \in [\frac{1}{2}, 1]$ and for $T \geq 3$ we have

$$N(\sigma, T) \ll T^{2(1-\sigma)} \log T \quad (1.18)$$

by the convexity principle concerning the order of $\zeta(s)$. This is called the **Density Hypothesis** (DH), and is a pleasing conjecture for the following reason: in various applications to the distribution of primes, DH is as powerful as RH but appears easier to prove. As an example, RH and the explicit formula (1.12) imply that

$$\psi(x+y) - \psi(x) = y + O_\epsilon(x^{\frac{1}{2}+\epsilon}) \quad (1.19)$$

and hence there is a prime in any interval $[x, x+y]$ for $y \gg_\epsilon x^{\frac{1}{2}+\epsilon}$. For more detail on this argument, see Section 10.5 of [72]. However, without RH we may write

$$\psi(x) = x - \sum_{|\gamma| \leq x^{\frac{1}{2}-\epsilon}} \frac{x^\rho}{\rho} + O_\epsilon(x^{\frac{1}{2}+\epsilon})$$

and hence for $x^{\frac{1}{2}+\epsilon} \ll y < x$,

$$\psi(x+y) - \psi(x) - y = - \sum_{|\gamma| \leq x^{\frac{1}{2}-\epsilon}} \frac{(x+y)^\rho - x^\rho}{\rho} + O_\epsilon(x^{\frac{1}{2}+\epsilon}).$$

Setting $T = x^{\frac{1}{2}-\epsilon}$, the sum is bounded by

$$y \sum_{|\gamma| \leq x^{\frac{1}{2}-\epsilon}} x^{\beta-1} \ll y \int_{\frac{1}{2}}^1 x^{\alpha-1} dN(\alpha, T) \ll y x^{-\frac{1}{2}} N\left(\frac{1}{2}, T\right) + y \log(x) \int_{\frac{1}{2}}^1 x^{\alpha-1} N(\alpha, T) d\alpha$$

which, using (1.14) to bound $N(\frac{1}{2}, T)$, DH to bound $N(\alpha, T)$, and a zero-free region estimate such as (1.13) to shorten the length of the integral, gives a sufficient bound that in turn implies (1.19).

Over the last hundred years, many bounds for $N(\sigma, T)$ have been proved³, and one of the most longstanding was by Ingham [69] in 1940 who showed that

$$N(\sigma, T) \ll T^{\frac{3(1-\sigma)}{(2-\sigma)}} (\log T)^5. \quad (1.20)$$

In a region very close to the critical line, this was improved by Selberg in [110], and then Jutila [77] to

$$N(\sigma, T) \ll_{\epsilon} T^{1-(1-\epsilon)(\sigma-\frac{1}{2})} \log T.$$

This is an improvement on (1.20) for $\sigma \in \left[\frac{1}{2}, \frac{1}{2} + (12 - \epsilon) \frac{\log \log T}{\log T}\right]$, as the power of T has increased but the power of the logarithm has decreased. The improvements introduced by Selberg and Jutila were based on multiplying $\zeta(s)$ by a mollifier, and then constructing a mollifier with more sophisticated coefficients than what had been previously used. Conrey has stated that he has improved Jutila's result to

$$N(\sigma, T) \ll_{\epsilon} T^{1-(\frac{8}{7}-\epsilon)(\sigma-\frac{1}{2})} \log T$$

by increasing the length of the mollifier [19]. The methods used to prove these results rely on the following theorem of Littlewood's [86] relating the zeros of a function to its logarithmic integral:

Lemma 1.2 (Littlewood). *Suppose that $\phi(s)$ is a function that is meromorphic in the domain $\mathcal{D} = \{\Re(s) \in [\alpha, \beta], \Im(s) \in [0, T]\}$ as well as being regular and non-zero on the line $\Re(s) = \beta$. Let $F(s) = \log \phi(s)$, supposing that such a function is regular in the neighbourhood of the line $\Re(s) = \beta$, to be defined from continuous variation along the path starting at $\beta + i\Im(s)$ to s parallel to the real axis. If this path crosses a zero or a pole of $\phi(s)$ then let $F(s) = \lim_{\epsilon \rightarrow +0} F(s + i\epsilon)$. Let $\nu(\sigma, T)$ denote the number of zeros of $\phi(s)$ minus the number of poles in the rectangle $\{s : \Re(s) \in (\sigma, \beta], \Im(s) \in (0, T]\}$. Then*

$$\int F(s) ds = -2\pi i \int_{\alpha}^{\beta} \nu(x, T) dx$$

with the first integral being taken around the boundary of $\mathcal{D}$ in the positive direction.

This lemma is essentially proved by applying Cauchy's residue theorem to the first integral around the boundary of $\mathcal{D}$.

³See Chapter 12 of [91] for an in-depth look at some of the earlier history.

Ingham's result (1.20) is still (apart from the logarithms) the best result in the range $\sigma \in (\frac{1}{2}, \frac{3}{4}]$, however more progress has been made closer to the line $\Re(s) = 1$. In fact, various mathematicians have been able to prove DH in expanding intervals on the right side of the critical strip. This began 1969 when Montgomery [90] improved (1.20) in the range $\sigma \in (\frac{4}{5}, 1)$, and proved DH for $\sigma \geq \frac{9}{10}$. Since then, the DH has been proved for in the following intervals:

	DH proved for $\sigma \geq$
Montgomery (1969), [90]	0.9
Huxley (1972), [63]	0.8333...
Ramachandra (1975), [103]	0.8076...
Forti and Viol (1973), [39]	0.8059...
Huxley (1973), [64]	0.8217...
Huxley (1975), [66]	0.8059...
Huxley (1975), [65]	0.8
Jutila (1976), [75]	0.7963...
Jutila (1977), [76]	0.7857...
Bourgain (2000), [13]	0.7812...

Table 1.1: Progress towards DH

Progress towards RH from another direction has been made throughout the centuries by showing that more and more zeta-zeros lie on the critical line, starting with Hardy [46] in 1914 who showed that there are an infinite number of critical zeros. Then, if we define

$$N_0(T) = \# \left\{ \gamma : \zeta \left(\frac{1}{2} + i\gamma \right) = 0, \ 0 < \gamma \leq T \right\} \quad (1.21)$$

counting the number of critical zeros (with multiplicity) up to height T , Hardy and Littlewood showed that $N_0(T) \gg T$ [48]. Selberg improved this by working with $M(s)\zeta(s)$ rather than $\zeta(s)$, where $M(s)$ is the square of the partial sum of the Dirichlet series of $\zeta(s)^{-\frac{1}{2}}$, showing in [109] that $N_0(T) \gg T \log T$ and hence a positive proportion of zeros are on the critical line. The main concept of their methods was to transform a mollified $\zeta(s)$ into a function that is real on the critical line and that is zero when $\zeta(s)$ is zero, then count the number of sign changes of the function moving up the critical line. It is an elegant idea, but only gives a very small positive proportion⁴. The proportion was improved to $1/3$ by Levinson [82] with a now widely used technique known as

⁴A value was given by S. Min in their thesis [89], showing the proportion was greater than $1/14,074,731 \approx 7 \times 10^{-8}$.

‘Levinson’s method’. The method in its original form has a reputation for being long and difficult (with its debut demonstration being over 50 pages) but the end result is much stronger numerically than Selberg’s method. However, Selberg’s has a greater flexibility that can be applied to other L -functions, including $GL(2)$ L -functions [11]. Levinson’s original method has since been vastly simplified by Young [123].

Many results have been proved using variations on Levinson’s method. Set

$$N_0^*(T) = \# \left\{ \gamma : \zeta \left(\frac{1}{2} + i\gamma \right) = 0, \zeta' \left(\frac{1}{2} + i\gamma \right) \neq 0, 0 < \gamma \leq T \right\}, \quad (1.22)$$

counting the simple critical zeros up to height T , and define

$$\lambda = \liminf_{T \rightarrow \infty} \frac{N_0(T)}{N(T)}, \quad \lambda^* = \liminf_{T \rightarrow \infty} \frac{N_0^*(T)}{N(T)}.$$

To name some more recent results, Conrey showed $\lambda \geq 0.4088$ and $\lambda^* \geq 0.401$ [20], which was improved to $\lambda \geq 0.4105$ and $\lambda^* \geq 0.4058$ by Bui, Conrey and Young [16], $\lambda \geq 0.410725$ and $\lambda^* \geq 0.405824$ by Robles, Roy, Zaharescu [107], $\lambda \geq 0.41729$ and most recently $\lambda^* \geq 0.40751$ by Pratt, Robles, Zaharescu, and Zeindler [100]. See also [4] for a numerically weaker result but one that uses a new method, and also [38].

An even stronger conjecture than RH is the **Linear Independence conjecture** (LI) for the Riemann zeta-function which states that the positive ordinates of non-trivial zeros of $\zeta(s)$ are linearly independent over $\mathbb{Q}$. The set of ordinates should be thought of as a multi-set i.e. if $\beta + i\gamma$ is a multiple zero of $\zeta(s)$, then γ should be a member of the set multiple times. However, if γ is in the set multiple times then the set is not linearly independent and therefore LI implies that all non-trivial zeros are simple. RH follows from LI as if $\beta + i\rho$ and $1 - \beta + i\rho$ are different zeros of $\zeta(s)$ then ρ appears twice in the multi-set of zeros. It seems that Wintner was the first person to mention LI (or at least briefly allude to an early form of it) in [121] in 1935. Since then, the conjecture developed into the form we know of now, and there are many other similar conjectures for other types of L -functions. For example, it is conjectured that the non-negative imaginary parts of the non-trivial zeros of Dirichlet L -functions corresponding to primitive characters are linearly independent over $\mathbb{Q}$.

Showing that the zeros are linearly independent would have a great many implications including that all non-trivial zeros must be simple, that there are no arithmetic progressions in the ordinates of zeros with three or more terms, and also the Riemann

Hypothesis. LI also has connections with the oscillations of the sum $\sum_{n \leq x} \mu(n)$ [70] and with prime number races [108]. The fact that RH is a consequence of LI (and it seems that RH should not necessarily imply LI), and proving RH is seen as being completely out of reach of modern mathematics, shows how impossibly unachievable proving LI could be. However, some mathematicians are unfazed by this and believe an even stronger statement: that the zeros of $\zeta(s)$ do not satisfy any algebraic relations at all and instead act like random transcendental numbers [87]. For a visual representation of this, see Steuding and Wegert's work on the first moment of $\zeta(s)$ in vertical arithmetic progressions, motivated by their beautiful phase portraits demonstrating “stochastic periodicity” [114].

Rather than trying to prove LI, many number theorists have worked on proving its consequences, such as that there are no three-term arithmetic progressions in the ordinates of the non-trivial zeros of $\zeta(s)$. Even this is extremely challenging. Putnam [101] proved that the set of zeros of $\zeta(s)$ does not contain any sequence of the form $\{\frac{1}{2} + ikb : k \in \mathbb{N}\}$ for non-zero, real b , and hence (by replacing b with a multiple of itself) each such sequence contains infinitely many non-zero points. Lapidus and van Frankenhuysen [81] managed to show that for a primitive character of modulus $q > 1$ and real $\alpha \neq 0$,

$$\# \left\{ 1 \leq k \leq T : L \left(\frac{1}{2} + i\alpha k, \chi \right) \neq 0 \right\} \gg_{\alpha, q} T^{\frac{5}{6}}$$

which was then improved to

$$\# \left\{ 1 \leq k \leq T : L \left(\frac{1}{2} + i(\alpha k + \beta), \chi \right) \neq 0 \right\} \gg_{\alpha, \beta, q} \frac{T}{\log T}$$

by Martin and Ng [87] for real α, β , which gives a better bound and extends the result to non-homogeneous progressions. They state that with minor modifications their method would give the same result for $\zeta(s)$ as well. Li and Radziwiłł considered the second moment of $\zeta(s)$ on vertical arithmetic progressions to improve Martin and Ng's result to a positive proportion. Precisely, at least $\frac{1}{3}$ of points in an arithmetic progression are non-zero [84].

1.3 Dirichlet L -functions

Dirichlet characters were first introduced by Dirichlet in 1837 [32] as a tool for studying primes in arithmetic progressions. For a positive integer q , a function $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ is a **Dirichlet character of modulus q** if for all $a, b \in \mathbb{Z}$:

- $\chi(ab) = \chi(a)\chi(b)$; i.e. χ is completely multiplicative.
- $\chi(a) = 0 \Leftrightarrow (a, q) > 1$,
- $\chi(a + q) = \chi(a)$; i.e. χ is periodic with period q .

These characters can be thought of as generalisations of the number 1, which is itself the unique Dirichlet character of modulus 1. From the three defining features of a Dirichlet character, one can derive that $|\chi(n)| \in \{0, 1\}$ for all integers n , and that there are precisely $\phi(q) = q \prod_{p|q} (1 - p^{-1})$ characters of modulus q . As $1 = \chi(1) = \chi(-1)^2$, the characters divide into two groups: **even** characters χ where $\chi(-1) = 1$, or **odd** characters χ where $\chi(-1) = -1$. The simplest character of modulus q

$$\chi_0(n) := \begin{cases} 1 & \text{if } (n, q) = 1, \\ 0 & \text{if } (n, q) > 1, \end{cases}$$

is called the principal character. An important sub-family of characters are the **primitive** characters, which are the non-principal characters where their modulus is the same as their smallest period. There are

$$\begin{aligned} \phi^*(q) &:= q \prod_{p||q} \left(1 - \frac{2}{p}\right) \prod_{p^2|q} \left(1 - \frac{1}{p}\right)^2 \\ &= \sum_{w|q} \mu\left(\frac{q}{w}\right) \phi(w) \end{aligned} \tag{1.23}$$

primitive characters of modulus q . For any character χ of modulus q , there exists a unique primitive character χ' of modulus q_0 for some factor q_0 of q that induces χ , meaning that $\chi(n) = \chi'(n)$ if $(n, q) = 1$, so it is natural to study just the primitive characters. It is not immediately clear that $\phi^*(q)$ is of a similar size to q , but the following lemma shall demonstrate that this is the case for any $q \not\equiv 2 \pmod{4}$.

Lemma 1.3. $\phi^*(q) = 0$ if $q \equiv 2 \pmod{4}$. If $q \not\equiv 2 \pmod{4}$ then there exists an absolute positive constant c such that

$$q \geq \phi^*(q) \gg \frac{q}{(\log \log q)^c}.$$

Proof. If $q \equiv 2 \pmod{4}$ then the fact that $\phi^*(q) = q \prod_{p|q} \left(1 - \frac{2}{p}\right) \prod_{p^2|q} \left(1 - \frac{1}{p}\right)^2$ implies that $\phi^*(q) = 0$. If $q \not\equiv 2 \pmod{4}$ then

$$\frac{q}{\phi^*(q)} = \prod_{p|q} \left(1 + \frac{2}{p} + O\left(\frac{1}{p^2}\right)\right).$$

We shall bound this by proving the stronger bound

$$\prod_{p|q} \left(1 + Kp^{-1+\frac{1}{\log \log q}}\right) \ll (\log \log q)^{c_K} \quad (1.24)$$

for any constant K and for an absolute constant c_K depending on K . To prove this, note that

$$\prod_{p|q} \left(1 + Kp^{-1+\frac{1}{\log \log q}}\right) \leq \prod_{p|q} \exp\left(Kp^{-1+\frac{1}{\log \log q}}\right) = \exp\left(K \sum_{p|q} p^{-1+\frac{1}{\log \log q}}\right).$$

Splitting the sum into the smaller and larger prime divisors,

$$\sum_{p|q} p^{-1+\frac{1}{\log \log q}} \leq \sum_{p \leq \log q} p^{-1+\frac{1}{\log \log q}} + \sum_{\substack{p|q \\ p \geq \log q}} p^{-1+\frac{1}{\log \log q}}.$$

If $p \leq \log q$ then $p^{-1+\frac{1}{\log \log q}} \ll p^{-1}$, and by Merten's estimates,

$$\sum_{p \leq \log q} \frac{1}{p} \ll \log \log \log q.$$

As there can be at most $\frac{\log q}{\log \log q}$ prime divisors of q greater than $\log q$,

$$\sum_{\substack{p|q \\ p \geq \log q}} p^{-1+\frac{1}{\log \log q}} \leq \frac{\log q}{\log \log q} (\log q)^{-1+\frac{1}{\log \log q}} \ll \frac{1}{\log \log q}.$$

Therefore, (1.24) holds and consequently

$$\frac{q}{\phi^*(q)} \leq \exp(O(\log \log \log q)) \ll (\log \log q)^c$$

for some absolute constant c . □

This lower bound can be sharpened, but for our purposes the size of the absolute constants do not matter.

For each Dirichlet character, one can define the **Dirichlet L -function** associated to χ

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for a complex variable s such that $\Re(s) > 1$. In fact, if χ is non-principal then the series is conditionally convergent for $\Re(s) > 0$ and defines a holomorphic function in this domain. If χ is principal, then the associated L -function behaves similarly to $\zeta(s)$, and has a simple pole at $s = 1$.

Dirichlet L -functions share many properties with $\zeta(s)$, such as having an Euler product

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (1.25)$$

as well as a functional equation, first given by Hurwitz in 1882 [62]. The functional equation is only valid for primitive characters, and differs depending on the character's parity. Define

$$\mathfrak{a} = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1, \end{cases}$$

and

$$\xi(s, \chi) = \left(\frac{\pi}{q}\right)^{-\frac{s+\mathfrak{a}}{2}} \Gamma\left(\frac{s+\mathfrak{a}}{2}\right) L(s, \chi)$$

then

$$\xi(1-s, \bar{\chi}) = \frac{i^{\mathfrak{a}} q^{\frac{1}{2}}}{\tau(\chi)} \xi(s, \chi) \quad (1.26)$$

where $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{m=1}^q \chi(m) e_q(m).$$

By taking its square, one can show that $|\tau(\chi)| = q^{\frac{1}{2}}$ when χ is primitive. Using the functional equation, one may derive various approximate functional equations of a similar shape to (1.9). The functional equation also demonstrates that $L(s, \chi)$ can be meromorphically continued to the entire complex plane.

A consequence of (1.25) that shall appear multiple times in the following chapters is that for the principal character χ_0 ,

$$L(s, \chi_0) = \zeta_q(s) = \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \zeta(s).$$

Hence by Lemma 1.24 $L(s, \chi_0) \ll q^{\epsilon} |\zeta(s)|$ for $\sigma > 1 - \frac{1}{\log \log q}$, and using a similar partial summation argument to (1.8) one can show that for $\sigma > 0$, $|t| \leq T$, and for any character χ of modulus q ,

$$L(s, \chi) \ll qT. \quad (1.27)$$

The zeros of Dirichlet L -functions are similar creatures to those of $\zeta(s)$, but with some interesting differences. Firstly (as is apparent from the different gamma functions in the functional equation) the trivial zeros for even primitive Dirichlet L -functions are found at $-2, -4, -6, \dots$ while they are found at $-1, -3, -5, \dots$ for the odd functions. Non-trivial zeros have symmetry in the critical line, but only in the real axis when χ is real. For this reason, the zero counting function for Dirichlet L -functions is normally defined to count the zeros with imaginary part between $-T$ and T , but to allow comparison with $\zeta(s)$ we divide by 2. In all of our results on zeros of Dirichlet L -functions, we are proving results on the zeros of $\prod_{\chi \pmod{q}}^* L(s, \chi)$, which are symmetric about the real axis, so either counting system gives the same result. Letting $N(T, \chi)$ denote half the number of zeros (with multiplicity) of $L(s, \chi)$ in the critical strip between $-T$ and T for χ of modulus q ,

$$N(T, \chi) = \frac{1}{2} \# \{ \rho \in \mathbb{C} : L(\rho, \chi) = 0, 0 \leq \Re(\rho) \leq 1, -T \leq \Im(\rho) \leq T \},$$

then

$$N(T, \chi) = \frac{T}{2\pi} \log \left(\frac{qT}{2\pi} \right) - \frac{T}{2\pi} + O(\log qT). \quad (1.28)$$

The proof of this is similar to the proof of (1.14), but for some minor differences. See Chapter 16 of [27] for details.

To find zero-free regions for Dirichlet L -functions of a fixed character, one can normally use the same results as for $\zeta(s)$. However, in the case that we allow q to vary some extra difficulties arise (see for example Chapter 14 of [27]) and a peculiar phenomenon occurs: the possibility of **exceptional zeros**. The classic zero-free region states that there is an effectively computable constant c such that there are no zeros of $\prod_{\chi \pmod{q}} L(s, \chi)$ with

$$\sigma \geq 1 - \frac{c}{\log(q(2 + |t|))} \quad (1.29)$$

except for at most one, that (supposing it exists) is real, simple, and corresponds to the non-principal real character. For a proof of this standard result, see for example Chapter 11 of [92]. We call this character an ‘exceptional character’. As for explicit bounds, it has been shown that for small⁵ q one may take c in (1.29) to be 0.178 with no exceptions [78]. For any fixed $q \geq 2$, this may be improved to show that apart from

⁵The result described is for $q \leq 400,000$. The author states that the larger q will be treated in a follow-up article.

at most one exception there are no zeros in the region with $c \leq 0.239$ and at most two zeros in the region with $c \leq 0.517$ [83].

Define for positive integers a, q ,

$$\pi(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} 1$$

to count the number of primes in the arithmetic progression of $a, a + q, a + 2q, \dots$ up to x . Using zero-free regions for $L(s, \chi)$ for characters χ of modulus q one can prove a prime number theorem for arithmetic progressions. However, the theoretical presence of an exceptional zero gives a theoretical extra main term. Specifically, suppose that $(a, q) = 1$ and there is no exceptional character of modulus q , then there exists a constant c such that

$$\pi(x; q, a) = \frac{\text{li}(x)}{\phi(q)} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right). \quad (1.30)$$

If there is an exceptional character χ_1 with exceptional zero β then

$$\pi(x; q, a) = \frac{\text{li}(x)}{\phi(q)} - \frac{\chi_1(a)\text{li}(x^\beta)}{\phi(q)} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right). \quad (1.31)$$

Clearly, if $(a, q) > 1$ then $\pi(x; q, a) \leq 1$. There are many proofs of (1.30) and (1.31). See, for example Chapter 11 of [92]. Clearly (1.30) and (1.31) only give asymptotic formulae when $q \ll (\log x)^A$ for some constant A , however it was shown that the asymptotic formula in (1.30) holds on average for larger q by Bombieri and Vinogradov [72]. Specifically

$$\sum_{q \leq Q} \max_{(a, q)=1} \left| \psi(x; q, a) - \frac{x}{\phi(q)} \right| \ll x(\log x)^{-A}$$

for any $A \geq 0$, where

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n),$$

$Q = x^{\frac{1}{2}}(\log x)^{-B}$ and the constant B depends on A . One can take Q to be even larger by introducing averaging over a as well, as shown by Barban, Davenport and Halberstam [72]. They proved that

$$\sum_{q \leq Q} \sum_{(a, q)=1} \left| \psi(x; q, a) - \frac{x}{\phi(q)} \right|^2 \ll x(\log x)^{-A},$$

for any $A \geq 0$, with $Q = x(\log x)^{-B}$ for a constant B depending on A .

Dirichlet L -functions have their own RH called the Generalised Riemann Hypothesis: that all non-trivial zeros lie on the critical line. The existence of exceptional zeros (also known as Landau–Siegel zeros) would disprove this conjecture, and the general consensus in the mathematical community seems to be that they do not exist. However, their existence would imply many remarkable results including the infinitude of twin primes [57].

Dirichlet L -functions are interesting in themselves, but are useful in proving results on primes in arithmetic progressions, such as Dirichlet’s classical result that there is an infinite number of primes in any arithmetic progression, also known as PNT for arithmetic progressions. As tools, their uses stretch further and can be applied to questions on class numbers using Dirichlet’s class number formula, which relates the number of equivalence classes of quadratic forms with discriminant q to $L(1, \chi)$ where χ is the real primitive non-principal character of modulus q .

To study the density of zeros of Dirichlet L -functions, we define

$$N(\sigma, T, \chi) = \frac{1}{2} \# \{ \rho \in \mathbb{C} : L(\rho, \chi) = 0, \sigma \leq \Re(\rho), -T \leq \gamma \leq T \}.$$

In Chapter 3, we shall focus on zero-density theorems that find bounds for

$$\sum_{\chi \pmod{q}}^* N(\sigma, T; \chi)$$

although it is also common to study zero-density for a fixed character χ (giving results similar to those for $\zeta(s)$) or to average over all primitive characters of modulus q for each $q \leq Q$. This extra averaging allows one to use the asymptotic large sieve to obtain even stronger bounds. See for example [26].

The Density Hypothesis for Dirichlet L -functions is the conjecture that for $\sigma \in [\frac{1}{2}, 1]$, any positive integer q , and for $T \geq 3$, we have

$$\sum_{\chi \pmod{q}}^* N(\sigma, T; \chi) \ll (qT)^{2(1-\sigma)} \log qT. \quad (1.32)$$

There have been many zero-density results, so we will only highlight a selection. Montgomery demonstrated (Chapter 12 [91]) that for $\sigma \in [\frac{1}{2}, \frac{4}{5}]$

$$\sum_{\chi \pmod{q}} N(\sigma, T; \chi) \ll (qT)^{\frac{3(1-\sigma)}{2-\sigma}} (\log qT)^9 \quad (1.33)$$

and for $\sigma \in [\frac{4}{5}, 1]$

$$\sum_{\chi \pmod{q}} N(\sigma, T; \chi) \ll (qT)^{\frac{2(1-\sigma)}{\sigma}} (\log qT)^{14}. \quad (1.34)$$

This does not prove DH in any region, but anticipated a flurry of activity proving zero-density results that are strong near $\sigma = 1$. The hypothesis (1.32) was proved for $\sigma \geq \frac{5}{6}$ by Balasubramanian and Ramachandra [3] and also by Jutila [74] (with different bounds), then for $\sigma \geq \frac{21}{26}$ by Jutila in [76]. Huxley and Jutila improved this to $\sigma \geq \frac{4}{5}$ in [67], before Heath-Brown pushed σ to $\geq \frac{15}{19}$ [55].

For a character χ , define

$$N_0(T, \chi) = \frac{1}{2} \# \left\{ \gamma : L\left(\frac{1}{2} + i\gamma, \chi\right) = 0, -T \leq \gamma \leq T \right\} \quad (1.35)$$

counting the number of critical zeros (with multiplicity) up to height T , and

$$N_0^*(T, \chi) = \frac{1}{2} \# \left\{ \gamma : L\left(\frac{1}{2} + i\gamma, \chi\right) = 0, L'\left(\frac{1}{2} + i\gamma, \chi\right) \neq 0, -T \leq \gamma \leq T \right\} \quad (1.36)$$

counting the simple critical zeros up to height T . Bauer [5] used Levinson's method to show that for any fixed character of modulus q , and sufficiently large T with $\log q = o(\log T)$

$$\frac{N_0(T, \chi)}{N(T, \chi)} \geq 0.365815$$

and

$$\frac{N_0^*(T, \chi)}{N(T, \chi)} \geq 0.356269.$$

This has been improved to

$$\frac{N_0(T, \chi)}{N(T, \chi)} \geq 0.4172$$

and

$$\frac{N_0^*(T, \chi)}{N(T, \chi)} \geq 0.4074$$

by Wu [122], still while $\log q = o(\log T)$. These results are similar to those proved in [100], however Wu's work holds for all Dirichlet L -functions (including the $q = 1$ case, which gives a result for $\zeta(s)$) and were published independently. Just like finding zero-free regions, finding results on proportions of critical zeros of Dirichlet L -functions becomes much harder when one allows the modulus q to vary. Note that the results stated above must take T to be extremely large compared to q to hold.

1.4 Moments of the Riemann zeta-function

As is often the case in mathematics, it is easier to deal with the average of a function rather than the function itself. There are multiple ways of considering the ‘average’ of $\zeta(s)$, but by far the most well known are the $2k$ th moments, defined as

$$I_k(T) := \frac{1}{T} \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^{2k} dt.$$

Just like in probability theory where the first moment of a function gives its expected value and the higher moments give variance, skewness, kurtosis, etc. each successive moment contains information about $\zeta(s)$, and so is of great interest to analytic number theorists.

The **Lindelöf Hypothesis** (LH), as conjectured by Lindelöf in 1908 [85], is that

$$\zeta \left(\frac{1}{2} + it \right) \ll_{\epsilon} t^{\epsilon}$$

or equivalently,

$$\forall k \in \mathbb{N}, \forall \epsilon > 0, I_k(T) \ll_{k,\epsilon} T^{\epsilon}.$$

They can be shown to be equivalent through a simple measure-theoretic argument, which we shall sketch. Clearly the first statement implies the second. Now assume that the second statement is true and for a contradiction that the first statement is not, then there exists an $\epsilon > 0$ and an infinite sequence t_j such that $|\zeta(\frac{1}{2} + it_j)| \gg t_j^{\epsilon}$. Due to the continuity of $|\zeta(\frac{1}{2} + it)|$ there is a small interval (of width $\delta > 0$, say) around each t_j such that for all t in that interval $|\zeta(\frac{1}{2} + it)| \gg t^{\epsilon}$ and so there is an infinite sequence of arbitrarily large T such that $\frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2k} dt \gg \delta T^{2k\epsilon}$. Taking k large enough gives the necessary contradiction.

Over the last century steady progress has been made in proving bounds on $|\zeta(\frac{1}{2} + it)|$, with the strongest to date being $|\zeta(\frac{1}{2} + it)| \ll_{\epsilon} |t|^{\frac{13}{84} + \epsilon}$ by Bourgain [14]. Given this steady march towards stronger bounds, it might seem like LH could be proved in the not too distant future. In fact Backlund proved [1] that LH is equivalent to the statement that for any fixed $\sigma > \frac{1}{2}$

$$N(\sigma, T+1) - N(\sigma, T) = o(\log T). \tag{1.37}$$

Note that this means that RH implies LH but is apparently not equivalent. Given that it is known by (1.14) that (1.37) is at most $O(\log T)$, this gives the impression that LH

is very close to being proved. However Hálász and Turán [45] showed that LH implies the density hypothesis for $\sigma \leq \frac{3}{4}$, and even that $N(\sigma, T) \ll T^\epsilon$ for $\sigma > \frac{3}{4}$. This would greatly improve the current state of knowledge on zeros of $\zeta(s)$, so perhaps a proof of LH is not as in reach as it might seem. Even more depressingly for those wishing to see a proof soon, to prove LH one has to bound $I_k(T)$ for all non-negative k , but the only range of k for which we have an acceptable bound is $k \in [0, 2]$, from Heap, Radziwiłł, and Soundararajan [52]. This bound is stronger than LH implies as they prove the bound $I_k(T) \ll (\log T)^{k^2}$. Conditional on RH, Soundararajan proved the upper bound $I_k(T) \ll_\epsilon (\log T)^{k^2+\epsilon}$ for all positive real k [113], which was subsequently improved by Harper to $I_k(T) \ll (\log T)^{k^2}$ [51] and this bound is conjectured to be sharp, as we shall discuss shortly.

More progress has been made in finding lower bounds. Ramachandra proved a sharp lower bound for all integers $k \geq 0$ for $I_k(T)$ (i.e. $I_k(T) \gg (\log T)^{k^2}$) in 1978 (see [104]) conditional on the Riemann Hypothesis. Since then a great deal of work has been published but to name a few key papers, Radziwiłł and Soundararajan have since removed the dependence on the Riemann Hypothesis for all real $1 < k$ in 2013 (see [102]), then in 2022 Heap and Soundararajan extended this range to $(\log T)^{-1/2} \leq k$ in [53].

Upper and lower bounds on $I_k(T)$ are good, but asymptotic formulae are better as they provide more information about $\zeta(s)$. Unfortunately, they are extremely hard to find, and so far they have only been proved for $k = 1$ by Hardy and Littlewood (1918) [47], who showed that

$$I_1(T) \sim \log T, \tag{1.38}$$

and for $k = 2$ by Ingham in 1926 in [68], who showed that

$$I_2(T) = \frac{1}{2\pi^2}(\log T)^4 + O((\log T)^3),$$

and improved (1.38) to

$$I_1(T) = \log\left(\frac{T}{2\pi}\right) + 2\gamma - 1 + O\left(T^{-\frac{1}{2}} \log T\right).$$

Heath-Brown and Huxley improved the error term of the second moment to $O\left(T^{-\frac{15}{22}+\epsilon}\right)$ in [58]. In [54], Heath-Brown found the lower order terms of the fourth moment to be

$$I_2(T) = \sum_{n=0}^4 c_n (\log T)^n + O\left(T^{-\frac{1}{8}+\epsilon}\right),$$

where $c_0, \dots, c_4$ are absolute constants but rather lengthy to write down explicitly. Conrey neatened up the main term by writing it as a residue of a certain function at $s = 0$ [21]. Motohashi then improved the error term to $O\left(T^{-\frac{1}{3}} \log^8 T\right)$ [96].

It has long been a folklore conjecture that $I_k(T) \sim c_k (\log T)^{k^2}$ for some set of constants c_k , but then Conrey and Ghosh made in [23] the more precise conjecture that

$$I_k(T) \sim \frac{a_k g_k}{\Gamma(k^2 + 1)} (\log T)^{k^2} \quad (1.39)$$

with

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m \geq 0} \left(\frac{\Gamma(m+k)}{m! \Gamma(k)}\right) p^{-m}$$

and g_k an integer when k is an integer. This agrees with the classic results of Hardy–Littlewood (with $g_1 = 1$) and Ingham (with $g_2 = 2$). Conrey and Ghosh continued to conjecture in [24] that $g_3 = 42$ and then Conrey and Gonek went on to conjecture in [25] that $g_4 = 24024$. It was then conjectured by Keating and Snaith in [79] using connections of the distribution of zeta-zeros to random matrix theory that

$$g_k = \frac{G^2(k+1) \Gamma(k^2 + 1)}{G(2k+1)}, \quad (1.40)$$

where $G(s)$ is the Barnes G -function. It is a convention that the $\Gamma(k^2 + 1)$ factor appears in the numerator of (1.40) as well as in the denominator of (1.39), so that g_k is an integer when k is an integer, and takes the form

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}.$$

Proving this would prove the Lindelöf Hypothesis.

As we have already mentioned, finding asymptotic formulae for moments can be very useful in proving results about the distribution of $\zeta(s)$. When trying to wring information out of $\zeta(s)$, it can be helpful to include an extra twist. In this case, the twist comes in the form of a suitably well behaved Dirichlet series with coefficients that may be specified. The **twisted** $2k$ th moment of the Riemann zeta-function is defined as

$$\frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} \left| M_N\left(\frac{1}{2} + it\right) \right|^2 dt,$$

where

$$M_N(s) = \sum_{n \leq N} \frac{a_n}{n^s}$$

for complex a_n .

Balasubramanian, Conrey, and Heath-Brown [2] showed that for $N = T^\theta$ with $\theta = \frac{1}{2} - \epsilon$, and $a_n \ll n^\epsilon$,

$$\begin{aligned} \frac{1}{T} \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 \left| M_N \left(\frac{1}{2} + it \right) \right|^2 dt \\ \sim \sum_{h,k \leq N} \frac{a_h \overline{a_k}(h,k)}{hk} \left(\log \left(\frac{T(h,k)^2}{2\pi hk} \right) + 2\gamma - 1 \right). \end{aligned} \quad (1.41)$$

In proving this result, the authors hit the so-called ‘half-barrier’: a natural obstacle that causes great difficulty when trying to take $\theta > \frac{1}{2}$. When finding an asymptotic formula for the twisted second moment, the standard method involves using the approximate functional equation (1.9) to re-write the integral as

$$\sum_{mn \leq T^{1+\epsilon}} \sum_{h,k \leq T^\theta} \frac{a_h \overline{a_k}}{(hkmn)^{\frac{1}{2}}} \int_0^T \left(\frac{mh}{nk} \right)^{-it} dt. \quad (1.42)$$

This can be separated into two parts: the diagonal terms with $mh = nk$, and the off-diagonal terms with $mh \neq nk$. The diagonal terms shall always contribute to the main term as $\frac{mh}{nk} = 1$ and the integral over t simplifies to T . We expect the integrand in the off-diagonal case to oscillate enough to give some form of cancellation, and this is the case when $\theta < \frac{1}{2}$. However when $\theta \geq \frac{1}{2}$ then $\frac{mh}{nk}$ can be close enough to 1 that there is not much oscillation and there is even a contribution to the main term. Finding this contribution to the main term can be very difficult, but using various ingenious methods it is possible. For example, in the case that $a_n = \mu(n)f(n)$ for $f(n)$ a specific type of smoothing function, Balasubramanian, Conrey, and Heath-Brown extended the length of the mollifier in (1.41) to $\theta < \frac{9}{17} - \epsilon$. This choice of twist is a natural choice to make, as it mollifies $\zeta(s)$ and is useful in applications. In fact, Balasubramanian, Conrey, and Heath-Brown used this result (among other things) to suggest that the proportion of critical zeros of $\zeta(s)$ is $\lambda \geq 0.38$ [2]. Conrey improved this to 0.4088 [20] by increasing the length of the mollifier to $\theta < \frac{4}{7} - \epsilon$. The other improvements to λ mentioned above come from specialising the mollifiers.

In [2], it is conjectured that the formula (1.41) is true of all $\theta < 1$, known as the $\theta = 1$ conjecture. Farmer [37] provided a heuristic argument as to why taking θ arbitrarily large continues to improve the mollifying effect, now known as the $\theta = \infty$ conjecture. Intuitively, verifying this conjecture would prove that almost all non-trivial

zeros are on the critical line. Bettin and Gonek [9] developed this line with an elegant proof that shows that in fact taking $\theta = \infty$ would actually prove RH.

Breaking the half-barrier is harder for general coefficients, as we no longer have specific knowledge about the coefficients that can be exploited. However, good estimates of sums of Kloosterman fractions can be enough to break the barrier. Duke, Friedlander, and Iwaniec's bound of bilinear forms with Kloosterman fractions in [33] is enough to take $\theta < \frac{48}{95} - \epsilon$. Bettin, Chandee, and Radziwiłł showed in [8] that the length of $M_N(s)$ in (1.41) can be as large as $\theta < \frac{17}{33} - \epsilon$ by using Bettin and Chandee's estimate for trilinear forms of Kloosterman fractions [7].

The asymptotic formula for the twisted fourth moment of the Riemann zeta-function is much more complex than the second, and was first derived by Gaggero Jara [41], up to a height of $\theta < \frac{4}{589} - \epsilon$. This was improved, first by Motohashi [97], then by Hughes and Young in [61]. Hughes and Young examined twists by individual terms of a Dirichlet polynomial, and could take the length of such a polynomial up to $\theta < \frac{1}{11} - \epsilon$. Bettin, Bui, Li, and Radziwiłł showed in [6] that by considering averaging over the terms in the Dirichlet polynomial, they could extend the asymptotic up to length $\theta < \frac{1}{4} - \epsilon$.

Approaching the fourth moment from a different direction that delves much more deeply into spectral theory, Motohashi and Bruggeman have proved results on both the twisted and un-twisted fourth moment. See [97] and [15].

To move beyond the twisted fourth moment will be challenging, but Conrey, Farmer, Keating, Rubinstein, and Snaith have provided a recipe giving a new heuristic for all of the main terms of moments of various different families of L -functions [22]. Their conjectures agree with the other conjectures previously mentioned, as well as numerical evidence.

1.5 Moments of Dirichlet L -functions

There are several different ways to consider the moments of Dirichlet L -functions, but perhaps the most common is in the q -aspect. The $2k$ th moment for $L(s, \chi)$ in the q -aspect is defined as

$$\frac{1}{\phi^*(q)} \sum_{\chi \pmod{q}}^* \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k}$$

and we are interested in the asymptotic behaviour of the sum as $q \rightarrow \infty$. In [98] Paley first proved that

$$\frac{1}{\phi^*(q)} \sum_{\chi \pmod{q}}^* \left| L\left(\frac{1}{2}, \chi\right) \right|^2 \sim \frac{\phi(q)}{q} \log q$$

as $q \rightarrow \infty$. Heath-Brown was the first to give an asymptotic formula for the fourth moment [56], proving

$$\frac{1}{\phi^*(q)} \sum_{\chi \pmod{q}}^* \left| L\left(\frac{1}{2}, \chi\right) \right|^4 = \frac{1}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q)^4 + O\left(\frac{2^{\omega(q)} q}{\phi^*(q)} (\log q)^3\right)$$

where $\omega(q)$ is the number of distinct prime factors of q . The error term here is smaller than the main term as long as q does not have too many distinct prime factors. Soundararajan proved the asymptotic formula for all large integers q [112], before Young managed to get the lower order terms (stating that they are computable, but without actually writing them) by getting a power saving in the error term, for q prime [124].

Iwaniec and Sarnak gave an asymptotic formula for the second moment with power savings in [73] which allowed them to give an asymptotic formula for the twisted second moment. For a Dirichlet polynomial

$$M_N(\chi) := \sum_{n \leq N} \frac{a_n \chi(n)}{n^{\frac{1}{2}}}$$

of length $N = q^\theta$ with $\theta \leq \frac{1}{2} - \epsilon$ with real coefficients $a_n \ll n^\epsilon$,

$$\frac{1}{\phi^*(q)} \sum_{\chi \pmod{q}}^+ \left| L\left(\frac{1}{2}, \chi\right) M_N(\chi) \right|^2 \sim \frac{\phi(q)}{q} \sum_{h,k \leq N} \frac{a_h a_k(h,k)}{hk} \log \frac{L^2(h,k)^2}{hk} \quad (1.43)$$

where the sum is over all even primitive characters and

$$\log L = \frac{1}{2} \log \left(\frac{q}{\pi} \right) + \frac{\Gamma'(\frac{1}{4})}{2\Gamma(\frac{1}{4})} + \gamma + \sum_{p|q} \frac{\log p}{p-1}.$$

Note how similar the resulting asymptotic is to the corresponding twisted moment of $\zeta(s)$. This is due to the fact that $\zeta(s)$ and Dirichlet L -functions fall into the same family of primitive L -functions, specifically the unitary family [22].

Also notice how the twist may be taken up to $\theta \leq \frac{1}{2} - \epsilon$. Again this is due to the half-barrier, although this time it is not due to an integral over t but instead to the cancellation that appears due to summing over the primitive characters. This

cancellation implies that the off-diagonal terms have a much smaller contribution when $\theta < \frac{1}{2} - \epsilon$.

The first move beyond the half-barrier in this context was by Duke, Friedlander, and Iwaniec who showed in [34] that the twisted moment in (1.43) could be bounded by $O_\epsilon(q^\epsilon)$ for some twists longer than $q^{\frac{1}{2}}$. Bui, Pratt, Robles, and Zaharescu broke the half-barrier in [18], extending the length of the asymptotic for general coefficients up to $q^{\frac{51}{101}-\epsilon}$ for a prime number q , and longer for more specialised coefficients. The reason that they could only take $\theta \leq \frac{51}{101} - \epsilon$ rather than up to Bettin, Chandee, and Radziwiłł's equivalent $\frac{17}{33} - \epsilon$ is due to a fundamental difference between the q -aspect and the t -aspect which we shall discuss more in Chapter 2.

Hough proved the existence of a computable asymptotic formula for the twisted fourth moment (again for a prime number q) with square-free coefficients up to height $\theta < \frac{1}{32} - \epsilon$ [60]. Zacharias then proved a result for the twisted fourth moment with cube-free coefficients for the smaller value $\theta \leq \frac{11}{8064} - \epsilon$ for prime moduli [125].

Turning our attention to moments in both the q -aspect and t -aspect simultaneously, Motohashi [95] has sketched the proof of the untwisted second moment for a single primitive character χ of modulus q ,

$$\int_0^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 dt = \frac{\phi(q)}{q} T \left(\log\left(\frac{qT}{2\pi}\right) + 2\gamma + 2 \sum_{p|q} \frac{\log p}{p-1} \right) + O\left(\left((qT)^{\frac{1}{3}} + q^{\frac{1}{2}}\right)(\log qT)^4\right).$$

See also Watt's work [120] for a bound for this integral but over short intervals in the t -aspect.

For the fourth moment, Montgomery [91] showed that

$$\sum_{\chi \pmod{q}}^* \int_0^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \ll \phi(q) T (\log qT)^4$$

which was improved to the asymptotic

$$\sum_{\chi \pmod{q}}^* \int_0^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \sim \frac{\phi^*(q)T}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q)^4 \quad (1.44)$$

by Rane [105], valid when $2^{\omega(q)} \leq \log q$. The lower order terms were given by Wang [119] but only for $2 \leq q \leq T$. Finally, Bui and Heath-Brown managed to extend the asymptotic (1.44) (without the lower order terms) to be valid for all $q, T \geq 2$ [17].

Harman, Watt, and Wong derived a bound for the twisted fourth moment of Dirichlet L -functions averaged over the primitive characters of modulus q and up to height T [50]. This bound is not uniform in q and T . See also [116] for a result on the fourth moment for a single character.

1.6 Thesis overview

In Chapter 2, we shall prove various asymptotic formulae for the twisted second moment of Dirichlet L -functions simultaneously in the q and the t -aspect. In Chapter 3, we shall use the theorems of Chapter 2 to prove two results on the zeros of Dirichlet L -functions. Specifically, we shall prove a zero-density theorem and a result on the proportion of critical zeros. In Chapter 4, we obtain an asymptotic formula for the twisted fourth moment of Dirichlet L -functions, again simultaneously in the q and t -aspect. Chapter 5 focuses on finding an asymptotic formula for the fourth moment of $\zeta(s)$ on a certain set of vertical arithmetic progressions.

Chapter 2

The twisted second moment of Dirichlet L -functions

The contents of this chapter are based on sections of the pre-print [31], which has been submitted for publication. For the purposes of making it suitable for the thesis format, they have been edited, re-written, and improved in the process.

In this chapter, we prove an asymptotic formula for the twisted second moment of Dirichlet L -functions, uniform in both the q and the t -aspects. First, we shall prove the result for a twist with general coefficients, then by specifying the coefficients we obtain a result for a longer twist. Finally, we show how we can take an even longer twist as t grows compared to q . We will demonstrate why the infamous half-barrier is difficult to overcome, and how to move beyond it in the context of Dirichlet L -functions.

2.1 Introduction

As mentioned previously, there is a fundamental difference between the q -aspect and the t -aspect when trying to find asymptotics of moments. The integral over t in sums such as (1.42) means that one need only consider the terms in these sums with $\left|\frac{an}{bm} - 1\right| \ll T^{\epsilon-1}$. This is called the ‘balanced’ case as $am \asymp bn$ and so are approximately the same size. In the q -aspect alone, the sums are not restricted to the balanced case and one must deal with the unbalanced case $am \not\asymp bn$. Dealing with the unbalanced case appears to be more challenging, and gives larger error terms.

When working with moments, it is often more convenient to work with what is

known as the ‘shifted’ moment, where one shifts the L -function off the critical line by a small amount. This allows the avoidance of some technical difficulties to do with poles and zeros, gives a more general result, as well as revealing more about the structure of the main term. These shifts are very helpful in some applications, as we shall see in Chapter 3. We shall present our results with shifts in Theorems 2.1, 2.2, and 2.3, then show how to remove them in Corollary 2.1.

As the primitive Dirichlet L -functions have different functional equations depending on whether their associated Dirichlet characters are odd or even, we split the sum over the characters into sums over the odds and the evens. The sum over the odd primitive characters and the sum over the even primitive characters are denoted as $\sum_{\chi \pmod{q}}^-$ and $\sum_{\chi \pmod{q}}^+$ respectively.

Theorem 2.1. *Let q be a positive integer and T a positive real number with $T \gg q^\epsilon$. Let ψ_t be a smooth real valued function supported on $t \in [T, 2T]$ with $\psi_t^{(j)} \ll T^\epsilon$. Let $\alpha, \beta \in \mathbb{C}$ satisfy $\alpha, \beta \ll \frac{\log \log qT}{\log qT}$. Suppose that $\kappa < 1/2 + 1/66$. For all $n \in \mathbb{N}$, $\alpha_n, \beta_n \in \mathbb{C}$ such that $\alpha_n, \beta_n \ll n^\epsilon$,*

$$\begin{aligned} & \frac{1}{\phi^*(q)T} \int \sum_{\chi \pmod{q}}^+ L(1/2 + \alpha + it, \chi) L(1/2 + \beta - it, \bar{\chi}) \sum_{a, b \leq (qT)^\kappa} \frac{\alpha_a \beta_b \chi(a) \bar{\chi}(b)}{a^{1/2+it} b^{1/2-it}} \psi_t dt \\ &= \frac{\hat{\psi}_0}{2T} \zeta_q(1 + \alpha + \beta) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b) = 1 \\ (abd, q) = 1}} \frac{\alpha_{ad} \beta_{bd}}{a^{1+\beta} b^{1+\alpha} d} + \frac{1}{2T} \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \zeta_q(1 - \alpha - \beta) \\ & \quad \times \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b) = 1 \\ (abd, q) = 1}} \frac{\alpha_{ad} \beta_{bd}}{a^{1-\alpha} b^{1-\beta} d} \int \frac{\Gamma\left(\frac{1/2-\alpha-it}{2}\right) \Gamma\left(\frac{1/2-\beta+it}{2}\right)}{\Gamma\left(\frac{1/2+\alpha+it}{2}\right) \Gamma\left(\frac{1/2+\beta-it}{2}\right)} \psi_t dt + O_\epsilon((qT)^{-\epsilon}). \end{aligned}$$

When summing over the odd primitive characters, the same main term arises but with $\frac{\Gamma\left(\frac{1/2-\alpha-it}{2}\right) \Gamma\left(\frac{1/2-\beta+it}{2}\right)}{\Gamma\left(\frac{1/2+\alpha+it}{2}\right) \Gamma\left(\frac{1/2+\beta-it}{2}\right)}$ replaced by $\frac{\Gamma\left(\frac{3/2-\alpha-it}{2}\right) \Gamma\left(\frac{3/2-\beta+it}{2}\right)}{\Gamma\left(\frac{3/2+\alpha+it}{2}\right) \Gamma\left(\frac{3/2+\beta-it}{2}\right)}$.

The main term is holomorphic in α and β and in the case that $\alpha = -\beta$, the above result should be considered as a limit.

To prove this theorem, we follow a similar structure to that of the proof of Bui, Pratt, Robles, and Zaharescu’s result in just the q -aspect [18], however the presence of the t -aspect causes some major differences.

A natural choice of mollifier is $M(s, \chi) = \sum_n \frac{\mu(n) f(n) \chi(n)}{n^s}$ where $f(n)$ is some

smoothing function. In this case, it is possible to exploit the properties of the Möbius function to get a smaller error term.

Theorem 2.2. *Suppose that the conditions of Theorem 2.1 hold, with the added assumption that $\alpha(n) = \mu(n)f(n)$ for some smooth bounded function $f(x)$ with $f'(x) \ll_\epsilon x^{-1+\epsilon}$. Then the conclusion of Theorem 2.1 holds for*

$$\kappa < \frac{1}{2} + \frac{5}{128}.$$

Finally, the following theorem gives a larger value of κ as T grows compared to q .

Theorem 2.3. *Suppose that the conditions of Theorem 2.2 hold. Define*

$$\gamma := \frac{\log T}{\log q}$$

then the conclusion of Theorem 2.1 holds for

$$\kappa < \frac{4 + \frac{3}{\gamma}}{7 \left(1 + \frac{1}{\gamma}\right)}.$$

Taking T arbitrarily large compared to q allows κ to grow up to a limit of $\frac{4}{7}$, recovering a result of Conrey's in [20], but for Dirichlet L -functions.

Using these theorems, we may recover the un-shifted asymptotic formula by letting the shifts tend to 0.

Corollary 2.1. *Suppose that the conditions of one of Theorems 2.1, 2.2, or 2.3 are satisfied. Then*

$$\begin{aligned} & \frac{1}{\phi^*(q)T} \int_T^{2T} \sum_{\chi \pmod{q}}^* L(1/2 + it, \chi) L(1/2 - it, \bar{\chi}) \sum_{a, b \leq (qT)^\kappa} \frac{\alpha_a \beta_b \chi(a) \bar{\chi}(b)}{a^{1/2+it} b^{1/2-it}} dt \\ & \sim \frac{\phi(q)}{q} \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad} \beta_{bd}}{abd} \left(\log \left(\frac{qT}{2\pi ab} \right) + 2\gamma - 1 + 2\log(2) + 2 \sum_{p|q} \frac{\log p}{p-1} \right) \end{aligned} \quad (2.1)$$

as $qT \rightarrow \infty$.

Proof. Let $I_{\alpha, \beta}(T)$ be the integral in Theorem 2.1. Then by using standard methods to remove the function ψ_t , and letting $\alpha = \beta$ be real, we see that as $qT \rightarrow \infty$, the main term in (2.1) tends to

$$\begin{aligned} \lim_{\alpha \rightarrow 0} I_{\alpha, \alpha}(T) &= \lim_{\alpha \rightarrow 0} \left(\zeta_q(1 + 2\alpha) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b)=1 \\ (abd, q)=1}} \frac{\alpha_{ad} \beta_{bd}}{(ab)^{1+\alpha} d} \right. \\ &\quad \left. + \frac{1}{2T} \left(\frac{q}{\pi} \right)^{-2\alpha} \int_T^{2T} G(\alpha, t) dt \zeta_q(1 - 2\alpha) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b)=1 \\ (abd, q)=1}} \frac{\alpha_{ad} \beta_{bd}}{(ab)^{1-\alpha} d} \right) \end{aligned} \quad (2.2)$$

where

$$G(\alpha, t) = \frac{\left| \Gamma \left(\frac{1/2 - \alpha + it}{2} \right) \right|^2}{\left| \Gamma \left(\frac{1/2 + \alpha + it}{2} \right) \right|^2} + \frac{\left| \Gamma \left(\frac{3/2 - \alpha + it}{2} \right) \right|^2}{\left| \Gamma \left(\frac{3/2 + \alpha + it}{2} \right) \right|^2}. \quad (2.3)$$

It is useful to note that $G(0, t) = 2$ and

$$\frac{\partial}{\partial \alpha} G(\alpha, t)|_{\alpha=0} = - \left(\frac{\Gamma' \left(\frac{1/2 + it}{2} \right)}{\Gamma \left(\frac{1/2 + it}{2} \right)} + \frac{\Gamma' \left(\frac{1/2 - it}{2} \right)}{\Gamma \left(\frac{1/2 - it}{2} \right)} + \frac{\Gamma' \left(\frac{3/2 + it}{2} \right)}{\Gamma \left(\frac{3/2 + it}{2} \right)} + \frac{\Gamma' \left(\frac{3/2 - it}{2} \right)}{\Gamma \left(\frac{3/2 - it}{2} \right)} \right). \quad (2.4)$$

Also

$$\frac{\Gamma' \left(\frac{1/2 \pm it}{2} \right)}{\Gamma \left(\frac{1/2 \pm it}{2} \right)} = \log \frac{1/2 \pm it}{2} + O(|t|^{-1})$$

for $|t| \gg 1$ (see for example Chapter 10 of [27]), hence by (2.4)

$$\begin{aligned} - \int_T^{2T} \frac{\partial}{\partial \alpha} G(\alpha, t)|_{\alpha=0} dt &= \int_T^{2T} \log \left(\frac{1}{4} + t^2 \right) + \log \left(\frac{9}{4} + t^2 \right) - 4 \log 2 dt + O(\log T) \\ &= 4T \log(2T) - 4T + O(\log T). \end{aligned}$$

As $\zeta_q(s) = \zeta(s) \prod_{p|q} (1 - p^{-s})$ and $\zeta(s) = \frac{1}{s-1} + \gamma + O(|1-s|)$ for s in a neighbourhood of 1, the limit (2.2) is equal to

$$\begin{aligned} &2\gamma \frac{\phi(q)}{q} \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b)=1 \\ (abd, q)=1}} \frac{\alpha_{ad} \beta_{bd}}{abd} + \lim_{\alpha \rightarrow 0} \frac{1}{2\alpha} \prod_{p|q} (1 - p^{-1-2\alpha}) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b)=1 \\ (abd, q)=1}} \frac{\alpha_{ad} \beta_{bd}}{(ab)^{1+\alpha} d} \\ &- \lim_{\alpha \rightarrow 0} \frac{1}{4T\alpha} \left(\frac{q}{\pi} \right)^{-2\alpha} \int G(\alpha, t) dt \prod_{p|q} (1 - p^{-1+2\alpha}) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b)=1 \\ (abd, q)=1}} \frac{\alpha_{ad} \beta_{bd}}{(ab)^{1-\alpha} d}, \end{aligned}$$

which we may calculate using L'Hôpital's rule, as

$$\left. \frac{d}{d\alpha} \frac{1}{2} \prod_{p|q} (1 - p^{-1-2\alpha}) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b)=1 \\ (abd, q)=1}} \frac{\alpha_{ad} \beta_{bd}}{(ab)^{1+\alpha} d} \right|_{\alpha=0}$$

$$= \frac{\phi(q)}{q} \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{abd} \left(\sum_{p|q} \frac{\log p}{p-1} - \frac{\log ab}{2} \right)$$

and

$$\begin{aligned} & \left. \frac{d}{d\alpha} \frac{1}{4T} \left(\frac{q}{\pi} \right)^{-2\alpha} \int_T^{2T} G(\alpha, t) dt \prod_{p|q} (1 - p^{-1+2\alpha}) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{(ab)^{1-\alpha}d} \right|_{\alpha=0} \\ &= \frac{\phi(q)}{q} \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{abd} \left(-\log \left(\frac{2qT}{\pi} \right) + 1 - \sum_{p|q} \frac{\log p}{p-1} + \frac{\log ab}{2} + O \left(\frac{\log T}{T} \right) \right). \end{aligned}$$

□

The proofs of Theorems 2.1, 2.2, and 2.3 are identical except for the treatment of the error term, which we shall deal with in Proposition 2.1. To be specific, the stronger assumptions in the different theorems allow us to find better bounds for the expression (2.16) which makes up the main error term.

2.2 Initial manipulations

Theorems 2.1, 2.2, and 2.3 have been proved for $\zeta(s)$ (see [8, 20]), and so we may assume that $q > 1$.

For the sake of simplicity, we focus on just the sum over even characters, then address the minor differences in proof needed for the odd sum in Section 2.3.4. To prove the theorems of this chapter, we need a few useful lemmas. The first of these is the approximate functional equation, and involves a function $G(s)$ that is chosen to have a number of helpful properties. We shall say $G(s)$ is a *helpful* function if it is even, entire, satisfies $G(0) = 1$ and the condition that for $|\sigma| \leq C$ and for all $A > 0$

$$G(\sigma + it) \ll_{A,C} (1 + |t|)^{-A}.$$

Helpful functions shall be used again in approximate functional equations in the fourth and fifth chapter.

Lemma 2.1 (Approximate Functional Equation). *Let χ be an even primitive character. Then we have the approximate functional equation*

$$\begin{aligned} L\left(\frac{1}{2} + \alpha + it, \chi\right) L\left(\frac{1}{2} + \beta - it, \bar{\chi}\right) &= \sum_{m,n \geq 1} \frac{\chi(m)\bar{\chi}(n)}{m^{1/2+\alpha+it}n^{1/2+\beta-it}} V_+(mn, t) \\ &+ \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \sum_{m,n \geq 1} \frac{\chi(m)\bar{\chi}(n)}{m^{1/2-\beta+it}n^{1/2-\alpha-it}} V_-(mn, t), \end{aligned}$$

where

$$V_{\pm}(x, t) = \frac{1}{2\pi i} \int_{(\epsilon)} X_{\pm}(s, t) \left(\frac{x\pi}{q}\right)^{-s} \frac{ds}{s}$$

and

$$X_{\pm}(s, t) = G(s) \frac{\Gamma(\frac{1/2 \pm (\alpha + it) + s}{2}) \Gamma(\frac{1/2 \pm (\beta - it) + s}{2})}{\Gamma(\frac{1/2 + \alpha + it}{2}) \Gamma(\frac{1/2 + \beta - it}{2})}$$

and $G(s)$ is a helpful function that is invariant under the transformations $\alpha \rightarrow -\beta$ and $\beta \rightarrow -\alpha$, and satisfies $G(\pm(\alpha + \beta)/2) = 0$.

The proof is standard. For example, see Theorem 5.3 of [72]. One does not need to specify $G(s)$ precisely, but an admissible choice is

$$G(s) = \frac{\left(\frac{\alpha+\beta}{2}\right)^2 - s^2}{\left(\frac{\alpha+\beta}{2}\right)^2} e^{s^2}.$$

Lemma 2.2. *For all $j, k, C \geq 0$,*

$$x^j t^k \frac{\partial^{j+k}}{\partial x^j \partial t^k} V_+(x, t) \ll_{j,k,C} \left(1 + \left|\frac{x}{qt}\right|\right)^{-C} \quad (2.5)$$

$$x^j t^k \frac{\partial^{j+k}}{\partial x^j \partial t^k} V_-(x, t) \ll_{j,k,C} |t|^{-\Re(\alpha+\beta)} \left(1 + \left|\frac{x}{qt}\right|\right)^{-C}. \quad (2.6)$$

The proof is a simple application of Stirling's approximation (1.16) applied to

$$\frac{\Gamma(\frac{1/2+\alpha+it+s}{2}) \Gamma(\frac{1/2+\beta-it+s}{2})}{\Gamma(\frac{1/2+\alpha+it}{2}) \Gamma(\frac{1/2+\beta-it}{2})} = t^s (1 + O((1 + |t|)^{-1})). \quad (2.7)$$

Similarly

$$\frac{\Gamma(\frac{3/2+\alpha+it+s}{2}) \Gamma(\frac{3/2+\beta-it+s}{2})}{\Gamma(\frac{3/2+\alpha+it}{2}) \Gamma(\frac{3/2+\beta-it}{2})} = t^s (1 + O((1 + |t|)^{-1})). \quad (2.8)$$

Lemma 2.3 (Orthogonality). *Suppose that $(m, q) = 1$, then*

$$\sum_{\chi \pmod{q}}^+ \chi(m) = \frac{1}{2} \sum_{\substack{uw=q \\ m \equiv \pm 1 \pmod{w}}} \mu(u) \phi(w).$$

The proof of this result is standard. See for example, (3.1) and (3.2) of [73].

Applying Lemma 2.3 to the approximate functional equation gives

$$\begin{aligned} & \sum_{\chi \pmod{q}}^+ L(1/2 + \alpha + it, \chi) L(1/2 + \beta - it, \bar{\chi}) \sum_{a, b \leq (qT)^\kappa} \frac{\alpha_a \beta_b \chi(a) \bar{\chi}(b)}{a^{1/2+it} b^{1/2-it}} = \\ & \sum_{w|q} \mu\left(\frac{q}{w}\right) \frac{\phi(w)}{2} \sum_{\substack{a, b, m, n \\ am \equiv \pm bn \pmod{w} \\ (abmn, q)=1}} \frac{\alpha_a \beta_b}{(ab)^{1/2} m^{1/2+\alpha} n^{1/2+\beta}} \left(\frac{bn}{am}\right)^{it} V_+(mn, t) \\ & + \sum_{w|q} \mu\left(\frac{q}{w}\right) \frac{\phi(w)}{2} \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \sum_{\substack{a, b, m, n \\ am \equiv \pm bn \pmod{w} \\ (abmn, q)=1}} \frac{\alpha_a \beta_b}{(ab)^{1/2} m^{1/2-\beta} n^{1/2-\alpha}} \left(\frac{bn}{am}\right)^{it} \times V_-(mn, t). \end{aligned}$$

Hence

$$\begin{aligned} & \int \sum_{\chi \pmod{q}}^* L(1/2 + \alpha + it, \chi) L(1/2 + \beta - it, \bar{\chi}) \sum_{a, b \leq (qT)^\kappa} \frac{\alpha_a \beta_b \chi(a) \bar{\chi}(b)}{a^{1/2+it} b^{1/2-it}} \psi_t dt \\ & = D^+ + O^+ + \left(\frac{q}{\pi}\right)^{-\alpha-\beta} (D^- + O^-) \end{aligned}$$

where

$$D^+ := \frac{\phi^*(q)}{2} \int \sum_{\substack{a, b \leq (qT)^\kappa \\ am = bn \\ (abmn, q)=1}} \frac{\alpha_a \beta_b}{(ab)^{1/2} m^{1/2+\alpha} n^{1/2+\beta}} V_+(mn, t) \psi_t dt$$

and

$$O^+ := \frac{1}{2} \sum_{w|q} \mu\left(\frac{q}{w}\right) \phi(w) \int \sum_{a, b, m, n}^w \frac{\alpha_a \beta_b}{(ab)^{1/2} m^{1/2+\alpha} n^{1/2+\beta}} \left(\frac{bn}{am}\right)^{it} V_+(mn, t) \psi_t dt$$

where we define

$$\sum_{a, b, m, n}^w = \sum_{\substack{a, b, m, n \\ am \equiv \pm bn \pmod{w} \\ am \neq bn \\ (abmn, q)=1}}.$$

The expressions D^- and O^- are obtained from D^+ and O^+ by substituting $\alpha, \beta \rightarrow -\beta, -\alpha$ and replacing V_+ by V_- . As the D^+ and O^+ cases are very similar to the D^- and O^- cases, we shall only demonstrate the details of the proof for the former.

2.3 The main terms

2.3.1 The diagonals

As the diagonals are made up of sums over the condition $am = bn$, by setting $d = (a, b)$ we may relabel $a = ad, b = bd, m = bn$, and $n = an$ with $(a, b) = 1$ and $(n, q) = 1$. Hence, by relabelling, D^+ is

$$\begin{aligned} &= \frac{\phi^*(q)}{2} \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b) = 1 \\ (abd, q) = 1}} \sum_{\substack{n \geq 1 \\ (q, n) = 1}} \frac{\alpha_{ad} \beta_{bd}}{a^{1+\beta} b^{1+\alpha} d n^{1+\alpha+\beta}} \int V_+(abn^2, t) \psi_t dt \\ &= \frac{\phi^*(q)}{2} \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b) = 1 \\ (abd, q) = 1}} \sum_{\substack{n \geq 1 \\ (q, n) = 1}} \frac{\alpha_{ad} \beta_{bd}}{a^{1+\beta} b^{1+\alpha} d n^{1+\alpha+\beta}} \frac{1}{2\pi i} \int \int_{(2)} X_+(s, t) \left(\frac{q}{abn^2\pi} \right)^s \psi_t \frac{ds}{s} dt \end{aligned}$$

and by writing the sum over n as a zeta-function and moving the contour back to $\Re(s) = \epsilon$ this is

$$= \frac{\phi^*(q)}{2} \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b) = 1 \\ (abd, q) = 1}} \frac{\alpha_{ad} \beta_{bd}}{a^{1+\beta} b^{1+\alpha} d} \frac{1}{2\pi i} \int \int_{(\epsilon)} X_+(s, t) \left(\frac{q}{ab\pi} \right)^s \zeta_q(1 + \alpha + \beta + 2s) \psi_t \frac{ds}{s} dt.$$

Note that we chose the contour of integration in the V -function to be $\Re(s) = 2$ at first so that the sum over n converges to the L -function, and then moved the contour back to $\Re(s) = \epsilon$ with the pole at $s = -(\alpha + \beta)/2$ being cancelled by the zero coming from $X_+(s, t)$. Similarly

$$D^- = \frac{\phi^*(q)}{2} \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b) = 1 \\ (abd, q) = 1}} \frac{\alpha_{ad} \beta_{bd}}{a^{1-\alpha} b^{1-\beta} d} \frac{1}{2\pi i} \int \int_{(\epsilon)} X_-(s, t) \left(\frac{q}{ab\pi} \right)^s \zeta_q(1 - \alpha - \beta + 2s) \psi_t \frac{ds}{s} dt.$$

2.3.2 The off-diagonals

The remaining terms (i.e. when $am \neq bn$) are the off-diagonals. The following lemma will allow us to show that the terms in the sum with am and bn sufficiently unbalanced will contribute a negligible error.

Lemma 2.4. *Suppose that $\psi_t : [T, 2T] \rightarrow \mathbb{R}$ is a smooth function in t with derivative*

$$\frac{d^K}{dt^K} \psi_t \ll_{K, \epsilon} T^\epsilon.$$

Then,

$$\int V_{\pm}(mn, t) \left(\frac{bn}{am} \right)^{it} \psi_t dt \ll_K \left| \log \frac{bn}{am} \right|^{-K} T^{1-K+\epsilon}.$$

This implies that this integral is negligibly small unless

$$1 - T^{\epsilon-1} < \frac{bn}{am} < 1 + T^{\epsilon-1}.$$

Proof. By (2.5), for $t \in [T, 2T]$

$$\frac{\partial^K}{\partial t^K} V_{\pm}(x, t) \psi_t dt \ll_{K,C} T^{-K+\epsilon} \left(1 + \frac{|x|}{qT} \right)^{-C}$$

for all $C > 0$. This implies that

$$\int V_{\pm}(mn, t) \left(\frac{bn}{am} \right)^{it} \psi_t dt \ll_K \left| \log \frac{bn}{am} \right|^{-K} T^{1-K+\epsilon}$$

which is $\ll_K T^{-K+\epsilon}$ unless $\left| \log \frac{bn}{am} \right| \ll T^{-1}$. Taking the Taylor expansion of $\log(1+x) = x + O(x^2)$ for $|x| < 1$, we see that this implies $1 - T^{\epsilon-1} < \frac{bn}{am} < 1 + T^{\epsilon-1}$. $\square$

It is now clear why the half-barrier is a problem. If $a, b \leq T^{\frac{1}{2}-2\delta}$ for some small $\delta > 0$, and $mn \ll T^{1+\delta}$ so that $abmn \ll T^{2-\delta}$, then either $bn = am$, or $\left| \frac{bn}{am} - 1 \right| \geq \frac{1}{am} \gg T^{1-\frac{\delta}{2}}$. This would mean (by taking $\epsilon < \frac{\delta}{2}$) that the off-diagonals may be bounded easily into an error term, and the only main term comes from the diagonal terms. However, if $a, b \geq T^{\frac{1}{2}}$ then there is a contribution to the main term.

In order to treat these cases, we introduce a dyadic partition of unity to the sums over m and n : let W be a smooth non-negative function supported in $[1, 2]$ such that

$$\sum_M W\left(\frac{m}{M}\right) = 1,$$

where M runs over a sequence of real numbers with $|\{M : X^{-1} \leq M \leq X\}| \ll \log X$. By the rapid decay of $V_{\pm}$, in (2.5) and (2.6) we may assume that $MN \ll (qT)^{1+\epsilon}$. We also split up the mollifying coefficients α_n, β_n dyadically, supposing that $\alpha_n(A)$ is supported on $n \in [A, 2A]$ and $\beta_n(B)$ is supported on $n \in [B, 2B]$ i.e. $\alpha_a = \sum_A \alpha_a(A)$ and by assumption $A, B \ll (qT)^{\kappa}$.

The following proposition allows us to deal with the contribution from the off-diagonal terms. We shall prove this proposition at the end of the chapter.

Proposition 2.1. *Let $T \gg q^{\epsilon}$ for a positive integer q , and let $w|q$ such that $\frac{q}{w}$ is square-free. Let ψ_t be defined as in Theorem 2.1 and $W(x)$ be smooth real valued*

functions supported on $[1, 2]$ such that for all $j \geq 0$, $W^{(j)}(x) \ll_\epsilon (qT)^\epsilon$. Let $\alpha, \beta \in \mathbb{C}$ satisfy $\alpha, \beta \ll \frac{\log \log qT}{\log qT}$. Suppose that $\kappa < \frac{17}{33}$. Suppose that for positive constants $1 \leq A, B \ll (qT)^\kappa$, $1 \leq MN \ll (qT)^{1+\epsilon}$ with $AM \asymp BN$, we have complex sequences $\alpha_a, \beta_b \in \mathbb{C}$ with support on $[A, 2A]$ and $[B, 2B]$ respectively such that $\alpha_n, \beta_n \ll n^\epsilon$.

Define $S_w^+(A, B, M, N)$ as

$$\int \sum_{a,b,m,n}^w \frac{\alpha_a(A)\beta_b(B)}{(ab)^{1/2}m^{1/2+\alpha}n^{1/2+\beta}} \left(\frac{bn}{am}\right)^{it} W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right) V_+(mn, t) \psi_t dt$$

then $S_w^+(A, B, M, N)$ is equal to

$$\sum_{\substack{f|q \\ (f,w)=1}} \mu(f) \frac{\phi(q/f)}{q} (\mathcal{M}_{w,f}^+(A, B, M, N) + \mathcal{M}_{w,f}^-(A, B, M, N)) + O_\epsilon(w^{-1}(qT)^{1-\epsilon}),$$

where

$$\begin{aligned} \mathcal{M}_{w,f}^+(A, B, M, N) = & \int \sum_{\substack{d, |r| \geq 1 \\ (d,q)=1 \\ (r,f)=1}} \sum_{\substack{a,b \\ (a,b)=1 \\ (ab,q)=1}} \frac{\alpha_{ad}(A)\beta_{bd}(B)}{(ab)^{1/2}d} \left(\frac{b}{a}\right)^{it} \frac{1}{2\pi i} \int_{(\epsilon)} \int X_+(s, t) \left(\frac{q}{\pi}\right)^s \\ & \left(\frac{abx - wr}{b}\right)^{-(1/2+s+\beta-it)} (bx)^{-(1/2+\alpha+s+it)} W\left(\frac{bx}{M}\right) W\left(\frac{abx - wr}{bN}\right) \psi_t dx \frac{ds}{s} dt \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_{w,f}^-(A, B, M, N) = & \int \sum_{\substack{d, |r| \geq 1 \\ (d,q)=1 \\ (r,f)=1}} \sum_{\substack{a,b \\ (a,b)=1 \\ (ab,q)=1}} \frac{\alpha_{ad}(A)\beta_{bd}(B)}{(ab)^{1/2}d} \left(\frac{b}{a}\right)^{it} \frac{1}{2\pi i} \int_{(\epsilon)} \int X_+(s, t) \left(\frac{q}{\pi}\right)^s \\ & \left(\frac{wr - abx}{b}\right)^{-(1/2+s+\beta-it)} (bx)^{-(1/2+\alpha+s+it)} W\left(\frac{bx}{M}\right) W\left(\frac{wr - abx}{bN}\right) \psi_t dx \frac{ds}{s} dt. \end{aligned}$$

Furthermore, suppose $\alpha_n = \mu(n)f(n)$ for some smooth bounded function $f(x)$ with $f'(x) \ll_\epsilon x^{-1+\epsilon}$. Then the same result holds for $\kappa < \frac{69}{128}$. Additionally, if we let $\gamma := \frac{\log T}{\log q}$ then the same result holds for $\kappa < \frac{4+\frac{3}{\gamma}}{7(1+\frac{1}{\gamma})}$.

By Proposition 2.1 we see that

$$O^+ = \sum_{A,B,M,N} O^+(A, B, M, N) + O_\epsilon((qT)^{1-\epsilon})$$

where

$$O^+(M, N) := \frac{1}{2} \sum_{w|q} \mu\left(\frac{q}{w}\right) \phi(w) \sum_{\substack{f|q \\ (f,w)=1}} \mu(f) \frac{\phi(q/f)}{q} \mathcal{M}_{w,f}^+(A, B, M, N)$$

$$+ \frac{1}{2} \sum_{w|q} \mu\left(\frac{q}{w}\right) \phi(w) \sum_{\substack{f|q \\ (f,w)}} \mu(f) \frac{\phi(q/f)}{q} \mathcal{M}_{w,f}^-(A, B, M, N).$$

We focus on the $\mathcal{M}^+$ terms first. Writing W in terms of its Mellin transform,

$$W(x) = \frac{1}{2\pi i} \int_{(c)} x^{-u} \tilde{W}(u) du,$$

we see that

$$\begin{aligned} \mathcal{M}_{w,f}^+(A, B, M, N) &= \frac{1}{(2\pi i)^3} \int \int_{(\epsilon)} \int_{(c_2)} \int_{(c_1)} X_+(s, t) \tilde{W}(u) \tilde{W}(v) \left(\frac{q}{\pi}\right)^s M^u N^v \\ &\quad \sum_{\substack{|r| \geq 1 \\ (r,f)=1}} \sum_{\substack{d,a,b \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}(A) \beta_{bd}(B)}{a^{1+\beta+s+v} b^{1+\alpha+s+u} d} \int x^{-(1/2+\alpha+it+s+u)} (x - wr/ab)^{-(1/2+\beta-it+s+v)} \\ &\quad \psi_t du dv dx \frac{ds}{s} dt. \end{aligned}$$

Now, we calculate the x -integral. If $r > 0$ then the integral over x is restricted to $x > wr/ab$ and if r is negative then we have $x > 0$. For absolute convergence, if $r > 0$, we impose the condition

$$\Re(\alpha + \beta + 2s + u + v) > 0, \quad \Re(\beta + s + v) < 1/2$$

and if $r < 0$ we impose the condition

$$\Re(\alpha + \beta + 2s + u + v) > 0, \quad \Re(\alpha + s + u) < 1/2.$$

Under these assumptions, the x -integral is equal to (see for example 17.43.21 and 17.43.22 of [43])

$$\left(\frac{w|r|}{ab}\right)^{-(\alpha+\beta+2s+u+v)} \times \begin{cases} \frac{\Gamma(\alpha+\beta+2s+u+v)\Gamma(1/2-\beta+it-s-v)}{\Gamma(1/2+\alpha+it+s+u)} & \text{if } r > 0, \\ \frac{\Gamma(\alpha+\beta+2s+u+v)\Gamma(1/2-\alpha-it-s-u)}{\Gamma(1/2+\beta-it+s+v)} & \text{if } r < 0, \end{cases}$$

and hence $\mathcal{M}_{w,f}^+(A, B, M, N)$ is

$$\begin{aligned} &= \frac{1}{(2\pi i)^3} \sum_{\substack{r \neq 0 \\ (r,f)=1}} \sum_{\substack{d,a,b \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha(A)_{ad} \beta(B)_{bd}}{d} \int \int_{(\epsilon)} \int_{(c_2)} \int_{(c_1)} X_+(s, t) \left(\frac{q}{\pi}\right)^s \tilde{W}(u) \tilde{W}(v) \\ &\quad w^{-(\alpha+\beta+2s+u+v)} a^{s+u-1+\alpha} b^{s+v-1+\beta} M^u N^v H_{u,v,t}^+(s) |r|^{-(\alpha+\beta+2s+u+v)} du dv \frac{ds}{s} dt \end{aligned}$$

where

$$H_{u,v,t}^+(s) = \Gamma(\alpha + \beta + 2s + u + v) \left(\frac{\Gamma(1/2 - \beta + it - s - v)}{\Gamma(1/2 + \alpha + it + s + u)} + \frac{\Gamma(1/2 - \alpha - it - s - u)}{\Gamma(1/2 + \beta - it + s + v)} \right).$$

In the $\mathcal{M}^-$ case, due to the extra minus sign in the x -integral, we arrive at the same result but with $H_{u,v,t}^+(s)$ replaced by

$$H_{u,v,t}^-(s) := \frac{\Gamma(1/2 - \alpha - it - s - u)\Gamma(1/2 - \beta + it - s - v)}{\Gamma(1 - \alpha - \beta - 2s - u - v)}.$$

Writing $H_{u,v,t}(s) := H_{u,v,t}^+(s) + H_{u,v,t}^-(s)$ and summing over $A, B \ll (qT)^\kappa$ in the dyadic decomposition allows us to write

$$O^+ = \sum_{M,N} O^+(M, N) + O_\epsilon((qT)^{1-\epsilon}) \quad (2.9)$$

where

$$O^+(M, N) := \frac{1}{2} \sum_{w|q} \mu\left(\frac{q}{w}\right) \phi(w) \sum_{\substack{f|q \\ (f,w)}} \mu(f) \frac{\phi(q/f)}{q} \frac{1}{(2\pi i)^3} \sum_{\substack{r \geq 1 \\ (r,f)=1}} \sum_{\substack{d,a,b \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{d} \\ \int \int_{(\epsilon)} \int_{(c_2)} \int_{(c_1)} X_+(s, t) \tilde{W}(u) \tilde{W}(v) \left(\frac{q}{\pi}\right)^s w^{-(\alpha+\beta+2s+u+v)} a^{s+u-1+\alpha} b^{s+v-1+\beta} \\ M^u N^v H_{u,v,t}(s) r^{-(\alpha+\beta+2s+u+v)} du dv \frac{ds}{s} dt.$$

It is important to understand the poles and zeros of $H_{u,v,t}(s)$.

Lemma 2.5. $H_{u,v,t}(s)$ has simple poles at $s = \frac{1}{2} - \alpha - it - u$ and $s = \frac{1}{2} - \beta + it - v$, each of residue 2, and a zero at $s = \frac{1-\alpha-\beta-u-v}{2}$.

Proof. Writing (for the sake of clarity) $x = \alpha + it + s + u$ and $y = \beta - it + s + v$, $H_{u,v,t}(s)$ is

$$\frac{\Gamma(x+y)[\Gamma(\frac{1}{2}-y)\Gamma(\frac{1}{2}+y) + \Gamma(\frac{1}{2}-x)\Gamma(\frac{1}{2}+x)]}{\Gamma(\frac{1}{2}+x)\Gamma(\frac{1}{2}+y)} + \frac{\Gamma(\frac{1}{2}-x)\Gamma(\frac{1}{2}-y)}{\Gamma(1-x-y)}$$

which has poles at $x, y = \frac{1}{2}$ i.e. if $s = \frac{1}{2} - \alpha - it - u$ or $\frac{1}{2} - \beta + it - v$. It is easy to check that these have residue 2. Also note that if $x + y = 1$ (with $x \neq \frac{1}{2} \neq y$) then the second fraction vanishes (as there is a pole in the denominator from $\Gamma(1-x-y)$) and the first fraction is (using the identity $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, see for example Chapter 10 of [27])

$$\frac{1}{\Gamma(\frac{1}{2}+x)\Gamma(\frac{1}{2}+y)} \left(\frac{\pi}{\sin(\pi(\frac{1}{2}+y))} + \frac{\pi}{\sin(\pi(\frac{1}{2}+x))} \right) = 0$$

as $\sin(\pi(\frac{1}{2}+y)) = \sin(\pi + \pi(\frac{1}{2}-x)) = -\sin(\pi(\frac{1}{2}-x)) = -\sin(\pi(\frac{1}{2}+x))$. $\square$

Returning to $O^+(M, N)$, we move the contours to replace the r -sum with a zeta-function. Choose $c_1 = 0$, $c_2 = \epsilon$ and move the s -contour to the right to $\Re(s) = \frac{1}{2} - \frac{\epsilon}{3}$, crossing a simple pole of $H_{u,v,t}(s)$ at $s = \frac{1}{2} - \beta + it - v$. Write $P^+(M, N)$ as the integral along the new line and $R^+(M, N)$ as the residue. We can then move the u contour in the residue to $\Re(u) = 2\epsilon$ which hits no poles and allows us to replace the r -sum with a zeta-function to give

$$R^+(M, N) = \frac{1}{2} \sum_{w|q} \mu\left(\frac{q}{w}\right) \phi(w) \sum_{\substack{f|q \\ (f,w)=1}} \mu(f) \frac{\phi(q/f)}{q} \sum_{\substack{d,a,b \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{d} \frac{1}{(2\pi i)^2} \\ \int_{(\epsilon)} \int_{(2\epsilon)} X_+(1/2 - \beta + it - v, t) \tilde{W}(u) \tilde{W}(v) M^u N^v \\ \left(\frac{q}{\pi}\right)^{1/2 - \beta + it - v} w^{-(\alpha - \beta + 1 + 2it + u - v)} a^{-1/2 - \beta + it - v + u + \alpha} b^{-1/2 + it} \\ \prod_{p|f} (1 - p^{-(\alpha - \beta + 2it + u - v + 1)}) \zeta(\alpha - \beta + 1 + 2it + u - v) \psi_t \frac{dudv}{1/2 - \beta + it - v} dt.$$

Using the following lemma, we can simplify R^+ and P^+ .

Lemma 2.6. *For any positive integer q and any complex variable s ,*

$$\sum_{w|q} \mu\left(\frac{q}{w}\right) \phi(w) \sum_{\substack{f|q \\ (f,w)=1}} \frac{\mu(f)\phi\left(\frac{q}{f}\right)}{q} w^{-s} \prod_{p|f} (1 - p^{-s}) = \phi^*(q) \prod_{p|q} (1 - p^{s-1}) q^{-s}. \quad (2.10)$$

Proof. If $p|f$ and $p|q/f$ then $p^2|q$. But $(f, w) = 1$ so $p^2|q/w \Rightarrow \mu(q/w) = 0$. Hence we may factorise $\phi(q/f) = \phi(q)/\phi(f)$, so defining the left hand side of (2.10) as M ,

$$\begin{aligned} M &= \frac{\phi(q)}{q} \sum_{w|q} \mu\left(\frac{q}{w}\right) \phi(w) \sum_{\substack{f|q \\ (f,w)=1}} \frac{\mu(f)}{\phi(f)} w^{-s} \prod_{p|f} (1 - p^{-s}) \\ &= \frac{\phi(q)}{q} q^{-s} \sum_{w|q} \mu(w) \phi\left(\frac{q}{w}\right) w^s \sum_{\substack{f|w \\ (f,q/w)=1}} \frac{\mu(f)}{\phi(f)} \prod_{p|f} (1 - p^{-s}) \\ &= \frac{\phi(q)}{q} q^{-s} \sum_{w|q} \mu(w) \phi\left(\frac{q}{w}\right) w^s \prod_{\substack{p|w \\ (p,q/w)=1}} \left(1 - \frac{1 - p^{-s}}{p - 1}\right). \end{aligned}$$

Given that

$$\phi\left(\frac{q}{w}\right) = \frac{q}{w} \prod_{p|q/w} \left(1 - \frac{1}{p}\right) = \frac{\phi(q)}{w} \prod_{\substack{p|q \\ (p,q/w)=1}} \left(1 - \frac{1}{p}\right)^{-1}$$

we see that

$$\begin{aligned}
M &= \frac{\phi(q)^2}{q} q^{-s} \sum_{w|q} \mu(w) w^{s-1} \prod_{\substack{p|w \\ (p,q/w)=1}} \left(1 - \frac{1-p^{-s}}{p-1}\right) \left(1 - \frac{1}{p}\right)^{-1} \\
&= \frac{\phi(q)^2}{q} q^{-s} \sum_{w|q} \mu(w) w^{s-1} \prod_{\substack{p|w \\ (p,q/w)=1}} \left(\frac{p^2 - 2p + p^{1-s}}{(p-1)^2}\right) \\
&= \frac{\phi(q)^2}{q} q^{-s} \prod_{p^2|q} (1 - p^{s-1}) \prod_{p||q} \left(1 - p^{s-1} \left(\frac{p^2 - 2p + p^{1-s}}{(p-1)^2}\right)\right)
\end{aligned}$$

as if $p^2|q$ and $p|w$ then either $\mu(w) = 0$ or $p|(w, q/w)$ so either the sum is empty (equal to zero) or the product is empty (equal to 1). Then rearranging gives that

$$\begin{aligned}
M &= \frac{\phi(q)^2}{q} q^{-s} \prod_{p^2|q} (1 - p^{s-1}) \prod_{p||q} \frac{\left(1 - \frac{2}{p}\right)}{\left(1 - \frac{1}{p}\right)^2} (1 - p^{s-1}) \\
&= \phi^*(q) \prod_{p||q} (1 - p^{s-1}) q^{-s},
\end{aligned}$$

as $\phi^*(q) = q \prod_{p||q} \left(1 - \frac{2}{p}\right) \prod_{p^2|q} \left(1 - \frac{1}{p}\right)^2$.

□

By Lemma 2.6,

$$R^+(M, N) = \frac{\phi^*(q)}{2} \sum_{\substack{d,a,b \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{d} \frac{1}{(2\pi i)^2} \int \int_{(\epsilon)} \int_{(2\epsilon)} X_+(1/2 - \beta + it - v, t)$$

$$\tilde{W}(u)\tilde{W}(v)M^uN^v \left(\frac{q}{\pi}\right)^{1/2-\beta+it-v} q^{-(\alpha-\beta+1+2it+u-v)} a^{-1/2-\beta+it-v+u+\alpha} b^{-1/2+it}$$

$$\prod_{p||q} (1 - p^{\alpha-\beta+2it+u-v}) \zeta(\alpha - \beta + 1 + 2it + u - v) \psi_t \frac{dudv}{1/2 - \beta + it - v} dt.$$

With the $P^+(M, N)$ term, we replace the r -sum with a zeta-function as before, apply Lemma 2.6 and shift the s -contour back to $\Re(s) = \epsilon$. This crosses the same pole at $s = 1/2 - \beta + it - v$, while the pole from the zeta-function at $s = (1 - \alpha - \beta - u - v)/2$ is cancelled out by the zero of $H_{u,v,t}(s)$ at this point. Denote the contribution from the first pole as $R^{+'}(M, N)$ and the new integral with the r -sum replaced as $P^{+'}(M, N)$

so

$$O^+(M, N) = P^{+'}(M, N) + R^{+'}(M, N) - R^+(M, N)$$

where $P^{+'}(M, N)$ is

$$\begin{aligned} & \frac{\phi^*(q)}{2(2\pi i)^3} \sum_{\substack{d,a,b \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{d} \int \int_{(\epsilon)} \int_{(\epsilon)} \int_{(0)} X_+(s, t) \tilde{W}(u) \tilde{W}(v) \left(\frac{q}{\pi}\right)^s q^{-(\alpha+\beta+2s+u+v)} \\ & a^{s+u-1+\alpha} b^{s+v-1+\beta} M^u N^v H_{u,v,t}(s) \prod_{p|q} (1 - p^{\alpha+\beta-1+2s+u+v}) \zeta(\alpha + \beta + 2s + u + v) \\ & \psi_t du dv \frac{ds}{s} dt. \end{aligned}$$

The difference between the two residue terms is in the u -contour i.e. integrating over $\Re(u) = 0, 2\epsilon$. Therefore $R^{+'}(M, N) - R^+(M, N)$ is the residue at $u = \beta - \alpha + v - 2it$ but this is cancelled by the zero from the $\prod(1 - p^{\alpha-\beta+2it+u-v})$ factor, i.e.

$$R^{+'}(M, N) - R^+(M, N) = 0.$$

Hence $O^+(M, N) = P^{+''}(M, N)$. By Lemma 4.3 in [18] we can remove the dyadic partition to obtain

$$\begin{aligned} \sum_{M,N} O^+(M, N) &= \frac{\phi^*(q)}{4\pi i} \sum_{\substack{d,a,b \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{d} \int \int_{(\epsilon)} X_+(s, t) \left(\frac{q}{\pi}\right)^s \\ & q^{-\alpha-\beta-2s} a^{s-1+\alpha} b^{s-1+\beta} H_{0,0,t}(s) \prod_{p|q} (1 - p^{\alpha+\beta-1+2s}) \zeta(\alpha + \beta + 2s) \psi_t \frac{ds}{s} dt. \end{aligned}$$

We can now write $H_{0,0,t}(s)$ as

$$\begin{aligned} H_{0,0,t}(s) &= \frac{\Gamma(\alpha + \beta + 2s)\Gamma(1/2 - \beta + it - s)}{\Gamma(1/2 + \alpha + it + s)} + \frac{\Gamma(\alpha + \beta + 2s)\Gamma(1/2 - \alpha - it - s)}{\Gamma(1/2 + \beta - it + s)} \\ &+ \frac{\Gamma(1/2 - \alpha - it - s)\Gamma(1/2 - \beta + it - s)}{\Gamma(1 - \alpha - \beta - 2s)} \end{aligned}$$

which can be rearranged (see Lemma 8.2 of [124] for details) to become

$$\pi^{1/2} \frac{\Gamma(\frac{\alpha+\beta+2s}{2})\Gamma(\frac{1/2-\alpha-it-s}{2})\Gamma(\frac{1/2-\beta+it-s}{2})}{\Gamma(\frac{1-\alpha-\beta-2s}{2})\Gamma(\frac{1/2+\alpha+it+s}{2})\Gamma(\frac{1/2+\beta-it+s}{2})}.$$

Hence,

$$H_{0,0,t}(s)X_+(s, t) = \pi^{1/2} X_-(-s, t) \frac{\Gamma(\frac{\alpha+\beta+2s}{2})}{\Gamma(\frac{1-\alpha-\beta-2s}{2})}. \quad (2.11)$$

Applying the functional equation (1.3)

$$\pi^{-(\alpha+\beta+2s)/2} \Gamma\left(\frac{\alpha + \beta + 2s}{2}\right) \zeta(\alpha + \beta + 2s)$$

$$= \pi^{-(1-\alpha-\beta-2s)/2} \Gamma\left(\frac{1-\alpha-\beta-2s}{2}\right) \zeta(1-\alpha-\beta-2s)$$

and the change of variable $s \rightarrow -s$ gives

$$\begin{aligned} O^+ &= -\frac{\phi^*(q)}{2} \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \sum_{\substack{d,a,b \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{a^{1-\alpha}b^{1-\beta}d} \frac{1}{2\pi i} \int \int_{(-\epsilon)} X_-(s,t) \left(\frac{\pi ab}{q}\right)^{-s} \\ &\quad \prod_{p|q} (1-p^{\alpha+\beta-1-2s}) \zeta(1-\alpha-\beta+2s) \psi_t \frac{ds}{s} dt + O((qT)^{1-\epsilon}) \\ &= -\frac{\phi^*(q)}{2} \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \sum_{\substack{d,a,b \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{a^{1-\alpha}b^{1-\beta}d} \frac{1}{2\pi i} \int \int_{(-\epsilon)} X_-(s,t) \left(\frac{\pi ab}{q}\right)^{-s} \\ &\quad \zeta_q(1-\alpha-\beta+2s) \psi_t \frac{ds}{s} dt + O((qT)^{1-\epsilon}). \end{aligned}$$

The O^- case is identical by replacing X_+ with X_- and the substitution $\alpha, \beta \rightarrow -\beta, -\alpha$.

2.3.3 Combining the main terms

We have shown that

$$\begin{aligned} &\frac{1}{\phi^*(q)T} \int \sum_{\chi \pmod{q}}^+ L(1/2 + \alpha + it, \chi) L(1/2 + \beta - it, \bar{\chi}) \sum_{a,b \leq (qT)^\kappa} \frac{\alpha_a \beta_b \chi(a) \bar{\chi}(b)}{a^{1/2+it} b^{1/2-it}} \psi_t dt \\ &= \frac{1}{\phi^*(q)T} \left(D^+ + O^+ + \left(\frac{q}{\pi}\right)^{-\alpha-\beta} (D^- + O^-) \right) + O_\epsilon((qT)^{-\epsilon}) \end{aligned}$$

where for instance $\left(\frac{q}{\pi}\right)^{-\alpha-\beta} D^- + O^-$ is equal to

$$\begin{aligned} &= \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \frac{\phi^*(q)}{2} \sum_{\substack{ad,bd \leq (qT)^\kappa \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{a^{1-\alpha}b^{1-\beta}d} \frac{1}{2\pi i} \\ &= \left(\int_{(\epsilon)} - \int_{(-\epsilon)} \right) \int X_-(s,t) \left(\frac{q}{ab\pi}\right)^s \zeta_q(1-\alpha-\beta+2s) \psi_t dt \frac{ds}{s} \\ &= \frac{\phi^*(q)}{2} \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \zeta_q(1-\alpha-\beta) \sum_{\substack{ad,bd \leq (qT)^\kappa \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad}\beta_{bd}}{a^{1-\alpha}b^{1-\beta}d} \int X_-(0,t) \psi_t dt, \end{aligned}$$

where the final line comes from the residue at $s = 0$. Note that the pole at $s = \frac{\alpha+\beta}{2}$ of the L -function is cancelled by the prescribed zero at that point from the function G . A similar expression holds for the sum of the other two terms, giving the result in Theorem 2.1 for the sum over even Dirichlet characters.

2.3.4 The odd characters

The odd characters go through almost identically, but with two differences: firstly we have to redefine the functions $X_{\pm}(s, T)$ because of the different functional equation for odd Dirichlet characters. Specifically

$$X_{\pm}(s, t) = G(s) \frac{\Gamma(\frac{3/2 \pm (\alpha + it) + s}{2}) \Gamma(\frac{3/2 \pm (\beta - it) + s}{2})}{\Gamma(\frac{3/2 + \alpha + it}{2}) \Gamma(\frac{3/2 + \beta - it}{2})}.$$

The relations (2.5) and (2.6) both still hold. The second difference is that summing over the odd primitive characters gives a different sum to the even characters i.e. for $(m, q) = 1$,

$$\sum_{\chi \pmod{q}}^{-} \chi(m) = \frac{1}{2} \sum_{\substack{uw=q \\ m \equiv +1 \pmod{w}}} \mu(u) \phi(w) - \frac{1}{2} \sum_{\substack{uw=q \\ m \equiv -1 \pmod{w}}} \mu(u) \phi(w).$$

This manifests itself in our definition of the function $H_{u,v,t}(s)$ at (2.9). In this setting we must redefine $H_{u,v,t}(s) := H_{u,v,t}^{+}(s) - H_{u,v,t}^{-}(s)$. The same method still works as our new $H_{u,v,t}(s)$ has zeros in the same positions, and no poles so there are no residue terms to deal with. To show at (2.11) that

$$H_{0,0,t}(s) X_{+}(s, t) = \pi^{1/2} X_{-}(-s, t) \frac{\Gamma(\frac{\alpha + \beta + 2s}{2})}{\Gamma(\frac{1 - \alpha - \beta - 2s}{2})}$$

we appeal to Lemma 8.4 of [124] instead of Lemma 8.2. Hence

$$\begin{aligned} & \frac{1}{\phi^{*}(q)T} \int \sum_{\chi \pmod{q}}^{-} L(1/2 + \alpha + it, \chi) L(1/2 + \beta - it, \bar{\chi}) \sum_{a, b \leq (qT)^{\kappa}} \frac{\alpha_a \beta_b \chi(a) \bar{\chi}(b)}{a^{1/2+it} b^{1/2-it}} \psi_t dt \\ &= \frac{\hat{\psi}_0}{2T} \zeta_q(1 + \alpha + \beta) \sum_{\substack{ad, bd \leq (qT)^{\kappa} \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad} \beta_{bd}}{a^{1+\beta} b^{1+\alpha} d} + \frac{1}{2T} \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \zeta_q(1 - \alpha - \beta) \\ & \quad \times \sum_{\substack{ad, bd \leq (qT)^{\kappa} \\ (a,b)=1 \\ (abd,q)=1}} \frac{\alpha_{ad} \beta_{bd}}{a^{1-\alpha} b^{1-\beta} d} \int \frac{\Gamma(\frac{3/2 - \alpha - it}{2}) \Gamma(\frac{3/2 - \beta + it}{2})}{\Gamma(\frac{3/2 + \alpha + it}{2}) \Gamma(\frac{3/2 + \beta - it}{2})} \psi_t dt + O((qT)^{-\epsilon}). \end{aligned}$$

Apart from these changes, the proof of the odd case follows through identically to that of the even case.

2.4 Proof of Proposition 2.1

In the course of the proof, it will be useful to trivially bound certain sums.

Lemma 2.7 (Trivial bound). *For integers f, h , and w , and for complex coefficients $\alpha_a, \beta_b \ll (qT)^\epsilon$*

$$\sum_{\substack{amh=\pm bnf+wr \\ a,b,m,n,r \asymp A,B,\frac{M}{h},\frac{N}{f},\frac{R}{w} \\ (ab,q)=1}} \alpha_a \beta_b \ll \frac{AR}{w} \left(1 + \frac{M}{fh}\right) (qT(AM+R))^\epsilon.$$

Proof. We bound this sum trivially by summing over a and r . Then we sum over $m \equiv \overline{ahwr} \pmod{f}$, of which there are $\ll 1 + M/fh$ possible values of m . Then we bound the sums over b and n using the divisor bound. $\square$

Taking $f = h = 1$, this would be sufficient to bound the contribution from the off-diagonal terms if $\kappa < \frac{1}{2}$.

To prove Proposition 2.1, we begin by writing the $am \equiv \pm bn \pmod{w}$ condition in the sum in $S_w^+(A, B, M, N)$ as $am = \pm bn + wr$. As $am \neq bn$, r must be non-zero, and by Lemma 2.4 we may assume that $|r| \ll AMw^{-1}T^{\epsilon-1}$, so we sum over $0 < |r| \leq Rw^{-1}$ where $R := AMT^{\epsilon-1}$. For brevity, define $F(a, b, m, n)$ to be

$$\int \frac{\alpha_a(A)\beta_b(B)}{(ab)^{1/2}m^{1/2+\alpha}n^{1/2+\beta}} \left(\frac{bn}{am}\right)^{it} W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right) V_+(mn, t) \psi_t dt$$

so that

$$S_w^+(A, B, M, N) = \sum_{0 < |r| \leq \frac{R}{w}} \sum_{\substack{a,b,m,n \\ am=\pm bn+wr \\ (abmn,q)=1}} F(a, b, m, n) + O_K((qT)^{-K}).$$

We then use Möbius inversion to remove the $(mn, q) = 1$ condition.

$$\sum_{\substack{a,b,m,n \\ am=\pm bn+wr \\ (abmn,q)=1}} F(a, b, m, n) = \sum_{\substack{f|q \\ (f,wr)=1}} \mu(f) \sum_{\substack{a,b,m,n \\ am=\pm bnf+wr \\ (abm,q)=1}} F(a, b, m, nf).$$

We may assume that $(f, wr) = 1$, as if $(f, rw) > 1$ then $(am, q) > 1$ which implies that the sum is empty. Applying the same trick to m gives

$$\begin{aligned} \sum_{\substack{f|q \\ (f,wr)=1}} \mu(f) \sum_{\substack{a,b,m,n \\ am=\pm bnf+wr \\ (abm,q)=1}} F(a, b, m, nf) \\ = \sum_{\substack{f|q \\ (f,wr)=1}} \sum_{\substack{h|q \\ (h,f)=1}} \mu(fh) \sum_{\substack{a,b,m,n \\ amh=\pm bnf+wr \\ (ab,q)=1}} F(a, b, mh, nf). \end{aligned}$$

We may ignore the terms where $(h, f) > 1$ as in this case $(h, f)|wr \Rightarrow (f, wr) > 1$. So, we may write

$$S_w^+(A, B, M, N) = \sum_{\substack{f|q \\ (f,w)=1}} \sum_{\substack{h|q \\ (h,f)=1}} \mu(fh) \sum_{\substack{0 < |r| \leq \frac{R}{w} \\ (r,f)=1}} \sum_{\substack{a,b,m,n \\ amh=\pm bnf+wr \\ (ab,q)=1}} F(a, b, mh, nf) \quad (2.12)$$

$$=: \sum_{\substack{f|q \\ (f,w)=1}} \sum_{\substack{h|q \\ (h,f)=1}} \mu(fh) S_{w,f,h}^+(A, B, M, N), \quad (2.13)$$

up to an error of size $O_K((qT)^{-K})$. Using Lemma 2.7, we see that

$$S_{w,f,h}^+(A, B, M, N) \ll \frac{(qT)^\epsilon T}{(ABMN)^{\frac{1}{2}}} \sum_{\substack{a,b,m,n \\ amh=\pm bnf+wr \\ (ab,q)=1}} |\alpha_a \beta_b| \ll (qT)^\epsilon \left(\frac{A}{w} + \frac{\sqrt{ABMN}}{wfh} \right)$$

and so

$$S_{w,f,h}^+(A, B, M, N) \ll \frac{(qT)^{\kappa+\epsilon}}{w} \left(1 + \frac{(qT)^{\frac{1}{2}}}{fh} \right). \quad (2.14)$$

Therefore, if $fh \geq (qT)^{\kappa-\frac{1}{2}+\epsilon}$ then Proposition 2.1 is trivially true. Hence we may assume in the following that $fh \leq (qT)^{\kappa-\frac{1}{2}+\epsilon}$, and without loss of generality $f \leq h$.

Let $d = (a, b)$ (and so $d|r$) which allows us to write the condition $amh = \pm bnf + wr$ as $m \equiv \left(\frac{ah}{d}\right) \frac{wr}{d} \pmod{\frac{bf}{d}}$ and so by Poisson summation $S_{w,f,h}^+(A, B, M, N)$ is

$$\begin{aligned} &= \sum_{\substack{d \geq 1 \\ (d,q)=1}} \sum_{\substack{0 < |r| \leq \frac{R}{dw} \\ (r,f)=1}} \sum_{\substack{a,b \\ (a,b)=d \\ (ab,q)=1}} \sum_{m \equiv \left(\frac{ah}{d}\right) wr \pmod{\frac{bf}{d}}} F\left(a, b, mh, \pm \frac{amh - wrd}{b}\right) \\ &= \sum_{\substack{d \geq 1 \\ (d,q)=1}} \sum_{\substack{0 < |r| \leq \frac{R}{dw} \\ (r,f)=1}} \sum_{\substack{a,b \\ (a,b)=d \\ (ab,q)=1}} \sum_{g \in \mathbb{Z}} e\left(-\frac{gwra\overline{h}/d}{bf/d}\right) \\ &\quad \times \int F\left(a, b, \frac{xbfh}{d}, \pm \frac{abx fh - wrd^2}{bd}\right) e(gx) dx. \end{aligned}$$

Making the change of variables $x \rightarrow hfx$, we separate the sum over g into the $g = 0$ case and the $g \neq 0$ case so that

$$\begin{aligned} S_{w,f,h}^+(A, B, M, N) &= \frac{1}{fh} (\mathcal{M}_{w,f}^+(A, B, M, N) + \mathcal{M}_{w,f}^-(A, B, M, N)) \\ &\quad + \mathcal{E}_{w,f,h}^\pm(A, B, M, N) \end{aligned} \quad (2.15)$$

where $\mathcal{M}_{w,f}^+(A, B, M, N)$ is the sum with $g = 0$ and the positive side of the $\pm$ sign, $\mathcal{M}_{w,f}^-(A, B, M, N)$ is the sum with $g = 0$ and the negative side of the $\pm$ sign, and with

$$\begin{aligned}
\mathcal{E}_{w,f,h}^{\pm}(A, B, M, N) &:= \frac{1}{hf} \sum_{\substack{d \geq 1 \\ (d,q)=1}} \sum_{\substack{0 < |r| \leq R/wd \\ (r,f)=1}} \sum_{\substack{0 < |g| \\ (ab,q)=1}} \sum_{\substack{(a,b)=1 \\ (ab,q)=1}} \frac{\alpha_{ad}(A)\beta_{bd}(B)}{(ab)^{1/2}d} \left(\frac{b}{a}\right)^{it} \\
&\quad \frac{e\left(\frac{-gwr\overline{ah}}{bf}\right)}{2\pi i} \int_{(\epsilon)} \int X_+(s, t) \left(\frac{q}{\pi}\right)^s \left(\pm \frac{abx - wr}{b}\right)^{-(1/2+s+\beta-it)} (bx)^{-(1/2+\alpha+s+it)} \\
&\quad W\left(\frac{bx}{M}\right) W\left(\pm \frac{abx - wr}{bN}\right) \psi_t e\left(\frac{gx}{hf}\right) dx \frac{ds}{s} dt. \quad (2.16)
\end{aligned}$$

Note that in $\mathcal{M}_{w,f}^+(A, B, M, N)$ and $\mathcal{M}_{w,f}^-(A, B, M, N)$ we may extend the sum over r to infinity, due to the support of the W functions. By (2.13), to find the main term of Proposition 2.1 we must show that

$$\sum_{\substack{f|q \\ (f,w)=1}} \frac{\mu(f)}{f} \sum_{\substack{h|q \\ (h,f)=1}} \frac{\mu(h)}{h} = \sum_{\substack{f|q \\ (f,w)=1}} \frac{\mu(f)\phi(q/f)}{q}. \quad (2.17)$$

Suppose for a contradiction that $p|(f, q/f)$ then $p^2|q$ and hence (as by assumption q/w is square-free) it must be the case that $p|w$. Hence p cannot divide f , so $(f, q/f) = 1$.

Therefore (2.17) holds, which gives the main term of Proposition 2.1.

We turn our attention to the term $\mathcal{E}_{w,f,h}^{\pm}$. By integrating by parts j times:

$$\int (bx)^{-(\alpha+s+it)} \left(\frac{\mp(abx - wr)}{b}\right)^{-(\beta+s-it)} W\left(\frac{bx}{M}\right) W\left(\frac{\mp(abx - wr)}{bN}\right) e\left(\frac{gx}{fh}\right) dx$$

is

$$O_{\epsilon,j} \left((qT)^{\epsilon} \frac{dM}{BN} \left| \frac{Bfh}{gdM} \right|^j \right)$$

for any fixed $j \geq 0$. Therefore, we may restrict the sum over g to $0 < |g| \leq Gfh/d$ where $G = \frac{B}{M}(qT)^{\epsilon}$.

We introduce the Mellin transform

$$\begin{aligned}
\gamma^{\pm}(a, b, r) &= \left(\frac{b}{a}\right)^{it} \left(\pm \frac{abx - wr}{b}\right)^{-(1/2+s+\beta-it)} (bx)^{-(1/2+\alpha+s+it)} \\
&\quad W\left(\frac{bx}{M}\right) W\left(\pm \frac{abx - wr}{bN}\right) \\
&= \frac{1}{(2\pi i)^3} \int_{(\epsilon)} \int_{(\epsilon)} \int_{(\epsilon)} a^{-u} b^{-v} r^{-z} \hat{\gamma}^{\pm}(u, v, z) du dv dz
\end{aligned}$$

where

$$\hat{\gamma}^{\pm}(u, v, z) = \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} a^{u-1} b^{v-1} r^{z-1} \gamma^{\pm}(a, b, r) da db dr.$$

By integration by parts j times,

$$\begin{aligned} \hat{\gamma}^\pm(u, v, z) &\ll_{j, \epsilon} \left(\frac{A}{d}\right)^{\Re(u)} \left(\frac{B}{d}\right)^{\Re(v)} \left(\frac{R}{dw}\right)^{\Re(z)} (MN)^{-\frac{1}{2}} \\ &\quad \times ((qT)^{-\epsilon}(1+|u|)(1+|v|)(1+|z|))^{-j} \end{aligned} \quad (2.18)$$

uniformly for $\Re(u), \Re(v), \Re(z) \geq \epsilon$ and fixed $j \geq 0$, and so we may restrict the integrals to $|u|, |v|, |z| \leq \epsilon$. Therefore, by partial summation,

$$\begin{aligned} \mathcal{E}_{w,f,h}^\pm &\ll_\epsilon \sum_{\substack{d \geq 1 \\ (d,q)=1}} \frac{(qT)^\epsilon}{hf\sqrt{AB}} \int_{(\epsilon)} \int_{(\epsilon)} \int_{(\epsilon)} \int_{(\epsilon)} \int_{x \asymp \frac{dM}{B}} |\hat{\gamma}^\pm(u, v, z)| |Z_d(x)| \\ &\quad \left| X_+(s, t) \left(\frac{q}{s}\right)^s \right| \psi_t du dv dz dx \frac{ds}{s} dt \end{aligned} \quad (2.19)$$

where

$$Z_d(x) = \sum_{\substack{0 < |r| \leq R/wd \\ (r,f)=1}} \sum_{0 < |g| \leq \frac{Gfh}{d}} \sum_{\substack{(a,b)=1 \\ (ab,q)=1}} \frac{\alpha_{ad}(A)\beta_{bd}(B)}{a^u b^v} r^{-z} e\left(\frac{gx}{hf}\right) e\left(\frac{-gwr\bar{a}h}{bf}\right). \quad (2.20)$$

By (2.18), the integral in x being supported for $x \asymp \frac{dM}{B} \asymp d\sqrt{\frac{MN}{AB}}$ and the rapid decay of $X(s, t)$

$$\mathcal{E}_{w,f,h}^\pm \ll_\epsilon (qT)^\epsilon \frac{T}{hf\sqrt{AB}} \sqrt{\frac{MN}{AB}} \frac{1}{\sqrt{MN}} \sum_d d \max_x |Z_d(x)| \quad (2.21)$$

$$\ll_\epsilon (qT)^\epsilon \frac{T}{hfAB} \sum_d d \max_x |Z_d(x)|. \quad (2.22)$$

In the general case of Theorem 2.1, we will apply an adapted version of Bettin and Chandee's Theorem 1 from [7]. With the added assumptions of Theorems 2.2 and 2.3, we will use other methods.

2.4.1 The general case

Theorem 2.4 (Bettin and Chandee). *Define for $A, M, N \geq 1, \theta \in \mathbb{Z} \setminus \{0\}$, and co-prime natural numbers f, h ,*

$$\mathcal{B}_{f,h,\theta}(M, N, A) := \sum_{\substack{a \asymp A \\ m \asymp M \\ n \asymp N \\ (m,nf)=1}} \alpha_m \beta_n \nu_a e\left(\theta \frac{amh}{fn}\right)$$

then

$$\mathcal{B}_{f,h,\theta}(M, N, A) \ll \|\alpha\| \|\beta\| \|\nu\| \left(1 + \frac{|\theta|A}{fhMN}\right)^{1/4} (f^{1/4} h^{1/2} AMN^{3/4})$$

$$+ f^{3/4} h^{1/2} A M^{1/2} N^{5/4} + f^{-2/5} h^{6/5} A M^{6/5} N^{1/10} + f^{1/5} h^{6/5} A^{2/5} M^{6/5} N^{7/10} \\ + f^{1/2} h^{3/5} A^{7/10} M^{3/5} N^{13/10} + f^{1/2} A N^{7/4} \Big)^{1/2}.$$

The proof of this theorem is an almost direct consequence of the original proof. The details of the exact differences can be found in the appendix. Therefore, substituting

- $\alpha_m \rightarrow \alpha_{ad}(A)$
- $\beta_n \rightarrow \beta_{bd}(B)$
- $\nu_a \rightarrow \sum_{rg=a} e \left(\frac{gx}{hf} \right) r^{-z}$
- $\theta \rightarrow w$

gives the bound

$$Z_d(x) \ll_{\epsilon} (qT)^{\epsilon} \frac{AB}{d^2} \sqrt{\frac{fh}{wT}} \left(1 + \frac{wABfhd^2}{fhABwTd^2} \right)^{1/4} \\ \times \left(f^{1/4} h^{1/2} \left(\frac{ABhf}{wTd^2} \right) \left(\frac{A}{d} \right) \left(\frac{B}{d} \right)^{3/4} + f^{3/4} h^{1/2} \left(\frac{ABhf}{wTd^2} \right) \left(\frac{A}{d} \right)^{1/2} \left(\frac{B}{d} \right)^{5/4} \right. \\ + f^{-2/5} h^{6/5} \left(\frac{ABhf}{wTd^2} \right) \left(\frac{A}{d} \right)^{6/5} \left(\frac{B}{d} \right)^{1/10} + f^{1/5} h^{6/5} \left(\frac{ABhf}{wTd^2} \right)^{2/5} \left(\frac{A}{d} \right)^{6/5} \left(\frac{B}{d} \right)^{7/10} \\ \left. + f^{1/2} h^{3/5} \left(\frac{ABhf}{wTd^2} \right)^{7/10} \left(\frac{A}{d} \right)^{3/5} \left(\frac{B}{d} \right)^{13/10} + f^{1/2} \left(\frac{ABhf}{wTd^2} \right) \left(\frac{B}{d} \right)^{7/4} \right)^{1/2} \\ \ll_{\epsilon} (qT)^{\epsilon} \frac{AB}{d^3} \sqrt{\frac{fh}{wT}} \left(\frac{AB^{\frac{7}{8}} f^{\frac{3}{4}} h^{\frac{5}{8}}}{\sqrt{wT}} + \frac{A^{\frac{3}{4}} B^{\frac{9}{8}} f^{\frac{7}{8}} h^{\frac{3}{4}}}{\sqrt{wT}} + \frac{A^{\frac{11}{10}} B^{\frac{11}{20}} f^{\frac{3}{10}} h^{\frac{11}{10}}}{\sqrt{wT}} \right. \\ \left. + \frac{A^{\frac{4}{5}} B^{\frac{11}{20}} f^{\frac{3}{10}} h^{\frac{4}{5}}}{(wT)^{\frac{1}{5}}} + \frac{A^{\frac{13}{20}} B f^{\frac{3}{5}} h^{\frac{13}{20}}}{(wT)^{\frac{7}{20}}} + \frac{A^{\frac{1}{2}} B^{\frac{11}{8}} f^{\frac{3}{4}} h^{\frac{1}{2}}}{\sqrt{wT}} \right). \quad (2.23)$$

Therefore, by (2.21),

$$\mathcal{E}_{w,f,h}^{\pm} \ll_{\epsilon} \sqrt{\frac{T}{hfw}} \left(\frac{AB^{\frac{7}{8}} f^{\frac{3}{4}} h^{\frac{5}{8}}}{\sqrt{wT}} + \frac{A^{\frac{3}{4}} B^{\frac{9}{8}} f^{\frac{7}{8}} h^{\frac{3}{4}}}{\sqrt{wT}} + \frac{A^{\frac{11}{10}} B^{\frac{11}{20}} f^{\frac{3}{10}} h^{\frac{11}{10}}}{\sqrt{wT}} \right. \\ \left. + \frac{A^{\frac{4}{5}} B^{\frac{11}{20}} f^{\frac{3}{10}} h^{\frac{4}{5}}}{(wT)^{\frac{1}{5}}} + \frac{A^{\frac{13}{20}} B f^{\frac{3}{5}} h^{\frac{13}{20}}}{(wT)^{\frac{7}{20}}} + \frac{A^{\frac{1}{2}} B^{\frac{11}{8}} f^{\frac{3}{4}} h^{\frac{1}{2}}}{\sqrt{wT}} \right) \\ \ll_{\epsilon} w^{-1} (qT)^{\epsilon} \left(AB^{\frac{7}{8}} f^{\frac{1}{4}} h^{\frac{1}{8}} + A^{\frac{3}{4}} B^{\frac{9}{8}} f^{\frac{3}{8}} h^{\frac{1}{4}} + A^{\frac{11}{10}} B^{\frac{11}{20}} f^{\frac{-1}{5}} h^{\frac{3}{5}} \right. \\ \left. + (wT)^{\frac{3}{10}} A^{\frac{4}{5}} B^{\frac{11}{20}} f^{\frac{-1}{5}} h^{\frac{3}{10}} + w^{\frac{3}{20}} T^{\frac{13}{20}} A^{\frac{13}{20}} B f^{\frac{1}{10}} h^{\frac{3}{20}} + A^{\frac{1}{2}} B^{\frac{11}{8}} f^{\frac{1}{4}} \right) \\ \ll_{\epsilon} w^{-1} (qT)^{\epsilon} \left((qT)^{\frac{15\kappa}{8}} f^{\frac{1}{4}} h^{\frac{1}{8}} + (qT)^{\frac{15\kappa}{8}} f^{\frac{3}{8}} h^{\frac{1}{4}} + (qT)^{\frac{33\kappa}{20}} f^{\frac{-1}{5}} h^{\frac{3}{5}} + (qT)^{\frac{27\kappa+6}{20}} \right)$$

$$+(qT)^{\frac{33\kappa+3}{20}} + (qT)^{\frac{15\kappa}{8}} f^{\frac{1}{4}} \Big),$$

as $A, B \ll (qT)^\kappa$ and $wfh \leq q$. By assumption $f \leq h$ and $fh \leq (qT)^{\kappa-\frac{1}{2}+\epsilon}$, so $f \leq (qT)^{\frac{\kappa-\frac{1}{2}+\epsilon}{2}}$. Substituting this in, we see that

$$\mathcal{E}_{w,f,h}^\pm \ll_\epsilon w^{-1} (qT)^{1-\epsilon}$$

as long as $\kappa < \frac{17}{33} - \delta$, for any small positive δ and where ϵ depends on δ . Therefore Proposition 2.1 follows in the general case.

2.4.2 Specialised coefficients

We now suppose that $\alpha_a(A) = \mu(a)f(a)W\left(\frac{a}{A}\right)$, where $f(x)$ is a smooth function with $f'(x) \ll_\epsilon x^{-1+\epsilon}$ and $W(x)$ is our standard dyadic smoothing function with support on $[1,2]$. For simplicity, we write $f_A(x) = f(x)W\left(\frac{x}{A}\right)a^{-u}$. With these conditions, we may apply a useful identity for the Möbius function called the Vaughan identity.

Let $U(s) = \sum_{n \leq W} \mu(n)n^{-s}$ for W a constant that we shall choose later. By comparing the coefficients of

$$\frac{1}{\zeta(s)} = \frac{1}{\zeta(s)}(1 - \zeta(s)U(s))^2 + 2U(s) - \zeta(s)U(s)^2$$

we see that

$$\mu(u) = c_1(u) + c_2(u) + c_3(u),$$

where

$$c_1(u) = \mu^2(u) \sum_{\substack{abc=u \\ a \geq W, b \geq W}} \mu(c)c_4(a)c_4(b) \text{ with } c_4(a) = -\mu^2(a) \sum_{\substack{ef=a \\ e \leq W}} \mu(e),$$

$$c_2(u) = \begin{cases} 2\mu(u) & \text{if } u \leq W, \\ 0 & \text{if } u > W, \end{cases}$$

and

$$c_3(u) = -\mu^2(u) \sum_{\substack{abc=u \\ a \leq W, b \leq W}} \mu(a)\mu(b).$$

This is known as the Vaughan identity, although it is not normally written with the extra $\mu^2(u)$ terms. We have added these in as we shall require $c_4(u)$ to be supported on square-free integers.

The Vaughan identity allows us to write

$$\alpha_{ad}(A)a^{-u} = c_1(ad)f_A(ad) + c_2(ad)f_A(ad) + c_3(ad)f_A(ad),$$

which in turn allows us to write

$$Z_d(x) = \sum_{i=1}^3 Z_d^{(i)}(x),$$

with

$$Z_d^{(i)}(x) = \sum_{\substack{0 < |r| \leq R/wd \\ (r,f)=1}} \sum_{\substack{0 < |g| \leq \frac{Gfh}{d} \\ (a,b)=1 \\ (ab,q)=1}} \sum_{\substack{(a,b)=1 \\ (ab,q)=1}} c_i(ad)f_A(ad)\beta_{bd}(B)r^{-z}e\left(\frac{gx}{hf}\right)e\left(\frac{-gwr\overline{ah}}{bf}\right),$$

where we have abused notation slightly to re-write $\beta_{bd}(B)b^{-v}$ as $\beta_{bd}(B)$. Letting $W = A^{\frac{1}{4}}$ means that $Z_d^{(2)}(x) = 0$ due to the support of $c_2(n)$ and the support of $f_A(x)$ not intersecting.

2.4.2.1 Bounding $Z_d^{(1)}(x)$

We may write $Z_d^{(1)}(x)$ as a linear combination of at most $O_\epsilon((qT)^\epsilon)$ sums which are

$$\ll \left| \sum_{d_1 d_2 d_3 = d} \sum_{\substack{a_1 \asymp A_1/d_1 \\ a_2 \asymp A_2/d_2 \\ a_3 \asymp A_3/d_3 \\ (a_1 a_2 a_3, q)=1}} \sum_{\substack{b \asymp B/d \\ (b, a_1 a_2 a_3 q)=1}} \sum_{\substack{0 < |h| \leq \frac{ABfh}{wTd^2} (qT)^\epsilon}} c_4(a_1 d_1) c_4(a_2 d_2) \mu(a_3 d_3) \right. \\ \left. f_A(a_1 a_2 a_3 d) \beta_{bd}(B) \nu_h e\left(-\frac{wk\overline{a_1 a_2 a_3 h}}{bf}\right) \right| \quad (2.24)$$

where $A_1 A_2 A_3 = A$ and

$$\nu_h = \sum_{\substack{rg=k \\ 0 \leq |r| < R/wd \\ 0 < |g| \leq Gfh/d}} r^{-z} e\left(\frac{xg}{fh}\right).$$

By the definition of $c_1(u)$ we may assume that $A_1, A_2 \geq \frac{W}{d}$ and without loss of generality $A_1 \leq A_2$. We group together some of the variables by writing

$$c_5(a'_2 d'_2) = \mu^2(a'_2 d'_2) \sum_{\substack{d_2 d_3 = d'_2 \\ a_2 a_3 = a'_2}} \mu(a_3 d_3) c_4(a_2 d_2)$$

to change the sum in (2.24) into a sum of size

$$\ll \left| \sum_{d_1 d_2 = d} \sum_{\substack{a_1 \asymp A_1/d_1 \\ a_2 \asymp A_2/d_2 \\ (a_1 a_2, q)=1 \\ (a_1 d_1, a_2 d_2)=1}} \sum_{\substack{b \asymp B/d \\ (b, a_1 a_2 q)=1}} \sum_{\substack{0 < |k| \leq \frac{ABfh}{wTd^2} (qT)^\epsilon}} c_4(a_1 d_1) c_5(a_2 d_2) f_A(a_1 a_2 d) \beta_{bd}(B) \right. \\ \left. \nu_k e \left(-\frac{w k \overline{a_1 a_2 h}}{b f} \right) \right| \quad (2.25)$$

with $W \ll A_1 \ll A_2 \ll A/W$ and $A_1 A_2 = A$. Let $A_1 = (qT)^{\kappa_1}$ and $A_2 = (qT)^{\kappa_2}$ so that $\kappa = \kappa_1 + \kappa_2$. Note that we may bound $c_4(n), c_5(n)$ by $O_\epsilon((qT)^\epsilon)$. To bound these sums, we appeal to an adapted version of Heath-Brown's Lemma 6 in [59].

Lemma 2.8 (Heath-Brown). *Suppose that $1 \leq U, K, S, T \leq x$, q, h, f are pairwise coprime positive integers, and $\alpha(s), \beta(t)$ are sequences with $|\alpha(s)|, |\beta(t)| \leq 1$ and support on $[S, 2S]$ and $[T, 2T]$ respectively. Define*

$$\Omega_4 := \sum_{\substack{u \asymp U \\ (u, qh)=1}} \sum_{|k| \asymp K} \left| \sum_{(s, uf)=1} \sum_{(t, uf)=1} \alpha(s) \beta(t) e \left(\frac{q k \overline{st h}}{u f} \right) \right|.$$

Then

$$\Omega_4 \ll_\epsilon x^\epsilon U K S T \left(\left(\frac{f}{K S} \right)^{1/4} + \left(\frac{U f}{K S T^2} \right)^{1/4} + \frac{1}{U^{1/4}} + \frac{1}{T^{1/2}} \right).$$

The original lemma does not include the f or h terms, but the proof with them included is virtually identical to the original, and is described in the appendix.

Making the substitutions

- $U \rightarrow \frac{B}{d}$
- $K \rightarrow \frac{R G f h}{w d^2} \asymp A B f h (w T)^{\epsilon-1} d^{-2}$
- $S \rightarrow \frac{A_1}{d_1}$
- $T \rightarrow \frac{A_2}{d_2}$

we find that (2.25) is

$$\ll_\epsilon (qT)^\epsilon \frac{(AB)^2 f h}{w T d^4} \left(\frac{d_1 d^2 w T}{A_1 A B h} + \frac{d_2 d^2 w T}{A_2 A^2 h} + \frac{d}{B} + \left(\frac{d_2}{A_2} \right)^2 \right)^{\frac{1}{4}}$$

and so

$$\mathcal{E}_{w, f, h}^\pm \ll_\epsilon (qT)^\epsilon \sum_d \sum_{d_1 d_2 = d} \frac{AB}{w d^3} \left(\frac{w T}{A_1 A B h} + \frac{w T}{A_2 A^2 h} + \frac{1}{B} + \left(\frac{1}{A_2} \right)^2 \right)^{\frac{1}{4}}$$

$$\begin{aligned}
&\ll_{\epsilon} w^{-1} \left(A_2^{\frac{1}{4}} A^{\frac{1}{2}} B^{\frac{3}{4}} (qT)^{\frac{1}{4}} + A_1^{\frac{1}{4}} A^{\frac{1}{4}} B (qT)^{\frac{1}{4}} + AB^{\frac{3}{4}} + A_1^{\frac{1}{2}} A^{\frac{1}{2}} B \right) \\
&\ll_{\epsilon} w^{-1} (qT)^{\epsilon} \left((qT)^{\frac{3\kappa}{2} - \frac{\kappa_1}{4} + \frac{1}{4}} + (qT)^{\frac{5\kappa}{4} + \frac{\kappa_1}{4} + \frac{1}{4}} + (qT)^{\frac{7\kappa}{4}} + (qT)^{\frac{3\kappa}{2} + \frac{\kappa_1}{2}} \right) \\
&\ll_{\epsilon} w^{-1} (qT)^{\frac{3\kappa}{2} - \frac{\kappa_1}{4} + \frac{1}{4} + \epsilon}
\end{aligned}$$

for $\kappa < \frac{69}{128}$ and $\kappa_1 \leq \kappa_2$. If $\kappa_1 \geq \kappa - \frac{39}{128}$ then this is

$$\ll_{\epsilon} w^{-1} (qT)^{\frac{5\kappa}{4} + \frac{167}{512} + \epsilon} \ll w^{-1} (qT)^{1-\epsilon'}$$

for $\kappa < \frac{69}{128} - \delta$, and for any small positive δ with ϵ small enough relative to δ .

When $\kappa_1 \leq \kappa - \frac{39}{128}$, we use a result by Matomäki to bound (2.25).

Lemma 2.9 (Matomäki). *Let $X_d = \sqrt{\frac{dK}{MC R h f}}$ and suppose that a bounded sequence $a(k, r)$ is supported on square-free r with $(r, d) = 1$. Then,*

$$\begin{aligned}
U &= \sum_{\substack{c \asymp C \\ (c, h)=1}} \sum_{\substack{m \asymp M \\ (m, cf)=1}} \left| \sum_{k \asymp K} \sum_{\substack{r \asymp R \\ (r, cf)=1}} a(k, r) e\left(\pm \frac{dkmrh}{cf}\right) \right| \\
&\ll_{\epsilon} (CMKR)^{\frac{1}{2} + \epsilon} \left[CM + MRK^{\frac{1}{2}} (K + R)^{\frac{1}{2}} + d^{\frac{7}{64}} (MChf)^{\frac{1}{2}} R^{\frac{3}{2}} (K + R)^{\frac{1}{2}} \right. \\
&\quad \times \left. \frac{(1 + X_d^{-1})^{\frac{7}{32}}}{1 + X_d} \left(1 + X_d + \sqrt{\frac{C}{hR^2M}} \right) \left(1 + X_d + \sqrt{\frac{K}{hfR}} \right) \right]^{\frac{1}{2}}.
\end{aligned}$$

This theorem is a slight adaptation of Lemma 11 of [88], with details of the difference in the proof in the appendix. Making the substitutions

- $C \rightarrow \frac{B}{d}$
- $d \rightarrow w$
- $K \rightarrow \frac{ABhf}{d^2(wT)^{1-\epsilon}}$
- $M \rightarrow \frac{A_2}{d_2}$
- $R \rightarrow \frac{A_1}{d_1}$
- $X_d \rightarrow \sqrt{\frac{wABhf}{(wT)^{1-\epsilon}ABhf}} \ll (qT)^{\epsilon} T^{-\frac{1}{2}}$

we find that (2.25) is

$$\ll_{\epsilon} (qT)^{\epsilon} \frac{AB}{d^2} \sqrt{\frac{hf}{wT}} \left[\frac{A_2 B}{dd_2} + \frac{A^{3/2} B^{1/2}}{d^2} \sqrt{\frac{hf}{wT}} \left(\frac{ABhf}{wTd^2} + \frac{A_1}{d_1} \right)^{1/2} \right]$$

$$\begin{aligned}
& + (wT)^{7/64} \frac{A_2^{1/2} B^{1/2} h^{1/2} f^{1/2} A_1^{3/2}}{d^{1/2} d_1^{3/2} d_2^{1/2}} \left(\frac{ABhf}{wTd^2} + \frac{A_1}{d_1} \right)^{1/2} \left(1 + \frac{Bd_1^2 d_2}{dA_1^2 A_2 h} \right)^{\frac{1}{2}} \\
& \times \left(1 + \frac{ABhfd_1}{wTd^2 A_1 hf} \right)^{1/2} \Bigg]^{\frac{1}{2}} \\
& \ll_{\epsilon} (qT)^{\epsilon} \frac{AB}{d^{\frac{5}{2}}} \sqrt{\frac{hf}{wT}} \left[A_2 B + A^{3/2} B^{1/2} \sqrt{\frac{hf}{wT}} \left(\frac{ABhf}{wT} + A_1 \right)^{1/2} \right. \\
& \left. + (wT)^{7/64} A^{1/2} A_1 B^{1/2} h^{1/2} f^{1/2} \left(\frac{ABhf}{wT} + A_1 \right)^{1/2} \left(1 + \frac{B}{AA_1 h} \right)^{\frac{1}{2}} \left(1 + \frac{A_2 B}{wT} \right)^{1/2} \right]^{\frac{1}{2}}
\end{aligned}$$

and so

$$\begin{aligned}
\mathcal{E}_{w,f,h}^{\pm} & \ll_{\epsilon} (qT)^{\epsilon} \frac{T^{\frac{1}{2}}}{w^{\frac{1}{2}} (hf)^{\frac{1}{2}}} \left[A_2 B + A^{3/2} B^{1/2} \sqrt{\frac{hf}{wT}} \left(\frac{ABhf}{wT} + A_1 \right)^{1/2} \right. \\
& \left. + (wT)^{7/64} A^{1/2} A_1 B^{1/2} h^{1/2} f^{1/2} \right. \\
& \left. \times \left(\frac{ABhf}{wT} + A_1 \right)^{1/2} \left(1 + \frac{B}{AA_1 h} \right)^{\frac{1}{2}} \left(1 + \frac{A_2 B}{wT} \right)^{1/2} \right]^{\frac{1}{2}} \\
& \ll_{\epsilon} w^{-1} (qT)^{\epsilon} \left((qT)^{\frac{\kappa}{2} + \frac{\kappa_2}{2} + \frac{1}{2}} + (qT)^{\frac{3\kappa}{2}} + (qT)^{\kappa + \frac{\kappa_1}{4} + \frac{1}{4}} \right. \\
& \left. + \left((qT)^{\kappa + \frac{\kappa_1}{2} + \frac{39}{128}} + (qT)^{\frac{\kappa}{2} + \frac{3\kappa_1}{4} + \frac{71}{128}} \right) \times \left(1 + (qT)^{\frac{\kappa}{4} + \frac{\kappa_2}{4} - \frac{1}{4}} \right) \right).
\end{aligned}$$

As $\frac{\kappa}{4} \leq \kappa_1 \leq \kappa - \frac{39}{128}$, and $\kappa < \frac{69}{128}$, this is

$$\begin{aligned}
& \ll_{\epsilon} w^{-1} (qT)^{\epsilon} \left((qT)^{\frac{7\kappa}{8} + \frac{1}{2}} + (qT)^{\frac{3\kappa}{2}} + (qT)^{\frac{5\kappa}{4} + \frac{89}{512}} + (qT)^{\frac{3\kappa}{2} + \frac{39}{256}} + (qT)^{\frac{5\kappa}{4} + \frac{167}{512}} \right. \\
& \left. + (qT)^{2\kappa - \frac{25}{256}} (qT)^{\frac{3\kappa}{2} + \frac{39}{256}} \right) \\
& \ll_{\epsilon} w^{-1} (qT)^{1-\epsilon}
\end{aligned}$$

for $\kappa \leq \frac{69}{128} - \delta$ and for any small positive δ , with ϵ small enough compared to δ .

2.4.2.2 Bounding $Z_d^{(3)}(x)$

We may write $Z_d^{(3)}(x)$ as a linear combination of at most $O_{\epsilon}((qT^{\epsilon}))$ sums which are

$$\ll \left| \sum_{d_1 d_2 = d} \sum_{\substack{a_1 \asymp A_1/d_1 \\ a_2 \asymp A_2/d_2 \\ (a_1 a_2, q) = 1}} \sum_{\substack{b \asymp B/d \\ (b, a_1 a_2 q) = 1}} \sum_{\substack{0 < |k| \leq \frac{ABfh}{wTd^2} (qT)^{\epsilon}}} \mu^2(a_1 d_1 a_2 d_2) c_6(a_1 d_1) f_A(a_1 a_2 d) \beta_{bd}(B) \right. \\
\left. \nu_k e \left(-\frac{w k \overline{a_1 a_2 h}}{b f} \right) \right| \quad (2.26)$$

with

$$c_6(n) = \sum_{\substack{xy=n \\ x,y \leq W}} \mu(x)\mu(y).$$

This means that we may assume $A_1 \leq W^2 = A^{1/2}$, otherwise the sums are empty. When $A_1 \gg A^{1/4}$ we use the same method as we did to bound $Z_d^{(1)}(x)$, but when $A_1 \ll A^{1/4}$ we shall apply the Weil bound for Kloosterman sums on the sum over a_2 . However before we do that, we must remove the $\mu^2(a_2)$ factor by using the inclusion-exclusion principle to introduce the values of a_2 that are not square-free. Specifically, for any arithmetic function $f(a)$,

$$\sum_a \mu^2(a)f(a) = \sum_n (-1)^{\omega(n)} \sum_a f(an^2),$$

where $\omega(n)$ is the number of prime factors of n , without multiplicity,

$$\omega(n) = \sum_{p|n} 1.$$

Define

$$Z_{d,n}^{(3)} := \sum_{d_1 d_2 = d} \sum_{\substack{a_1 \asymp A_1/d_1 \\ a_2 \asymp A_2/nd_2 \\ (a_1 a_2, q)=1 \\ (a_1 n, d_1 a_2 d_2)=1}} \sum_{\substack{b \asymp B/d \\ (b, a_1 a_2 n q)=1}} \sum_{0 < |k| \leq \frac{ABfh}{wTd^2} (qT)^\epsilon} \mu^2(a_1 d_1 a_2 d_2) c_6(a_1 d_1) f_A(a_1 a_2 n^2 d) \\ \beta_{bd}(B) \nu_k e \left(-\frac{w k \overline{a_1 a_2 n^2 h}}{bf} \right) \Big|, \quad (2.27)$$

so that

$$Z_d^{(3)}(x) \ll \sum_{\substack{n \ll A_2^{\frac{1}{2}} \\ (n, q)=1}} |Z_{d,n}^{(3)}|.$$

By Lemma 2.8,

$$Z_d^{(3)}(x) \ll_\epsilon (qT)^\epsilon \frac{(AB)^2 fh}{wTd^4 n^2} \left(\frac{d_1 d^2 wT}{A_1 ABh} + \frac{d_2 d^2 wT n^2}{A_2 A^2 h} + \frac{d}{B} + \left(\frac{d_2}{A_2 n} \right)^2 \right)^{\frac{1}{4}}$$

and so we find that

$$\frac{T}{hfAB} \sum_d \sum_{n \geq X} d |Z_{d,n}^{(3)}| \ll w^{-1} h^{-\frac{1}{4}} X^{-1} (qT)^{\frac{3\kappa}{2} - \frac{\kappa_1}{4} + \frac{1}{4} + \epsilon} + w^{-1} (qT)^{1-\epsilon}$$

and therefore

$$\mathcal{E}_{w,f,h}^\pm \ll w^{-1} (qT)^{1-\epsilon} + \frac{T}{hfAB} \sum_d \sum_{n \leq X} d |Z_{d,n}^{(3)}|,$$

where $X := h^{-\frac{1}{4}}(qT)^{\frac{3\kappa}{2} - \frac{\kappa_1}{4} - \frac{3}{4} + \epsilon}$.

We now apply Weil's bound for Kloosterman sums to $Z_{d,n}^{(3)}$. This implies that

$$\sum_{\substack{\frac{A_2}{n^2 d_2} \leq a_2 \leq \frac{A_2}{n^2 d_2} + x \\ (a_2, bq)=1}} e\left(\frac{-wk\overline{a_1 a_2} h n^2}{bf}\right) \ll_{\epsilon} (bf)^{1/2+\epsilon} (wk, bf) \left(1 + \frac{A_2}{bf d_2 n^2}\right)$$

and hence (with partial summation over a_2), we arrive at the bound

$$Z_{d,n}^{(3)}(x) \ll_{\epsilon} (qT)^{\epsilon} \frac{A_1}{d_1} \left(\frac{Bf}{d_2}\right)^{1/2} \left(1 + \frac{A_2 d}{Bf d_2 n^2}\right) \sum_{0 < |k| \leq \frac{ABfh}{wTd^2} (qT)^{\epsilon}} \sum_{b \asymp B/d} (k, bf)$$

which is

$$\ll_{\epsilon} \frac{(qT)^{\epsilon}}{d^3} \left(\frac{A_1 A B^{\frac{5}{2}} f^{\frac{3}{2}} h}{wT} + \frac{A^2 B^{\frac{3}{2}} f^{\frac{1}{2}} h}{wT n^2} \right). \quad (2.28)$$

Therefore

$$\begin{aligned} \frac{T}{hfAB} \sum_d \sum_{n \leq X} d |Z_{d,n}^{(3)}| &\ll_{\epsilon} (qT)^{\epsilon} \frac{1}{wfhAB} \left(X A_1 A B^{\frac{5}{2}} f^{\frac{3}{2}} h + A^2 B^{\frac{3}{2}} f^{\frac{1}{2}} h \right) \\ &\ll_{\epsilon} w^{-1} X f^{\frac{1}{2}} (qT)^{\frac{3\kappa}{2} + \kappa_1 + \epsilon} \ll_{\epsilon} w^{-1} f^{\frac{1}{4}} (qT)^{3\kappa + \frac{3\kappa_1}{4} - \frac{3}{4} + \epsilon} \end{aligned}$$

because $f \leq h$. As we may assume that $f \leq (qT)^{\frac{\kappa}{2} - \frac{1}{4} + \epsilon}$ and $\kappa_1 \leq \frac{\kappa}{4}$ then this is

$$\ll_{\epsilon} w^{-1} (qT)^{\frac{53\kappa}{16} - \frac{13}{16}} \ll_{\epsilon} w^{-1} (qT)^{1-\epsilon}$$

as $\kappa < \frac{69}{128}$. This concludes the proof of Proposition 2.1 when the coefficients are specialised.

2.4.3 T large compared to q

Now assume that

$$\kappa < \frac{4 + \frac{3}{\gamma}}{7 \left(1 + \frac{1}{\gamma}\right)}$$

and we still have the assumption that $\alpha_a(A) = \mu(a)f(a)W\left(\frac{a}{A}\right)$ as above. We use the Vaughan identity again, but now without forcing our variables to be square-free. Specifically, we write

$$\mu(u) = c_1(u) + c_2(u) + c_3(u),$$

where

$$c_1(u) = \sum_{\substack{abc=u \\ a \geq W, b \geq W}} \mu(c) c_4(a) c_4(b) \text{ with } c_4(a) = - \sum_{\substack{ef=a \\ e \leq W}} \mu(e),$$

$$c_2(u) = \begin{cases} 2\mu(u) & \text{if } u \leq W, \\ 0 & \text{if } u > W, \end{cases}$$

and

$$c_3(u) = - \sum_{\substack{abc=u \\ a \leq W, b \leq W}} \mu(a)\mu(b).$$

We repeat the same method as for the specialised coefficients but instead of bounding $Z_d^{(1)}(x)$ and $Z_d^{(3)}$ using results by Heath-Brown and Matomäki, we use a different bound by Deshouillers and Iwaniec that is essentially Lemma 1 of [30]. The complete details of the proof are in the appendix. As we are no longer using Lemma 2.9, we do not require our variables to be square-free.

Lemma 2.10 (Deshouillers and Iwaniec). *Let $C, D, U, V \geq 1$, h, f, w to be pairwise co-prime positive integers, and $|c(u, v)| \leq 1$. Then*

$$\begin{aligned} H(C, D, U, V) &:= \sum_{\substack{c \asymp C \\ (c, h)=1}} \sum_{\substack{d \asymp D \\ (d, cf)=1}} \left| \sum_{u \asymp U} \sum_{v \asymp (v, cf)=1} c(u, v) e\left(\frac{uvw\overline{vdh}}{cf}\right) \right| \\ &\ll_{\epsilon} (CDUV)^{\frac{1}{2}+\epsilon} \left((CD)^{\frac{1}{2}} + (U+V)^{\frac{1}{4}} [fCD(wU+hV)(fC+hV^2) \right. \\ &\quad \left. + hwD^2UV^2]^{\frac{1}{4}} \right). \end{aligned}$$

Making the substitutions

- $C \rightarrow \frac{B}{d},$
- $D \rightarrow \frac{A_2}{d_2},$
- $U \rightarrow \frac{ABhf}{d^2(wT)^{1-\epsilon}},$
- $V \rightarrow \frac{A_1}{d_1}$

means that we may bound (2.25) by

$$\begin{aligned} &\ll_{\epsilon} (qT)^{\epsilon} \left(\frac{A^2 B^2 hf}{d^4 wT} \right)^{\frac{1}{2}} \left(\left(\frac{A_2 B}{dd_2} \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left(\frac{ABhf}{d^2 wT} + \frac{A_1}{d_1} \right)^{\frac{1}{4}} \left[\frac{fA_2 B}{dd_2} \left(\frac{ABfh}{d^2 T} + \frac{hA_1}{d_1} \right) \left(\frac{fB}{d} + \frac{hA_1^2}{d_1^2} \right) + \frac{h^2 w A_2^2 AB f A_1^2}{d^4 wT} \right]^{\frac{1}{4}} \right) \\ &\ll_{\epsilon} \frac{(qT)^{\epsilon}}{d^2} \frac{AB(hf)^{\frac{1}{2}}}{(wT)^{\frac{1}{2}}} \left(A_2^{\frac{1}{2}} B^{\frac{1}{2}} + \left(\frac{ABhf}{wT} + A_1 \right)^{\frac{1}{4}} \right) \end{aligned}$$

$$\times \left[fhA_2B \left(\frac{ABf}{T} + A_1 \right) (fB + hA_1^2) + \frac{fh^2A^3B}{T} \right]^{\frac{1}{4}} \Bigg).$$

Hence by (2.21), $\mathcal{E}_{w,f,h}^\pm$ is

$$\begin{aligned} &\ll_\epsilon (qT)^\epsilon \frac{T^{\frac{1}{2}}}{(whf)^{\frac{1}{2}}} \left((qT)^{\frac{\kappa}{2} + \frac{\kappa_2}{2}} + \left((qT)^{\frac{\kappa}{2}} \left(\frac{hf}{wT} \right)^{\frac{1}{4}} + (qT)^{\frac{\kappa_1}{4}} \right) \right. \\ &\times \left[\frac{f^3h(qT)^{4\kappa+\kappa_2}}{T} + \frac{f^2h^2(qT)^{4\kappa+\kappa_1}}{T} + f^2h(qT)^{3\kappa} + fh^2(qT)^{2\kappa+2\kappa_1} + \frac{fh^2(qT)^{4\kappa}}{T} \right]^{\frac{1}{4}} \Bigg) \\ &\ll_\epsilon (qT)^\epsilon \frac{T^{\frac{1}{2}}}{(qwhf)^{\frac{1}{2}}} \left((qT)^{\frac{\kappa}{2} + \frac{\kappa_2}{2}} + \left[\frac{f^4h^2(qT)^{6\kappa+\kappa_2}}{wT^2} + \frac{f^3h^3(qT)^{6\kappa+\kappa_1}}{wT^2} \right. \right. \\ &+ \frac{f^3h^2(qT)^{5\kappa}}{wT} + \frac{f^2h^3(qT)^{4\kappa+2\kappa_1}}{wT} + \frac{f^2h^3(qT)^{6\kappa}}{wT^2} + \frac{f^3h(qT)^{5\kappa}}{T} + \frac{f^2h^2(qT)^{4\kappa+2\kappa_1}}{T} \\ &\left. \left. + f^2h(qT)^{3\kappa+\kappa_1} + fh^2(qT)^{2\kappa+3\kappa_1} + \frac{fh^2(qT)^{4\kappa+\kappa_1}}{T} \right]^{\frac{1}{4}} \right) \\ &\ll_\epsilon \left(\frac{1}{whf} \right)^{\frac{1}{2}} (qT)^{\frac{\kappa}{2} + \frac{\kappa_2}{2} + \frac{1}{2} + \epsilon} + (qT)^\epsilon \left[wf^2(qT)^{6\kappa+\kappa_2} + wfh(qT)^{6\kappa+\kappa_1} \right. \\ &+ \frac{wf}{q}(qT)^{5\kappa+1} + \frac{wh}{q}(qT)^{4\kappa+2\kappa_1+1} + wh(qT)^{6\kappa} + \frac{w^2f}{hq}(qT)^{5\kappa+1} + \frac{w^2}{q}(qT)^{4\kappa+2\kappa_1+1} \\ &\left. + \frac{w^2}{hq^2}(qT)^{3\kappa+\kappa_1+2} + \frac{w^2}{fq^2}(qT)^{2\kappa+3\kappa_1+2} + \frac{w^2}{fq}(qT)^{4\kappa+\kappa_1+1} \right]^{\frac{1}{4}}. \end{aligned}$$

As $whf \leq q$, and since $f \leq h$ then $f \leq q^{\frac{1}{2}}$, we see that this is at most

$$\begin{aligned} &\ll_\epsilon w^{-1}(qT)^{\frac{\kappa}{2} + \frac{\kappa_2}{2} + \frac{1}{2} + \epsilon} + w^{-1}(qT)^\epsilon \left[q(qT)^{6\kappa+\kappa_2} + q(qT)^{6\kappa+\kappa_1} \right. \\ &+ (qT)^{5\kappa+1} + (qT)^{4\kappa+2\kappa_1+1} + (qT)^{6\kappa} + q(qT)^{5\kappa+1} + q(qT)^{4\kappa+2\kappa_1+1} \\ &\left. + (qT)^{3\kappa+\kappa_1+2} + (qT)^{2\kappa+3\kappa_1+2} + q(qT)^{4\kappa+\kappa_1+1} \right]^{\frac{1}{4}}. \end{aligned}$$

Suppose that $\frac{\kappa}{4} \leq \kappa_1 \leq \frac{\kappa}{2}$ and so $\kappa_2 = \kappa - \kappa_1$, which means in this case the error term simplifies down to

$$\ll_\epsilon w^{-1}q^{\frac{1}{4}}(qT)^{\frac{27\kappa}{16} + \epsilon} + w^{-1}(qT)^{\frac{7\kappa}{8} + \frac{1}{2} + \epsilon}.$$

Then, we may bound the sums arising from $Z_d^{(1)}(x)$ and $Z_d^{(3)}$ in the regimes where $\frac{\kappa}{4} \leq \kappa_1 \leq \frac{\kappa}{2}$ by $\ll_\epsilon w^{-1}q^{\frac{1}{4}}(qT)^{\frac{27\kappa}{16} + \epsilon} + w^{-1}(qT)^{\frac{7\kappa}{8} + \frac{1}{2} + \epsilon}$, and using (2.28) for $Z_d^{(3)} \ll Z_{d,1}^{(3)}$ when $\kappa_1 \leq \frac{\kappa}{4}$ we find that we may bound $\mathcal{E}_{w,f,h}^\pm$ by at most

$$\ll_\epsilon w^{-1}q^{\frac{1}{4}} \left((qT)^{\frac{7\kappa}{8} + \frac{1}{2} + \epsilon} + (qT)^{\frac{7\kappa}{4} + \epsilon} \right).$$

This is $\ll_\epsilon w^{-1}(qT)^{1-\epsilon}$ for $\kappa < \frac{4+\frac{1}{\gamma}}{7(1+\frac{1}{\gamma})} - \delta$, for any small positive δ which concludes the proof of Proposition 2.1.

Chapter 3

Applications of the twisted second moment

The contents of this chapter are based on sections of the pre-print [31], which has been submitted for publication. For the purposes of making it suitable for the thesis format, they have been edited, re-written, and improved in the process.

This chapter splits neatly into two sections to prove a zero-density theorem, and to prove a result on the critical zeros of Dirichlet L -functions. These are consequences of the results in Chapter 2. By considering a two-piece mollifier, as found in (for example) [16, 100, 122] or even a four-piece mollifier such as in [107], the results of this chapter could be made stronger. However, for the sake of simplicity and clarity we only work with a one-piece mollifier.

It appears that Montgomery's bound (1.33)

$$\sum_{\chi \pmod{q}} N(\sigma, T; \chi) \ll (qT)^{\frac{3(1-\sigma)}{2-\sigma}} (\log qT)^9$$

is the strongest bound for σ close to $\frac{1}{2}$, and has been for over 50 years. In this chapter, we will prove a result that improves this bound very close to the critical line. In the second half of the chapter we will prove a result on the proportion of critical zeros of $\prod_{\chi \pmod{q}}^* L(s, \chi)$ up to height T with $\log q = O(\log T)$. In this range both q and T may be varied with great flexibility.

3.1 A zero-density theorem

We shall prove the following theorem.

Theorem 3.1. *Suppose that q is a positive integer and T is a positive real number with $T \gg q^\epsilon$, then for $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{K \log \log qT}{\log qT}$ and any $\delta > 0$*

$$\sum_{\chi \pmod{q}}^* N(\sigma, T; \chi) \ll_{\epsilon, \delta} (qT)^{1 - (\frac{69}{128} - \delta)(2\sigma - 1)} \log qT. \quad (3.1)$$

The bound (3.1) is an improvement on (1.33) for

$$\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{31 \log \log qT}{\log qT}$$

and using Theorem 2.3 as T grows compared to q , the range of improvement can be pushed to a limit of

$$\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{42 \log \log qT}{\log qT}. \quad (3.2)$$

To see this, note that

$$(qT)^{1 - \kappa(2\sigma - 1)} \log qT \ll (qT)^{\frac{3(1 - \sigma)}{2 - \sigma}} (\log qT)^9$$

if

$$1 - \kappa(2\sigma - 1) \leq \frac{3(1 - \sigma)}{2 - \sigma} + 8L$$

where $L = \frac{\log \log qT}{\log qT}$. This rearranges to

$$2\kappa\sigma^2 + \sigma(2 - 5\kappa + 8L) + 2\kappa - 1 - 16L \leq 0$$

which is true when

$$\frac{1}{2} \leq \sigma \leq \frac{5\kappa - 2 - 8L + \sqrt{4 - 12\kappa + 9\kappa^2 + L(32 + 48\kappa L) + 64L^2}}{4\kappa}.$$

We can write this upper bound as

$$\frac{5\kappa - 2 - 8L + \sqrt{(2 - 3\kappa + L(\frac{16 + 24\kappa}{2 - 3\kappa} - \delta))^2 + 2\delta(2 - 3\kappa)L - L^2 \left((\frac{16 + 24\kappa}{2 - 3\kappa} - \delta)^2 - 64 \right)}}{4\kappa}$$

for any small positive δ . For large enough qT (and hence small enough L), this upper bound is greater than

$$\frac{5\kappa - 2 - 8L + 2 - 3\kappa + \frac{16 + 24\kappa}{2 - 3\kappa}L - \delta L}{4\kappa} = \frac{1}{2} + \frac{12L}{2 - 3\kappa} - \frac{\delta L}{4\kappa}$$

$$= \frac{1}{2} + \frac{(12 - \delta')L}{2 - 3\kappa}$$

by re-scaling δ . Making the substitution $\kappa = \frac{69}{128}$ gives that

$$(qT)^{1-\kappa(2\sigma-1)} \log qT \ll (qT)^{\frac{3(1-\sigma)}{2-\sigma}} (\log qT)^9$$

if

$$\frac{1}{2} \leq \sigma \leq \frac{1}{2} + (31.35 - \delta) \frac{\log \log qT}{\log qT}$$

for any small positive δ . Taking $\kappa = \frac{4}{7}$ gives the bound (3.2). This also implies that if one could prove the asymptotic formula in Theorem 2.1 holds for $\kappa > \frac{2}{3}$, then this would immediately give an improvement on Montgomery's zero-density formula for an interval of positive width.

In order to prove Theorem 3.1, we define

$$N(\sigma, 2T, T; \chi) := N(\sigma, 2T; \chi) - N(\sigma, T; \chi)$$

and prove that

$$\sum_{\chi \pmod{q}}^* N(\sigma, 2T, T; \chi) \ll_{\epsilon, \delta} (qT)^{1 - (\frac{69}{128} - \delta)(2\sigma-1)} \log qT.$$

This is sufficient, as then (3.1) follows by a standard dyadic argument. A little care needs to be taken with the low-lying zeros, specifically those with height $\ll q^\epsilon$. We use the fact that $N(T+1, \chi) - N(T, \chi) \ll \log qT$ for an implied absolute constant (for a proof of this, see Proposition 5.7 of [72]), to bound the number of these zeros.

To prove zero-density theorems for $\zeta(s)$, Theorem 9.16 of [115] is very useful. To prove Theorem 3.1 we will use a version of this theorem, adapted for the context of Dirichlet L -functions, as a little extra care must be taken because of the extra sum over the q characters.

Suppose that for each Dirichlet character χ of modulus q we have a mollifier

$$M_\chi(s) = \sum_{a \leq X} \frac{\alpha_a \chi(a)}{a^s}$$

where $\alpha_a = \mu(a)$ for $1 \leq a \leq Y \leq X$ and $|\alpha_a| \leq 1$ for all a . These mollifiers allow us to use the following lemma, which we shall prove later.

Lemma 3.1. *Say $l(\sigma)$ is a positive non-increasing function for $\sigma \geq \frac{1}{2}$ with $l(\frac{1}{2}) = 1$, bounded derivative and $(qT)^{1-l(\sigma)} \leq Y \leq X \leq (qT)^A$. Suppose*

$$\sum_{\chi \pmod{q}}^* \int_T^{2T} |L(\sigma + it, \chi) M_\chi(\sigma + it) - 1|^2 dt \ll (qT)^{l(\sigma)}$$

uniformly for $\sigma > \frac{1}{2}$, then

$$\sum_{\chi \pmod{q}}^* N(\sigma, 2T, T; \chi) \ll (qT)^{l(\sigma)} \log qT$$

uniformly for $\frac{1}{\log qT} < \sigma - \frac{1}{2} \ll \frac{\log \log qT}{\log qT}$.

As Theorem 3.1 is trivially true when $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{1}{\log(qT)}$, this reduces the proof of Theorem 3.1 to finding a bound for

$$\sum_{\chi \pmod{q}}^* \int_T^{2T} |L(\sigma + it, \chi) M_\chi(\sigma + it) - 1|^2 dt$$

which is equal to

$$\begin{aligned} \sum_{\chi \pmod{q}}^* \int_T^{2T} |L(\sigma + it, \chi) M_\chi(\sigma + it)|^2 dt - 2\Re \left(\int_T^{2T} |L(\sigma + it, \chi) M_\chi(\sigma + it)|^2 dt \right) \\ + \phi^*(q)T. \end{aligned}$$

We treat the two integrals separately, and show that their main terms cancel with the $\phi^*(q)T$ term when a suitable mollifier is chosen subject to the constraint that $\alpha_n = \mu(n)f(n)$ for a real smooth function $f(n)$ satisfying the conditions in Theorem 2.2. To do this, we choose

$$\alpha_n = \frac{\mu(n)}{n^{\sigma - \frac{1}{2}}} \kappa_\delta(n) \tag{3.3}$$

where

$$\kappa_\delta(n) = \begin{cases} 1 & 1 \leq n \leq (qT)^{\kappa - \delta}, \\ \frac{\log((qT)^\kappa/n)}{\delta \log qT} & (qT)^{\kappa - \delta} < n < (qT)^\kappa, \\ 0 & (qT)^\kappa \leq n, \end{cases}$$

for a small fixed $\delta > 0$. This then gives the following two results, from which Theorem 3.1 immediately follows by using standard methods to remove the function ψ_t .

Lemma 3.2. *For $\frac{1}{\log(qT)} \leq \sigma - \frac{1}{2} \ll \frac{\log \log(qT)}{\log qT}$ and $\kappa_\delta(n)$ defined as above for a fixed $\delta > 0$,*

$$\int_{\chi \pmod{q}} \sum^* \left| L(\sigma + it, \chi) \sum_{a \leq (qT)^\kappa} \frac{\mu(a) \kappa_\delta(a) \chi(a)}{a^{\sigma+it}} \right|^2 \psi_t dt = \hat{\psi}_0 \phi^*(q) + O_\delta((qT)^{1+(\kappa-\delta)(1-2\sigma)}) . \quad (3.4)$$

Lemma 3.3. For $\sigma \geq \frac{1}{2}$,

$$\int_{\chi \pmod{q}} \sum^* L(\sigma + it, \chi) M(\sigma + it, \chi) \psi_t dt = \hat{\psi}_0 \phi^*(q) + O_\epsilon((qT)^{\epsilon+\kappa(1-\sigma)}) . \quad (3.5)$$

3.1.1 Proof of Lemma 3.2

To show (3.4), we use Theorem 2.2, taking $\alpha = \beta = \sigma - \frac{1}{2}$ and $\beta_n = \alpha_n$ as defined in (3.3) to find

$$\begin{aligned} & \frac{1}{\phi^*(q)T} \int_{\chi \pmod{q}} \sum^* |L(\sigma + it, \chi)|^2 \sum_{a,b \leq (qT)^\kappa} \frac{\alpha_a \alpha_b \chi(a) \bar{\chi}(b)}{a^{1/2+it} b^{1/2-it}} \psi_t dt \\ &= \frac{\hat{\psi}_0}{T} \zeta_q(2\sigma) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a,b)=1 \\ (abd,q)=1}} \frac{\mu(ad) \mu(bd) \kappa_\delta(ad) \kappa_\delta(bd)}{(abd)^{2\sigma}} + \frac{1}{2T} \left(\frac{q}{\pi}\right)^{1-2\sigma} \zeta_q(2-2\sigma) \\ & \quad \times \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a,b)=1 \\ (abd,q)=1}} \frac{\mu(ad) \mu(bd) \kappa_\delta(ad) \kappa_\delta(bd)}{abd^{2\sigma}} \int G(\alpha, t) \psi_t dt + O_\epsilon((qT)^{-\epsilon}) \end{aligned}$$

where $G(\alpha, t)$ is defined in (2.3). By Stirling's approximation (1.16), $G(\alpha, t) \ll |t|^{1-2\sigma}$ so we are aiming to show that

$$S_1 := \zeta_q(2\sigma) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a,b)=1 \\ (abd,q)=1}} \frac{\mu(ad) \mu(bd) \kappa_\delta(ad) \kappa_\delta(bd)}{(abd)^{2\sigma}} = 1 + O((qT)^{(\kappa-\delta)(1-2\sigma)}) \quad (3.6)$$

and that

$$S_2 := \zeta_q(2-2\sigma) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a,b)=1 \\ (abd,q)=1}} \frac{\mu(ad) \mu(bd) \kappa_\delta(ad) \kappa_\delta(bd)}{abd^{2\sigma}} = O((qT)^{(2\sigma-1)(1-\kappa)}) . \quad (3.7)$$

Focussing first on S_1 , as $\sigma > \frac{1}{2}$, $\zeta_q(2\sigma)$ has a convergent Dirichlet series and so

$$\zeta_q(2\sigma) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a,b)=1 \\ (abd,q)=1}} \frac{\mu(ad) \mu(bd) \kappa_\delta(ad) \kappa_\delta(bd)}{(abd)^{2\sigma}} = \sum_{\substack{a,b,d,\lambda \\ (a,b)=1 \\ (abd\lambda,q)=1}} \frac{\mu(ad) \mu(bd) \kappa_\delta(ad) \kappa_\delta(bd)}{(abd\lambda)^{2\sigma}}$$

$$= \sum_{\substack{am=bn \\ (abmn, q)=1}} \frac{\mu(a)\mu(b)\kappa_\delta(a)\kappa_\delta(b)}{(abmn)^\sigma}$$

by relabelling $ad = a, bd = b, a\lambda = n$, and $b\lambda = m$. Let $c(n) = \sum_{am=n} \mu(a)\kappa_\delta(a)$, so by Möbius inversion

$$c(n) = \begin{cases} 1 & n = 1, \\ 0 & 1 < n \leq (qT)^{\kappa-\delta}. \end{cases}$$

Hence,

$$S_1 = 1 + O\left(\sum_{n \geq (qT)^{\kappa-\delta}} \frac{c(n)^2}{n^{2\sigma}}\right).$$

To extend the sum in the error term by a sum over all natural numbers, we use a method known as *Rankin's trick*. This method is essentially applying the inequality

$$\sum_{n \geq N} f(n) \leq \sum_{n=1}^{\infty} f(n) \left(\frac{n}{N}\right)^\delta$$

which holds for any positive function $f(n)$ and any positive δ . Specifically in our case, we set $w = 1 + \frac{A}{\log qT}$ with $A > 0$ such that $2\sigma - w > 0$

$$\begin{aligned} \sum_{n \geq (qT)^{\kappa-\delta}} \frac{c(n)^2}{n^{2\sigma}} &\leq \sum_{n=1}^{\infty} \frac{c(n)^2}{n^{2\sigma}} \left(\frac{n}{(qT)^{\kappa-\delta}}\right)^{2\sigma-w} \ll (qT)^{(\kappa-\delta)(w-2\sigma)} \sum_{n=1}^{\infty} \frac{c(n)^2}{n^w} \\ &\ll (qT)^{(\kappa-\delta)(1-2\sigma)} \sum_{n=1}^{\infty} \frac{c(n)^2}{n^w} \end{aligned}$$

as $(qT)^{\frac{A}{\log qT}} = e^A$. For $z > 0$, define

$$K_n(z) = \begin{cases} \log \frac{(qT)^z}{n} & n \leq (qT)^z, \\ 0 & n > (qT)^z. \end{cases}$$

Therefore

$$\kappa_\delta(n) = \frac{K_n(\kappa) - K_n(\kappa - \delta)}{\delta \log qT}$$

and so

$$c(n)^2 \ll_\delta \frac{\left(\sum_{d|n} \mu(d) K_d(\kappa)\right)^2 + \left(\sum_{d|n} \mu(d) K_d(\kappa - \delta)\right)^2}{(\log qT)^2}. \quad (3.8)$$

We use a result by Motohashi (see [94], proof of Lemma 5) that

$$\sum_{n=1}^{\infty} \left(\sum_{d|n} \mu(d) K_d(z)\right)^2 n^{-w} \ll z(\log qT)^2. \quad (3.9)$$

so by (3.8) and (3.9),

$$\sum_{n=1}^{\infty} \frac{c(n)^2}{n^w} \ll_{\delta} 1$$

and hence

$$S_1 = 1 + O_{\delta} \left((qT)^{(\kappa-\delta)(1-2\sigma)} \right). \quad (3.10)$$

To bound S_2 , we note that by a Mellin formula, for $1 \leq n \leq (qT)^z$

$$\log \frac{(qT)^z}{n} = \frac{1}{2\pi i} \int_{(1)} \left(\frac{(qT)^z}{n} \right)^u \frac{du}{u^2} \quad (3.11)$$

and so

$$\kappa_{\delta}(n) = \frac{1}{\delta 2\pi i \log qT} \int_{(1)} \frac{(qT)^{\kappa u} - (qT)^{(\kappa-\delta)u}}{n^u} \frac{du}{u^2}.$$

This means that

$$S_2 = \frac{\zeta_q(2-2\sigma)}{(\delta 2\pi i \log qT)^2} \int_{(1)} \int_{(1)} \sum_{\substack{a,b,d \\ (a,b)=1 \\ (abd,q)=1}} \frac{\mu(ad)\mu(bd)}{a^{1+u}b^{1+v}d^{2\sigma+u+v}} \frac{f(u)f(v)}{u^2v^2} du dv$$

where

$$f(u) = (qT)^{\kappa u} - (qT)^{(\kappa-\delta)u}.$$

By considering Euler products

$$\begin{aligned} \sum_{\substack{a,b,d \\ (a,b)=1 \\ (abd,q)=1}} \frac{\mu(ad)\mu(bd)}{a^{1+u}b^{1+v}d^{2\sigma+u+v}} &= \prod_{p \nmid q} \left(1 + \frac{1}{p^{2\sigma+u+v}} - \frac{1}{p^{1+u}} - \frac{1}{p^{1+v}} \right) \\ &= \frac{\zeta_q(2\sigma+u+v)}{\zeta_q(1+u)\zeta_q(1+v)} A(u,v) \end{aligned}$$

where $A(u,v)$ is equal to the product

$$\begin{aligned} &\prod_{p \nmid q} \left(1 - \frac{1}{p^{2\sigma+u+v}} \right) \left(1 - \frac{1}{p^{1+u}} \right)^{-1} \left(1 - \frac{1}{p^{1+v}} \right)^{-1} \left(1 + \frac{1}{p^{2\sigma+u+v}} - \frac{1}{p^{1+u}} - \frac{1}{p^{1+v}} \right) \\ &= \prod_{p \nmid q} \left(1 - \frac{1}{p^{1+u}} \right)^{-1} \left(1 - \frac{1}{p^{1+v}} \right)^{-1} \left(\left(1 - \frac{1}{p^{1+u}} \right) \left(1 - \frac{1}{p^{1+v}} \right) + O\left(p^{-\frac{3}{2}}\right) \right) \\ &= \prod_{p \nmid q} \left(1 + O\left(p^{-\frac{3}{2}}\right) \right), \end{aligned}$$

as long as $\Re(u), \Re(v) > -\frac{1}{6}$. Then

$$\prod_{p \nmid q} \left(1 + O\left(p^{-\frac{3}{2}}\right) \right) \ll \exp \left(\sum_{p \nmid q} O\left(p^{-\frac{3}{2}}\right) \right) \ll 1.$$

Hence $A(u, v)$ converges absolutely for $\Re(u), \Re(v) > -\frac{1}{6}$, and

$$S_2 = \frac{\zeta_q(2-2\sigma)}{(\delta 2\pi i \log qT)^2} \int_{(1)} \int_{(1)} \frac{\zeta_q(2\sigma+u+v)A(u, v)f(u)f(v)}{\zeta_q(1+u)\zeta_q(1+v)u^2v^2} du dv.$$

The double integral has no singularities for $\Re(u), \Re(v) > 0$ so we may move the contours of integration to $\Re(u) = \Re(v) = \gamma := \frac{(2\sigma-1)(1-\kappa)}{2\kappa}$. On these contours

$$f(u)f(v) \ll (qT)^{(2\sigma-1)(1-\kappa)}, \quad \zeta_q(2\sigma+u+v) \ll \zeta_q(2\sigma)$$

and (by comparing Euler products) we can see that for $\Re(u) > 0$

$$|\zeta_q(1+u)|^{-1} \leq \frac{q}{\phi(q)} |\zeta(1+u)|^{-1} \leq \frac{q\zeta(2+2\gamma)}{\phi(q)\zeta(1+\gamma)} \ll \frac{q}{\phi(q)} \gamma,$$

as $\zeta^{-1}(1+\gamma) = \gamma(1+O(\gamma))$. With these bounds, we can see

$$\begin{aligned} S_2 &\ll \frac{q^2 |\zeta_q(2-2\sigma)| \zeta_q(2\sigma) (qT)^{(2\sigma-1)(1-\kappa)} \gamma^2}{\phi(q)^2 (\log qT)^2} \left(\int_0^\infty \frac{1}{\gamma^2 + t^2} dt \right)^2 \\ &\ll \frac{q^2 |\zeta_q(2-2\sigma)| \zeta_q(2\sigma) (qT)^{(2\sigma-1)(1-\kappa)}}{\phi(q)^2 (\log qT)^2} \\ &\ll \frac{\phi_{2-2\sigma}(q) \phi_{2\sigma}(q)}{\phi(q)^2 (2\sigma-1)^2 (\log qT)^2} (qT)^{(2\sigma-1)(1-\kappa)} \\ &\ll \frac{\phi_{2-2\sigma}(q) \phi_{2\sigma}(q)}{\phi(q)^2} (qT)^{(2\sigma-1)(1-\kappa)} \end{aligned}$$

as by assumption $2\sigma-1 \gg \frac{1}{\log qT}$. By examining the product formula of $\phi_z(q)$ we see that

$$\frac{\phi_{2-2\sigma}(q) \phi_{2\sigma}(q)}{\phi(q)^2} = \prod_{p|q} \left(\frac{(p^{2-2\sigma}-1)(p^{2\sigma}-1)}{(p-1)^2} \right) = \prod_{p|q} \left(\frac{1-p^{2\sigma-2}-p^{-2\sigma}+p^{-2}}{1-2p^{-1}+p^{-2}} \right) \leq 1$$

and hence

$$S_2 \ll (qT)^{(2\sigma-1)(1-\kappa)}.$$

3.1.2 Proof of Lemma 3.3

Suppose that χ is a primitive character of conductor q , $\sigma \in [1/2, 1]$, $t \in [T, 2T]$ and define

$$A := \sum_n \frac{\chi(n)}{n^{\sigma+it}} e^{-\frac{n}{(qT)^2}}.$$

Then, as

$$e^{-x} = \frac{1}{2\pi i} \int_{(1)} \Gamma(s) x^{-s} ds,$$

The sum A may be written as

$$\frac{1}{2\pi i} \int_{(1)} (qT)^{2s} \Gamma(s) L(\sigma + it + s, \chi) ds.$$

Moving the contour of integration to have real part $-1 + \epsilon$, we hit a pole at $s = 0$. The integral at the new contour may be bounded by the exponential decay of the Gamma function, and by the functional equation for Dirichlet L -functions (1.26),

$$L(\sigma - 1 + \epsilon + it, \chi) \ll (qT)^{\frac{3}{2} - \sigma - \epsilon} |L(2 - \sigma - \epsilon - it, \bar{\chi})| \ll (qT)^{1 - \epsilon}$$

so that

$$\int_{(-1+\epsilon)} (qT)^{2s} \Gamma(s) L(\sigma + it + s, \chi) ds \ll_{\epsilon} (qT)^{-1+\epsilon}.$$

Hence

$$L(\sigma + it, \chi) = A + O_{\epsilon}((qT)^{-1+\epsilon}). \quad (3.12)$$

By (3.12) and the orthogonality of primitive Dirichlet characters,

$$\int \sum_{\chi \pmod{q}}^* L(\sigma + it, \chi) M(\sigma + it, \chi) \psi_t dt$$

is equal to

$$\sum_{w|q} \mu\left(\frac{q}{w}\right) \phi(w) \sum_{\substack{n \leq (qT)^{2+\epsilon} \\ a \leq (qT)^{\kappa} \\ an \equiv 1 \pmod{w} \\ (an, q) = 1}} \frac{\alpha_a e^{-\frac{n}{(qT)^2}}}{(an)^{\sigma}} \int (an)^{-it} \psi_t dt + O_{\epsilon}((qT)^{\epsilon + \kappa(1-\sigma)}).$$

If $an > 1$ then by integration by parts K times

$$\int (an)^{it} \psi_t dt \ll_K (\log an)^{-K} T^{1+\epsilon-K}$$

so we may make the error term arbitrarily small. When $an = 1$ then the integral is just $\hat{\psi}_0$. Hence

$$\int \sum_{\chi \pmod{q}}^* L(\sigma + it, \chi) M(\sigma + it, \chi) \psi_t dt = \phi^*(q) \hat{\psi}_0 + O_{\epsilon}((qT)^{\epsilon + \kappa(1-\sigma)}).$$

3.1.3 Proof of Lemma 3.1

Write $f_{\chi}(s) = L(s, \chi) M_{\chi}(s) - 1$. And note that for $\Re(s) > 1$, by the Möbius formula

$$f_{\chi}(s) = \sum_{2 \leq n \leq Y} \frac{\chi(n) \sum_{a|n} \mu(a)}{n^s} + \sum_{n \geq Y} \frac{\chi(n) \sum_{a|n} \alpha_a}{n^s} \ll \sum_{n \geq Y} \frac{\tau(n)}{n^{\Re(s)}} \ll Y^{1+\epsilon-\Re(s)}. \quad (3.13)$$

Also, by the bound $L(\sigma + it, \chi) \ll qT$ for $\sigma \geq \frac{1}{2}$, $|t| \leq T$ by (1.27), and trivially bounding $M_\chi(\sigma + it) \ll (qT)^\epsilon X^{1-\sigma}$ we see that for all $\Re(s) > \frac{1}{2}$

$$f_\chi(s) \ll (qT)^K \quad (3.14)$$

for some fixed absolute K . Let $h_\chi(s) = 1 - f_\chi(s)^2$, then by applying Littlewood's lemma in the form of Lemma 1.2, one can find that for $\sigma > \frac{1}{2} + \frac{1}{\log qT}$

$$\begin{aligned} \frac{N(\sigma, 2T, T; \chi)}{\log qT} &\ll \int_T^{2T} |\log h_\chi(\sigma + it)| + |\log h_\chi(2 + it)| dt \\ &\quad + \int_{\frac{1}{2}}^2 |\arg(h_\chi(z + i2T))| + |\arg(h_\chi(z + iT))| dz. \end{aligned}$$

Lemma 1.1 applied to the second integral, along with (3.14) shows that

$$\int_{\frac{1}{2}}^2 |\arg(h_\chi(z + i2T))| + |\arg(h_\chi(z + iT))| dz \ll \log qT.$$

As $\log |h_\chi(s)| \leq \log(1 + |f_\chi(s)|^2) \leq |f_\chi(s)|^2$ we see by (3.13) that

$$\int_T^{2T} \log |h_\chi(2 + it)| dt \ll_\epsilon TY^{-2+\epsilon}$$

and

$$\int_T^{2T} \log |h_\chi(\sigma + it)| dt \ll \int |f_\chi(\sigma + it)|^2 dt.$$

Putting this all together

$$\sum_{\chi \pmod{q}}^* N(\sigma, 2T, T; \chi) \ll (qT)^{l(\sigma)} \log(qT) + \phi^*(q)(\log qT)^2.$$

As $T \gg q^\epsilon$, $l(\sigma) - 1 \ll \sigma - \frac{1}{2}$, and $\sigma - \frac{1}{2} \ll \frac{\log \log qT}{\log qT}$, we see that

$$\phi^*(q)(\log qT)^2 \ll (qT)^{l(\sigma)} \log qT$$

and the lemma follows.

3.2 Zeros on the critical line

In this section, we describe Levinson's method for finding proportions of zeros on the critical line, then how it can be applied with Theorem 2.2 to find a result on the proportion of simple critical zeros of Dirichlet L -functions.

3.2.1 Levinson's method

As mentioned in the introduction, Levinson's method in its original form is considered long and complicated, but essentially the main idea breaks down into three sections: reducing the problem to one of calculating a certain twisted second moment, calculating that moment, and optimising various aspects of the calculation. The method has been innovated over time, see for example [123], where Young vastly simplifies the work needed to evaluate the second moment, cutting the necessary calculations down from over 30 pages in Levinson's original paper to less than six. Also see the appendix of [26] for a very clear demonstration of the necessary ideas in Levinson's method for any L -function.

In this subsection, we will follow the process in [26] to demonstrate the set-up of Levinson's method in the context of Dirichlet L -functions, to prove the following theorem.

Theorem 3.2 (Levinson's theorem). *Let $L := \log qT$, $Q(x)$ be a real linear polynomial with $Q(0) = 1$, and $P(x)$ a polynomial with $P(0) = 0$ and $P(1) = 1$. For a Dirichlet character χ of modulus q , define*

$$V_\chi(s) = Q\left(-\frac{1}{L} \frac{d}{ds}\right) L(s, \chi)$$

and

$$M(s, \chi) = \sum_{a \leq X} \frac{\chi(a)\mu(a)}{a^s} P\left(\frac{\log \frac{X}{a}}{\log X}\right).$$

Then for any positive constant R ,

$$\frac{N_0^*(2T, T, q)}{N(2T, T, q)} \geq 1 - \frac{1}{R} \log \left(\frac{1}{\phi^*(q)T} \int_T^{2T} \sum_{\chi \pmod{q}}^* \left| V_\chi\left(\frac{1}{2} - \frac{R}{L} + it\right) M\left(\frac{1}{2} + it\right) \right|^2 dt \right) + o(1),$$

where

$$N_0^*(2T, T, q) = \sum_{\chi \pmod{q}}^* N_0^*(2T, \chi) - N_0^*(T, \chi)$$

and

$$N(2T, T, q) = \sum_{\chi \pmod{q}}^* N(2T, \chi) - N(T, \chi).$$

One can find a slightly higher proportion of zeros on the critical line by allowing $Q(x)$ to be a polynomial of degree greater than 1 (but still subject to some conditions, see [20] for details), however this improvement comes by sacrificing the restriction to counting simple zeros.

To prove this theorem, suppose that χ is a primitive even Dirichlet character of conductor q (due to the difference in functional equation, odd and even characters must be treated separately) so by (1.26)

$$L(s, \chi) \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \eta_\chi = L(1-s, \bar{\chi}) \left(\frac{q}{\pi}\right)^{\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \bar{\eta}_\chi, \quad (3.15)$$

where

$$\frac{\bar{\eta}_\chi}{\eta_\chi} = \frac{\tau(\chi)}{q^{\frac{1}{2}}}.$$

The idea is to use (3.15), and to choose functions $Y(s)$ and $G(s, \chi)$ such that

$$\begin{aligned} Y(s)L(s, \chi) \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \eta_\chi &= G(s, \chi) \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \eta_\chi \\ &\quad + G(1-s, \bar{\chi}) \left(\frac{q}{\pi}\right)^{\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \bar{\eta}_\chi \end{aligned} \quad (3.16)$$

where $Y(s) = Y(1-s)$. This would imply that for s on the critical line

$$Y(s)L(s, \chi) \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \eta_\chi = 2\Re \left(G(s, \chi) \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \eta_\chi \right)$$

and so, as $\left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \eta_\chi \neq 0$,

$$\begin{aligned} Y(s)L(s, \chi) = 0 &\Leftrightarrow \Re \left(G(s, \chi) \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \eta_\chi \right) = 0 \\ &\Leftrightarrow G(s, \chi) = 0 \\ &\text{or } \left(\arg G(s, \chi) \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \eta_\chi \equiv \frac{\pi}{2} \pmod{\pi} \text{ and } G(s, \chi) \neq 0 \right). \end{aligned}$$

Every time the argument of $G(s, \chi) \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \eta_\chi$ increases by π , it must have been equal to $\frac{\pi}{2}$ at least once. Let $\mathcal{C}$ be the contour from $\frac{1}{2} + iT$ to $\frac{1}{2} + i2T$, bypassing each zero of $G(s, \chi)$ on the right hand side. Define

$$N_0(2T, T, \chi) = \# \left\{ \gamma : L\left(\frac{1}{2} + i\gamma, \chi\right) = 0, T \leq \gamma \leq 2T \right\}$$

counting multiplicity, and

$$N_0(2T, T, Y) = \# \left\{ \gamma : Y\left(\frac{1}{2} + i\gamma\right) = 0, T \leq \gamma \leq 2T \right\}$$

then it follows that

$$N_0(2T, T, Y) + N_0(2T, T, \chi) \geq \frac{1}{\pi} \Delta_C \arg G(s, \chi) \left(\frac{q}{\pi} \right)^{\frac{s}{2}} \Gamma \left(\frac{s}{2} \right).$$

To calculate the right hand side, note that

$$\Delta_C \arg \left(\frac{q}{\pi} \right)^{\frac{s}{2}} = \Delta_C \frac{t}{2} \log \frac{q}{\pi} = \frac{T}{2} \log \frac{q}{\pi}$$

and by Stirling's approximation (1.16)

$$\begin{aligned} \Delta_C \arg \Gamma \left(\frac{s}{2} \right) &= \Delta_C \Im \left(\log \Gamma \left(\frac{s}{2} \right) \right) \\ &= \Im \left(\left(\frac{1}{4} + iT \right) \log \left(\frac{1}{4} + iT \right) \right. \\ &\quad \left. - \left(\frac{1}{4} + \frac{iT}{2} \right) \log \left(\frac{1}{4} + \frac{iT}{2} \right) - \frac{iT}{2} + O(1) \right) \\ &= \frac{T}{2} \log(T) + \frac{T}{2} \log(2) - \frac{T}{2} + O(1), \end{aligned}$$

so by (1.28),

$$\frac{1}{\pi} \Delta_C \arg G(s, \chi) \left(\frac{q}{\pi} \right)^{\frac{s}{2}} \Gamma \left(\frac{s}{2} \right) = N(2T, T, \chi) + \frac{1}{\pi} \Delta_C \arg G(s, \chi) + O(\log qT).$$

At this point, it is useful to make some choices about $Y(s)$ and $G(s, \chi)$ so that we may get as tight an inequality as possible. We will also choose our functions to make the calculations a little easier. Ideally we would like $Y(s)$ to have few zeros, and if we can show that $N_0(2T, T, Y) \ll \log qT$ then we can absorb their contribution into the error term. We will let

$$G(s, \chi) = \sum_{n=1}^{\infty} \frac{a_{\chi}(n)}{n^s}$$

for some complex coefficients $a_{\chi}(n) \ll n^{\epsilon}$ such that $G(s, \chi)$ converges absolutely when $\Re(s) > 1 + 2\epsilon$. This choice means that it is easier to calculate $\Delta_C \arg G(s, \chi)$.

Let $\mathcal{R}$ be the rectangle with $\mathcal{C}$ as its left side and $[2 + iT, 2 + 2iT]$ as its right side. Label the north side of the rectangle $\mathcal{R}_N$, the east side $\mathcal{R}_E$ and the south side $\mathcal{R}_S$ so that

$$\Delta_C \arg G(s, \chi) = \Delta_{\mathcal{R}} \arg G(s, \chi) - \Delta_{\mathcal{R}_N} \arg G(s, \chi) - \Delta_{\mathcal{R}_E} \arg G(s, \chi) - \Delta_{\mathcal{R}_S} \arg G(s, \chi).$$

The fact that $G(s, \chi)$ converges absolutely on $\mathcal{R}_E$ means that using standard techniques (see for example the proof of Proposition 5.8 in [72]) one can bound the contributions from the north, east, and south sides by at most $O(\log qT)$. Hence

$$\Delta_C \arg G(s, \chi) = \Delta_{\mathcal{R}} \arg G(s, \chi) + O(\log qT).$$

Since

$$\Delta_{\mathcal{R}} \arg G(s, \chi) = -2\pi N_{\mathcal{R}}(G(s, \chi))$$

where $N_{\mathcal{R}}(G(s, \chi))$ is the number of zeros of $G(s, \chi)$ inside $\mathcal{R}$, and the minus sign comes from the direction of integration around $\mathcal{R}$, we find that

$$N_0(2T, T, \chi) \geq N(2T, T, \chi) - 2N_{\mathcal{R}}(G(s, \chi)) + O(\log qT).$$

We are now close to being able to reduce this problem to one of calculating the twisted second moment. We introduce the twist by letting $M(s, \chi)$ be a Dirichlet polynomial and noting that

$$N_{\mathcal{R}}(G(s, \chi)) \leq N_{\mathcal{R}}(G(s, \chi)M(s, \chi)).$$

We want $M(s, \chi)$ to smooth out the behaviour of $G(s, \chi)$, as well as obeying some necessary restrictions and without introducing too many of its own zeros. Suppose

$$M(s, \chi) = \sum_{m \leq X} \frac{b_{\chi}(m)}{m^s} \quad (3.17)$$

with $\log X \ll \log qT$, $b_{\chi}(m) \ll m^{\epsilon}$ and $b_{\chi}(1) = 1$. This means that we can write

$$\log M(s, \chi) = \sum_{m=2}^{\infty} \frac{c_{\chi}(m)}{m^s}$$

(using the Taylor expansion $\log(1+x) = x - \frac{x^2}{2} + \dots$) with $c_{\chi}(m) \ll m^{2\epsilon}$, so the series converges absolutely for $\Re(s) \geq 2$ say, and hence

$$\int_{3+iT}^{3+2iT} \log M(s) ds \ll \sum_{m=2}^{\infty} \frac{|c_{\chi}(m)|}{m^3 \log m} \ll 1. \quad (3.18)$$

This fact shall be useful in a moment after we apply Littlewood's lemma, this time in a slightly different form to that presented in Theorem 1.2. The form useful to us shows that

$$\Re \left(\frac{1}{2\pi i} \int_{\partial \mathcal{D}} \log (G(s, \chi)M(s, \chi)) ds \right) = \sum_{\rho \in \mathcal{D}} \text{dist}(\rho)$$

where the argument of the complex logarithm is defined by continuous variation going counter-clockwise, as long as $G(s, \chi)M(s, \chi)$ is a regular function on the rectangle $\mathcal{D}$ and does not vanish on the boundary $\partial \mathcal{D}$. The sum runs over the zeros ρ of $G(s, \chi)M(s, \chi)$ and $\text{dist}(\rho)$ is the horizontal distance of each ρ to the left side of $\mathcal{D}$. We take $\mathcal{D}$ wider than $\mathcal{R}$, with the left side on the line $\Re(s) = \sigma < \frac{1}{2}$ and the right

side on the line $\Re(s) = 3$. The precise value of σ will be chosen later to get an optimal result. Littlewood's lemma shows that

$$N_{\mathcal{R}}(G(s, \chi)M(s, \chi)) \leq \left(\frac{1}{2} - \sigma\right)^{-1} \Re \left(\frac{1}{2\pi i} \int_{\partial \mathcal{D}} \log(G(s, \chi)M(s, \chi)) ds \right).$$

By (3.18) and applying the same logic to $G(s, \chi)$, the integral along the right side of $\mathcal{D}$ is of size $\ll 1$. The real parts of the integrals along the horizontal sides of $\mathcal{D}$ are integrals of $\arg(G(z + iT, \chi)M(z + iT, \chi))$ and $\arg(G(z + 2iT, \chi)M(z + 2iT, \chi))$. As $M(s, \chi) \ll X^{1-s+\epsilon}$ for $\Re(s) \geq 0$, these integrals are bounded by $\ll \log(TX) \ll \log qT$. Therefore,

$$\begin{aligned} N_{\mathcal{R}}(G(s, \chi)M(s, \chi)) &\leq \left(\frac{1}{2} - \sigma\right)^{-1} \frac{1}{2\pi i} \int_T^{2T} \log |(G(\sigma + it, \chi)M(\sigma + it, \chi))| ds \\ &\quad + O(\log qT) \end{aligned}$$

and a general form of Levinson's theorem follows. However, to prove the version in Theorem 3.2, we need to make some choices about $Y(s)$ and the Dirichlet series $G(s, \chi)$. Recall that we need these functions to satisfy (3.16), with $Y(s) = Y(1-s)$ and $N_0(2T, T, Y) \ll \log qT$. A tempting choice would be to simply take $Y(s) = 2$ and $G(s, \chi) = L(s, \chi)$ so that (3.16) is satisfied immediately by (1.26), the functional equation for $L(s, \chi)$. However there are more optimal choices, with a good source being linear combinations of $L(s, \chi)$ and its derivatives. See [20] for details. By taking the logarithmic derivative of (3.15) and substituting it back into (3.15), one can see that for any constant λ

$$G(s, \chi) = L(s, \chi) + \lambda L'(s, \chi)$$

and

$$Y(s) = 2 - \lambda \left(\log\left(\frac{q}{2}\right) + \frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \right)$$

are acceptable choices. In fact, these are similar to the original functions used by Levinson. As well as giving us the extra variable λ over which to optimise, having $G(s, \chi)$ in this form is useful as it is relatively simple to apply the twisted second moment results, Theorems 2.1, 2.2, and 2.3, to it. Also, note that for a fixed λ and for q, T large, by Stirling's approximation (1.16) for $\frac{\Gamma'(s)}{\Gamma(s)}$ we can see that $Y(s)$ will have no zeros. Also, we can see that if ρ is a zero of order 2 or more then $G(s, \chi) = 0$. In general, if we were to choose a linear combination of $L^{(k)}(s, \chi)$ for $k = 0, 1, \dots, n$ then we could count the proportion of zeros of order n or less on the critical line.

Turning our attention to the mollifier, we see that

$$b_\chi(m) = \chi(m)\mu(m)P\left(\frac{\log \frac{X}{m}}{\log X}\right)$$

for a polynomial with $P(0) = 0$ and $P(1) = 1$ is a suitable choice for the coefficients in (3.17). Introducing the polynomial provides more variables over which to optimise.

The same procedure should be followed for odd characters χ , but with the different functional equation. To complete the proof of Theorem 3.3, we let $\sigma = \frac{1}{2} - \frac{R}{L}$.

3.2.2 Critical zeros of Dirichlet L -functions

We demonstrate a second application of Theorem 2.2, using it in combination with Levinson's method to prove a result on the proportion of simple zeros on the critical line.

Theorem 3.3. *Suppose that q is a positive integer and $T \gg q^\epsilon$ for some fixed positive ϵ . For $\kappa < 1/2 + 5/128$ we have as $qT \rightarrow \infty$*

$$\frac{N_0^*(2T, T, q)}{N(2T, T, q)} \geq 1 - \frac{1}{R} \log(c(P, Q, R)) + o(1),$$

where

$$c(P, Q, R) = 1 + \frac{1}{\kappa} \int_0^1 \int_0^1 e^{2Rv} \left(\frac{d}{dx} e^{R\kappa x} P(x+u) Q(v+\kappa x) \Big|_{x=0} \right)^2 dudv,$$

$R > 0$ is a positive constant, $P(x)$ is a polynomial with $P(0) = 0, P(1) = 1$, and $Q(x)$ is a real linear polynomial with $Q(0) = 1$.

By choosing R, P , and Q optimally, we arrive at the following corollary.

Corollary 3.1. *For sufficiently large qT with $T \gg q^\epsilon$*

$$\frac{N_0^*(T, q)}{N(T, q)} \geq 0.382.$$

Proof. The optimisation process can be found in Section 4 of [20]. This method demonstrates that the optimal choice for $P(x)$ is of the form

$$P(x) = \frac{e^{rx} - e^{sx}}{e^r - e^s}$$

for r, s constants. While this is not a polynomial, it may be uniformly approximated by real polynomials. Choosing

$$Q(x) = 1 - 1.035x, \quad R = 1.179$$

gives

$$1 - \frac{1}{R} \log(c(P, Q, R)) = 0.382156$$

and hence

$$\frac{N_0^*(2T, T, q)}{N(2T, T, q)} \geq 0.382$$

for large enough qT . By a simple dyadic decomposition argument, we can obtain that

$$\frac{N_0^*(T, q)}{N(T, q)} \geq 0.382.$$

□

Informally, this means that for any integer q with $\log q \ll \log T$, at least 38.2% of zeros up to height T of the primitive Dirichlet L -functions of modulus q lie on the critical line. In the regime in which T is large compared to q , by replacing Theorem 2.2 with Theorem 2.3 in the proof of Theorem 3.3, we may improve this ratio up to

$$\frac{N_0^*(2T, T, q)}{N(2T, T, q)} \geq 0.401$$

recovering Conrey's result in [20]. This is an improvement on previous results while $\log q \asymp \log T$ however when $\log q = o(\log T)$ then the results of Wu in [122] supersede this.

To prove Theorem 3.3 we shall follow the elegant reformulation of Levinson's method by Young in [123], but in the context of families of Dirichlet L -functions. Some extra care has to be taken because of the addition of the q -aspect.

3.2.3 Proof of Theorem 3.3

Assume the conditions of Theorem 3.3 with $P(x) = \sum_{i \geq 1} a_i x^i$, and for convenience we shall write $P[a] = P\left(\frac{\log \frac{X}{a}}{\log X}\right)$. By Theorem 3.2

$$\frac{N_0^*(2T, T, q)}{N(2T, T, q)} \geq 1 - \frac{1}{R} \log \left(\frac{1}{\phi^*(q)T} \int_T^{2T} \sum_{\chi \pmod{q}}^* \left| V_\chi \left(\frac{1}{2} - \frac{R}{L} + it \right) M \left(\frac{1}{2} + it \right) \right|^2 dt \right)$$

$$+ o(1) \quad (3.19)$$

as $qT \rightarrow \infty$.

Let

$$I(\alpha, \beta) = \sum_{\chi \pmod{q}}^* \int_{\mathbb{R}} L\left(\frac{1}{2} + \alpha + it, \chi\right) L\left(\frac{1}{2} + \beta - it, \chi\right) \left| M\left(\frac{1}{2} + it, \chi\right) \right|^2 \psi_t dt$$

for $\alpha, \beta \ll L^{-1}$, and for a smooth function ψ_t supported on $[T, 2T]$. We will arrive at the integral in (3.19) by evaluating

$$Q\left(-\frac{1}{L} \frac{d}{d\alpha}\right) Q\left(-\frac{1}{L} \frac{d}{d\beta}\right) I(\alpha, \beta)$$

at $\alpha = \beta = -R/L$ and by using standard techniques to remove ψ_t .

First, however, we manipulate $I(\alpha, \beta)$ into a more convenient form to apply differentiation. Applying Theorem 2.2 for $X = (qT)^\kappa$ with $\kappa < 69/128$ we see that

$$\begin{aligned} \frac{I(\alpha, \beta)}{\phi^*(q)T} &= \frac{\hat{\psi}_0}{T} \zeta_q(1 + \alpha + \beta) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b) = 1 \\ (abd, q) = 1}} \frac{\mu(ad)P[ad]\mu(bd)P[bd]}{a^{1+\beta}b^{1+\alpha}d} \\ &\quad + \frac{\hat{\psi}_0}{T} \left(\frac{qT}{\pi}\right)^{-\alpha-\beta} (1 + O(L^{-1})) \zeta_q(1 - \alpha - \beta) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b) = 1 \\ (abd, q) = 1}} \frac{\mu(ad)P[ad]\mu(bd)P[bd]}{a^{1-\alpha}b^{1-\beta}d} \\ &\quad + O_\epsilon((qT)^{-\epsilon}), \end{aligned}$$

as the ratio of gamma functions in the integral is $t^{-\alpha-\beta}(1 + O(t^{-1})) = T^{-\alpha-\beta}(1 + O(L^{-1}))$ for $t \in [T, 2T]$ by (2.7) and (2.8).

Let

$$S(\alpha, \beta) = \zeta_q(1 + \alpha + \beta) \sum_{\substack{ad, bd \leq (qT)^\kappa \\ (a, b) = 1 \\ (abd, q) = 1}} \frac{\mu(ad)P[ad]\mu(bd)P[bd]}{a^{1+\beta}b^{1+\alpha}d}$$

so

$$\begin{aligned} \frac{I(\alpha, \beta)}{\phi^*(q)T} &= \frac{\hat{\psi}_0}{T} \left(S(\alpha, \beta) + (qT)^{-\alpha-\beta} S(-\beta, -\alpha)(1 + O(L^{-1})) \right) + O_\epsilon((qT)^{-\epsilon}) \\ &= \frac{\hat{\psi}_0}{T} \left(S(\alpha, \beta) + (qT)^{-\alpha-\beta} S(-\beta, -\alpha) \right) + O(L^{-1}) \end{aligned}$$

as long as $S(-\beta, -\alpha) \ll 1$ which we shall show is the case in the following lemma.

Lemma 3.4. *Uniformly on any fixed annuli such that $\alpha, \beta \asymp L^{-1}$, $|\alpha + \beta| \gg L^{-1}$*

$$S(\alpha, \beta) = \frac{1}{(\alpha + \beta) \log X} \frac{d^2}{dx dy} X^{\alpha x + \beta y} \int_0^1 P(x+u)P(y+u)du|_{x=y=0} + O_\epsilon(L^{-1+\epsilon}).$$

By Lemma 3.4

$$S(\alpha, \beta) + S(-\beta, -\alpha) = \int_0^1 2P'(t)P(t)dt + O_\epsilon(L^{-1+\epsilon}) = 1 + O_\epsilon(L^{-1+\epsilon})$$

so

$$\begin{aligned} S(\alpha, \beta) + (qT)^{-\alpha-\beta} S(-\beta, -\alpha) &= [S(\alpha, \beta) + S(-\beta, -\alpha)] + \left((qT)^{-\alpha-\beta} - 1\right) S(-\beta, -\alpha) \\ &= 1 + \left((qT)^{-\alpha-\beta} - 1\right) S(-\beta, -\alpha) + O_\epsilon(L^{-1+\epsilon}). \end{aligned}$$

Also,

$$\frac{1}{\kappa} \int_0^1 (qT)^{-v(\alpha+\beta)} dv = \frac{1 - (qT)^{-\alpha-\beta}}{(\alpha + \beta) \log X}$$

so up to an error of at most $O_\epsilon(L^{-1+\epsilon})$, $((qT)^{-\alpha-\beta} - 1)S(-\alpha, -\beta)$ is equal to

$$\frac{1}{\kappa} \frac{d^2}{dx dy} X^{-\beta x - \alpha y} \int_0^1 \int_0^1 (qT)^{-v(\alpha+\beta)} P(x+u)P(y+u)dudv|_{x=y=0}$$

and consequently

$$\begin{aligned} \frac{I(\alpha, \beta)}{\hat{\psi}_0 \phi^*(q)} &= 1 + \frac{1}{\kappa} \frac{d^2}{dx dy} X^{-\beta x - \alpha y} \int_0^1 \int_0^1 (qT)^{-v(\alpha+\beta)} P(x+u)P(y+u)dudv|_{x=y=0} \\ &\quad + O_\epsilon(L^{-1+\epsilon}). \end{aligned} \quad (3.20)$$

By (3.20), we can evaluate

$$Q\left(-\frac{1}{L} \frac{d}{d\alpha}\right) Q\left(-\frac{1}{L} \frac{d}{d\beta}\right) I(\alpha, \beta)|_{\alpha=\beta=-\frac{R}{L}} \quad (3.21)$$

as both $I(\alpha, \beta)$ and the main term of (3.20) are holomorphic with respect to α and β (as long as they are small enough), then the derivatives in (3.21) can be calculated as integrals around $-\frac{R}{L}$ of radii $\asymp L^{-1}$. As the error term is uniformly $\ll O_\epsilon(L^{-1+\epsilon})$ on these contours, the same error terms hold for the derivatives of $I(\alpha, \beta)$. A little care should be taken here, as Lemma 3.4 is dependent on the condition that $|\alpha + \beta| \gg L^{-1}$, however given that both $I(\alpha, \beta)$ and the main term of (3.20) are holomorphic, we may ignore this condition as the error term must also be holomorphic in this region and so the maximum modulus principle implies that the error term in the region without the $|\alpha + \beta| \gg L^{-1}$ condition is still at most $O_\epsilon(L^{-1+\epsilon})$.

To evaluate the main term, notice that

$$\begin{aligned} Q\left(-\frac{1}{L}\frac{d}{d\alpha}\right)(X^{-\alpha y}(qT)^{-\alpha v})\Big|_{\alpha=-\frac{R}{L}} &= Q(v+\kappa y)(qT)^{-\alpha(v+\kappa y)}\Big|_{\alpha=-\frac{R}{L}} \\ &= Q(v+\kappa y)e^{R(v+\kappa y)} \end{aligned}$$

so by applying (3.21) to (3.20), and using standard methods to remove the function ψ_t , we arrive at the inequality in Theorem 3.3.

All that remains is to prove Lemma 3.4.

Proof of Lemma 3.4. For $1 \leq a \leq X$ and $i \in \mathbb{N}$,

$$\frac{i!}{(\log X)^i} \frac{1}{2\pi i} \int_{(1)} \left(\frac{X}{a}\right)^v \frac{dv}{v^{i+1}} = \begin{cases} \left(\frac{\log \frac{X}{a}}{\log X}\right)^i & \text{if } 1 \leq a \leq X \\ 0 & \text{if } a > X \end{cases}$$

hence $S(\alpha, \beta)$ is equal to

$$\frac{\zeta_q(1+\alpha+\beta)}{(2\pi i)^2} \sum_{i,j} \frac{a_i a_j i! j!}{(\log X)^{i+j}} \int_{(1)} \int_{(1)} X^{u+v} \sum_{\substack{a,b,d \\ (a,b)=1 \\ (abd,q)=1}} \frac{\mu(ad)\mu(bd)}{a^{1+\beta+u} b^{1+\alpha+v} d^{1+u+v}} \frac{du}{u^{i+1}} \frac{dv}{v^{j+1}}.$$

By considering Euler products in a similar fashion to that in Section 3.1.1,

$$\sum_{\substack{a,b,d \\ (a,b)=1 \\ (abd,q)=1}} \frac{\mu(ad)\mu(bd)}{a^{1+\beta+u} b^{1+\alpha+v} d^{1+u+v}} = \frac{\zeta_q(1+u+v) A_{\alpha,\beta}(u,v)}{\zeta_q(1+\alpha+v) \zeta_q(1+\beta+u)} \quad (3.22)$$

where $A_{\alpha,\beta}(u,v)$ is an absolutely convergent Euler product for $\Re(u), \Re(v) > -\frac{1}{6}$ say.

Hence, by writing $\zeta_q(1+u+v)$ as its Dirichlet series,

$$S(\alpha, \beta) = \sum_{i,j} \frac{a_i a_j i! j!}{(\log X)^{i+j}} \zeta_q(1+\alpha+\beta) J_{i,j}(\alpha, \beta) \quad (3.23)$$

where

$$J_{i,j}(\alpha, \beta) = \frac{1}{(2\pi i)^2} \sum_{n=1}^{\infty} \frac{1}{n} \int_{(1)} \int_{(1)} \left(\frac{X}{n}\right)^{u+v} \frac{A_{\alpha,\beta}(u,v) \prod_{p|q} (1-p^{-1-u-v})}{\zeta_q(1+\alpha+v) \zeta_q(1+\beta+u)} \frac{du dv}{u^{i+1} v^{j+1}}.$$

For $n > X$ we may move the contours far to the right to make the error negligibly small, otherwise we move first the contour of the integral over v to $\mathcal{C}^\alpha$ which is the vertical line $\Re(v) = -\Re(\alpha)$ with a small box removed to the left around the origin.

Specifically, for the v -integral let

$$\mathcal{C}_1^\alpha = \left(-\Re(\alpha) - i\infty, -\Re(\alpha) - i(\log X)^{10} \right]$$

$$\begin{aligned}
\mathcal{C}_2^\alpha &= \left[-\Re(\alpha) - i(\log X)^{10}, -(\log \log X)^{-1} - i(\log X)^{10} \right] \\
\mathcal{C}_3^\alpha &= \left[-(\log \log X)^{-1} - i(\log X)^{10}, -(\log \log X)^{-1} + i(\log X)^{10} \right] \\
\mathcal{C}_4^\alpha &= \left[-(\log \log X)^{-1} + i(\log X)^{10}, -\Re(\alpha) + i(\log X)^{10} \right] \\
\mathcal{C}_5^\alpha &= \left[-\Re(\alpha) + i(\log X)^{10}, -\Re(\alpha) + i\infty \right]
\end{aligned}$$

so $\mathcal{C}^\alpha = \cup_{i=1}^5 \mathcal{C}_i^\alpha$. This crosses a pole at $v = 0$. Then, in the resulting two integrals we move the u contour left to $\mathcal{C}^\beta$ (the same contour as $\mathcal{C}^\alpha$ but with α replaced by β), crossing another pole at $u = 0$. Consequently,

$$J_{i,j}(\alpha, \beta) = R + I_1 + I_2 + I_3 + O_A((qT)^{-A})$$

where R is the double residue at $u = v = 0$ and the other terms are the resulting integrals from moving the contours which we shall bound into an error term. Explicitly

$$\begin{aligned}
R &= \sum_{1 \leq n \leq X} \frac{1}{n} \text{Res}_{u=v=0} \left(\left(\frac{X}{n} \right)^{u+v} \frac{A_{\alpha,\beta}(u, v) \prod_{p|q} (1 - p^{-1-u-v})}{\zeta_q(1 + \alpha + v) \zeta_q(1 + \beta + u) u^{i+1} v^{j+1}} \right), \\
I_1 &= \frac{1}{2\pi i} \sum_{1 \leq n \leq X} \frac{1}{n} \int_{\mathcal{C}^\alpha} \left(\frac{X}{n} \right)^v \frac{1}{\zeta_q(1 + \alpha + v)} R_\beta(v) \frac{dv}{v^{j+1}}, \\
I_2 &= \frac{1}{2\pi i} \sum_{1 \leq n \leq X} \frac{1}{n} \int_{\mathcal{C}^\beta} \left(\frac{X}{n} \right)^u \frac{1}{\zeta_q(1 + \beta + u)} R_\alpha(u) \frac{du}{u^{i+1}},
\end{aligned}$$

where

$$R_\beta(v) = \text{Res}_{u=0} \left(\left(\frac{X}{n} \right)^u \frac{A_{\alpha,\beta}(u, v) \prod_{p|q} (1 - p^{-1-u-v})}{\zeta_q(1 + \beta + u) u^{i+1}} \right)$$

and similarly for $R_\alpha(u)$. The final integral is

$$I_3 = \frac{1}{(2\pi i)^2} \sum_{1 \leq n \leq X} \frac{1}{n} \int_{\mathcal{C}^\alpha} \int_{\mathcal{C}^\beta} \left(\frac{X}{n} \right)^{u+v} \frac{A_{\alpha,\beta}(u, v) \prod_{p|q} (1 - p^{-1-u-v})}{\zeta_q(1 + \alpha + v) \zeta_q(1 + \beta + u)} \frac{dudv}{u^{i+1} v^{j+1}}.$$

As $A_{\alpha,\beta}(u, v)$ converges absolutely on the contours $\mathcal{C}^\alpha$ and $\mathcal{C}^\beta$, and by (1.24)

$$\prod_{p|q} (1 - p^{-1-u-v}) \ll_\epsilon L^\epsilon,$$

therefore

$$I_3 \ll_\epsilon \sum_{1 \leq n \leq X} \frac{L^\epsilon}{n} \int_{\mathcal{C}^\alpha} \left(\frac{X}{n} \right)^{\Re(v)} \frac{\prod_{p|q} |1 - p^{-1-\alpha-v}|}{|\zeta(1 + \alpha + v)|} \frac{dv}{|v|^{j+1}}$$

$$\int_{\mathcal{C}^\beta} \left(\frac{X}{n}\right)^{\Re(u)} \frac{\prod_{p|q} |1 - p^{-1-\beta-u}|}{|\zeta(1+\beta+u)|} \frac{du}{|u|^{i+1}}.$$

We focus on the v -integral first, and the u -integral will be bounded almost identically.

By (1.24), we see $\prod_{p|q} |1 - p^{-1-\alpha-v}| \ll_\epsilon L^\epsilon$ on all of $\mathcal{C}^\alpha$. On $\mathcal{C}_1^\alpha$ and $\mathcal{C}_5^\alpha$

$$\left(\frac{X}{n}\right)^{-\Re(\alpha)} \ll 1, \quad \zeta(1+\alpha+v)^{-1} \ll \log |v|$$

so

$$\int_{\mathcal{C}_1^\alpha \cup \mathcal{C}_5^\alpha} \left(\frac{X}{n}\right)^{\Re(v)} \frac{dv}{|\zeta(1+\alpha+v)||v|^{j+1}} \ll \int_{(\log X)^{10}}^\infty \frac{\log t}{t^{j+1}} dt \ll L^{-9i}.$$

On $\mathcal{C}_2^\alpha$ and $\mathcal{C}_4^\alpha$, $\zeta(1+\alpha+v)^{-1} \ll \log X$ so

$$\int_{\mathcal{C}_2^\alpha \cup \mathcal{C}_4^\alpha} \left(\frac{X}{n}\right)^{\Re(v)} \frac{dv}{|\zeta(1+\alpha+v)||v|^{j+1}} \ll \frac{1}{(\log \log X) L^{10i-1}}.$$

Finally, on $\mathcal{C}_3^\alpha$ using the bound

$$\zeta(1+\alpha - (\log \log X)^{-1} + it)^{-1} \ll \begin{cases} (\log \log X)^{-1} & \text{for } |t| \leq (\log \log X)^{-1}, \\ \log |t| & \text{for } |t| \geq (\log \log X)^{-1}, \end{cases}$$

we find that

$$\int_{\mathcal{C}_3^\alpha} \left(\frac{X}{n}\right)^{\Re(v)} \frac{dv}{|\zeta(1+\alpha+v)||v|^{j+1}} \ll_\epsilon \left(\frac{X}{n}\right)^{-\frac{1}{\log \log X}} L^\epsilon.$$

By bringing all of the error terms together

$$\int_{\mathcal{C}^\alpha} \left(\frac{X}{n}\right)^{\Re(v)} \frac{1}{|\zeta(1+\alpha+v)|} \frac{dv}{|v|^{j+1}} \ll_\epsilon L^\epsilon \left(L^{-9} + \left(\frac{X}{n}\right)^{-\frac{1}{\log \log X}} \right). \quad (3.24)$$

Therefore, by treating the integral over u in the same fashion,

$$I_3 \ll_\epsilon \sum_{n \leq X} \frac{L^\epsilon}{n} \left(L^{-18} + \left(\frac{X}{n}\right)^{-\frac{2}{\log \log X}} \right).$$

As

$$\sum_{1 \leq n \leq X} n^{-1+\frac{2}{\log \log X}} \ll \int_1^X t^{-1+\frac{2}{\log \log X}} dt \ll X^{\frac{2}{\log \log X}} \log \log X$$

we see that $I_3 \ll_\epsilon L^\epsilon$.

To bound I_1 , we first must bound the residue at $u = 0$. Consider the contour integral of radius $\asymp L^{-1}$,

$$R_\beta(v) = \frac{1}{2\pi i} \oint \left(\frac{X}{n}\right)^u \frac{A_{\alpha,\beta}(u,v) \prod_{p|q} (1 - p^{-1-u-v})}{\zeta_q(1+\beta+u) u^{i+1}} du. \quad (3.25)$$

On this contour,

$$A_{\alpha,\beta}(u,v) \ll 1, \quad \left(\frac{X}{n}\right)^u \ll 1, \quad \prod_{p|q} (1 - p^{-1-u-v}) \ll_{\epsilon} L^{\epsilon}$$

and by the Taylor expansion of $\zeta(1 + \beta + u)$ and (1.24),

$$\zeta_q^{-1}(1 + \beta + u) \ll \prod_{p|q} |1 + p^{-1-\beta-u}| |\beta + u| \ll_{\epsilon} L^{-1+\epsilon}$$

and hence the residue (3.25) is $\ll_{\epsilon} L^{i+\epsilon-1}$. Then by (3.24),

$$I_1 \ll_{\epsilon} \sum_{n \leq X} \frac{L^{i+\epsilon-1}}{n} \left(L^{-9} + \left(\frac{X}{n}\right)^{-\frac{1}{\log \log X}} \right) \ll_{\epsilon} L^{i+\epsilon-1}.$$

Similarly, $I_2 \ll_{\epsilon} L^{j+\epsilon-1}$.

We are left with R . Suppose that $\alpha = \beta = u = v = s > 0$, then by (3.22)

$$\begin{aligned} A_{s,s}(s,s) &= \zeta_q(1+2s) \sum_{\substack{a,b,d \\ (a,b)=1 \\ (abd,q)=1}} \frac{\mu(ad)\mu(bd)}{(abd)^{1+2s}} \\ &= \sum_{\substack{a,b,d,n \\ (a,b)=1 \\ (abdn,q)=1}} \frac{\mu(ad)\mu(bd)}{(abdn)^{1+2s}} \end{aligned}$$

by the Dirichlet series of the L -function. Relabelling $a = ad$, $b = bd$, $m = bn$, $n = an$, we may write the sum as

$$\sum_{\substack{am=bn \\ (abmn,q)=1}} \frac{\mu(a)\mu(b)}{(abmn)^{1/2+s}} = 1$$

by the Möbius formula. Hence $A_{s,s}(s,s) = 1$ for all $s > 1$. As the Euler product converges absolutely at the origin, $A_{0,0}(0,0) = \lim_{s \rightarrow 0} A_{s,s}(s,s) = 1$. Therefore, by taking the Taylor expansion around the origin, if $u \asymp v \asymp L^{-1}$,

$$A_{\alpha,\beta}(u,v) = A_{0,0}(0,0) + O(L^{-1}) = 1 + O(L^{-1}).$$

In its current form, the product over the prime divisors of q is a nuisance, so we will simplify it using Taylor approximation. As

$$\frac{d}{dz} \prod_{p|q} (1 - p^{-1-z}) = \prod_{p|q} (1 - p^{-1-z}) \sum_{p|q} \frac{\log p}{p^{1+z} - 1},$$

using a simple inductive argument one can see that for $l \geq 1$

$$\left| \frac{d^l}{dz^l} \prod_{p|q} (1 - p^{-1-z}) \right|_{z=0} \ll_l \prod_{p|q} (1 - p^{-1}) \sum_{p|q} \frac{(\log p)^l}{p}.$$

Then

$$\sum_{p|q} \frac{(\log p)^l}{p} \ll (\log q)^{l-1} \sum_{p|q} \frac{\log p}{p}$$

and similarly to the proof of Lemma 1.3

$$\begin{aligned} \sum_{p|q} \frac{\log p}{p} &\leq \sum_{p \leq \log q} \frac{\log p}{p} + \sum_{\substack{p|q \\ p \geq \log q}} \frac{\log p}{p} \\ &\ll \log \log q + 1. \end{aligned}$$

Therefore,

$$\left. \frac{d^l}{dz^l} \prod_{p|q} (1 - p^{-1-z}) \right|_{z=0} \ll_l (\log q)^{l+\epsilon-1} \prod_{p|q} (1 - p^{-1})$$

and hence by Taylor expansion

$$\prod_{p|q} (1 - p^{-1-z}) = \prod_{p|q} (1 - p^{-1}) (1 + O_\epsilon(L^{\epsilon-1})). \quad (3.26)$$

Thus,

$$\frac{\prod_{p|q} (1 - p^{-1-u-v})}{\zeta_q(1 + \beta + u)\zeta_q(1 + \alpha + v)} = \frac{\prod_{p|q} (1 - p^{-1})}{\zeta(1 + \beta + u)\zeta(1 + \alpha + v)} (1 + O_\epsilon(L^{\epsilon-1}))$$

and so by the Taylor expansion $\zeta(1 + s)^{-1} = s(1 + O(s))$, R is equal to

$$\sum_{1 \leq n \leq X} \frac{\prod_{p|q} (1 - p^{-1})^{-1}}{n(2\pi i)^2} \left(\oint \left(\frac{X}{n} \right)^u \frac{\beta + u}{u^{i+1}} du \right) \left(\oint \left(\frac{X}{n} \right)^v \frac{\alpha + v}{v^{j+1}} dv \right) (1 + O(L^{-1})).$$

Now we are in a position to find the residue. As

$$\begin{aligned} \frac{1}{2\pi i} \oint \left(\frac{X}{n} \right)^u \frac{\beta + u}{u^{i+1}} du &= \frac{d}{dx} \frac{e^{\beta x}}{2\pi i} \oint \left(\frac{Xe^x}{n} \right)^u \frac{du}{u^{i+1}} \Big|_{x=0} \\ &= \frac{1}{i!} \frac{d}{dx} e^{\beta x} \left(x + \log \frac{X}{n} \right)^i \Big|_{x=0}, \end{aligned}$$

we obtain

$$\begin{aligned} R &= \frac{1}{i!j!} \frac{d^2}{dx dy} e^{\alpha y + \beta x} \sum_{1 \leq n \leq X} \frac{1}{n} \left(x + \log \frac{X}{n} \right)^i \left(y + \log \frac{X}{n} \right)^j \Big|_{x=y=0} \\ &\quad + O_\epsilon(L^{i+j+\epsilon-2}). \end{aligned}$$

We apply the Euler–Maclaurin formula term by term. For any $l \geq 0$,

$$\sum_{n \leq X} \frac{1}{n} \left(\log \frac{X}{n} \right)^l = \int_1^X t^{-1} \left(\log \frac{X}{t} \right)^l dt + O((\log X)^l)$$

$$= (\log X)^{l+1} \int_0^1 u^l du + O((\log X)^l)$$

by the change of variable $t = X^{1-u}$. Note that the largest error term will come from when $l = i + j$, however this still gives an error of at most $O(L^{i+j-2})$. Therefore, up to an error of $O_\epsilon(L^{i+j+\epsilon-2})$, the double residue R is equal to

$$\begin{aligned} &= \frac{\log X \prod_{p|q} (1 - p^{-1})^{-1}}{i!j!} \frac{d^2}{dx dy} e^{\alpha y + \beta x} \int_1^X (x + u \log X)^i (y + u \log X)^j du \Big|_{x=y=0} \\ &= \frac{(\log X)^{i+j-1} \prod_{p|q} (1 - p^{-1})^{-1}}{i!j!} \frac{d^2}{dx dy} X^{\alpha y + \beta x} \int_1^X (x + u)^i (y + u)^j du \Big|_{x=y=0} \end{aligned}$$

as

$$\frac{d}{dx} e^{\beta x} (x + u \log X)^i \Big|_{x=0} = (\log X)^{i-1} \frac{d}{dx} X^{\beta x} (x + u)^i \Big|_{x=0}.$$

The total error term from $R + I_1 + I_2 + I_3$ is at most

$$\ll_\epsilon L^\epsilon + L^{\epsilon+i+j-2} + L^{\epsilon+i-1} + L^{\epsilon+j-1}$$

and as (by (3.26) and the Taylor expansion of $\zeta(1 + \alpha + \beta)$),

$$\zeta_q(1 + \alpha + \beta) = \frac{1}{\alpha + \beta} \prod_{p|q} (1 - p^{-1}) (1 + O_\epsilon(L^{\epsilon-1}))$$

then by (3.23) we may conclude the proof of Lemma 3.4. □

Chapter 4

The twisted fourth moment of Dirichlet L -functions

In this chapter, we prove Theorem 4.1 which gives an asymptotic formula for the twisted fourth moment of Dirichlet L -functions in both the q and the t -aspects.

4.1 Introduction

The fourth moment is much more complex than the second moment, however we may use a similar structure of proof to derive an asymptotic formula. Just as in Chapter 2, we shall use the approximate functional equation to write the moment as several Dirichlet series with diagonal and off-diagonal terms. We then apply Poisson summation to the off-diagonals to isolate the main term contribution before using estimates of sums involving Kloosterman sums to bound the rest. At a number of points, we shall skip over details that were already discussed rigorously in the second chapter. While the results in Chapter 2 on the second moment hold for all natural numbers q , Theorem 4.1 only holds when q is a prime or powerful number, where a powerful number is an integer such that whenever a prime $p|q$ it is the case that $p^2|q$.

Theorem 4.1. *Let q be a positive integer that is either a prime or a powerful number. Let T be a real number with $T \gg q^\epsilon$, ψ_t be a smooth real valued function supported on $[T, 2T]$ with $\psi_t^{(j)} \ll T^\epsilon$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ satisfy $\alpha, \beta, \gamma, \delta \ll \frac{1}{\log qT}$. Suppose that $\kappa < \frac{1}{4}$. Let α_n be a series of real numbers supported on square-free integers $n \leq (qT)^\kappa$.*

that are co-prime to q , and satisfies the bound $\alpha_n \ll n^\epsilon$. Define

$$Z_{\alpha,\beta,\gamma,\delta,a,b}(s) = A_{\alpha,\beta,\gamma,\delta}(s)B_{\alpha,\beta,\gamma,\delta,a,b}(s),$$

where

$$A_{\alpha,\beta,\gamma,\delta}(s) = \frac{\zeta_q(1+\alpha+\gamma+s)\zeta_q(1+\alpha+\delta+s)\zeta_q(1+\beta+\gamma+s)\zeta_q(1+\beta+\delta+s)}{\zeta_q(2+\alpha+\gamma+\beta+\delta+2s)},$$

$$B_{\alpha,\beta,\gamma,\delta,a,b}(s) = \prod_{p^v \parallel a} \left(\frac{\sum_{j=0}^{\infty} \tau_{\alpha,\beta}(p^j) \tau_{\gamma,\delta}(p^{j+v}) p^{-j(s+1)}}{\sum_{j=0}^{\infty} \tau_{\alpha,\beta}(p^j) \tau_{\gamma,\delta}(p^j) p^{-j(s+1)}} \right) \\ \times \prod_{p^v \parallel b} \left(\frac{\sum_{j=0}^{\infty} \tau_{\alpha,\beta}(p^{j+v}) \tau_{\gamma,\delta}(p^j) p^{-j(s+1)}}{\sum_{j=0}^{\infty} \tau_{\alpha,\beta}(p^j) \tau_{\gamma,\delta}(p^j) p^{-j(s+1)}} \right),$$

and $\tau_{\alpha,\beta}(n) = \sum_{n_1 n_2 = n} n_1^{-\alpha} n_2^{-\beta}$. Then defining

$$I_{\alpha,\beta,\gamma,\delta}^+(T) := \frac{1}{\phi^*(q)T} \int_{\chi \pmod{q}}^+ L\left(\frac{1}{2} + \alpha + it, \chi\right) L\left(\frac{1}{2} + \beta + it, \chi\right) \\ L\left(\frac{1}{2} + \gamma - it, \bar{\chi}\right) L\left(\frac{1}{2} + \delta - it, \bar{\chi}\right) \sum_{a,b \leq (qT)^\kappa} \frac{\alpha_a \alpha_b \chi(a) \bar{\chi}(b)}{a^{1/2+it} b^{1/2-it}} \psi_t dt,$$

we have

$$I_{\alpha,\beta,\gamma,\delta}^+(T) = \frac{1}{2} \sum_d \sum_{\substack{a,b \\ (a,b)=1}} \frac{\alpha_{ad} \alpha_{bd}}{abd} \int \psi_t \left(Z_{\alpha,\beta,\gamma,\delta,a,b}(0) \right. \\ + Z_{-\gamma-\delta,-\alpha,-\beta,a,b}(0) \left(\frac{q}{\pi} \right)^{-\alpha-\beta-\gamma-\delta} \frac{\Gamma(\frac{1/2-\alpha-it}{2}) \Gamma(\frac{1/2-\beta-it}{2}) \Gamma(\frac{1/2-\gamma+it}{2}) \Gamma(\frac{1/2-\delta+it}{2})}{\Gamma(\frac{1/2+\alpha+it}{2}) \Gamma(\frac{1/2+\beta+it}{2}) \Gamma(\frac{1/2+\gamma-it}{2}) \Gamma(\frac{1/2+\delta-it}{2})} \\ + Z_{-\gamma,\beta,-\alpha,\delta,a,b}(0) \left(\frac{q}{\pi} \right)^{-\alpha-\gamma} \frac{\Gamma(\frac{1/2-\alpha-it}{2}) \Gamma(\frac{1/2-\gamma+it}{2})}{\Gamma(\frac{1/2+\alpha+it}{2}) \Gamma(\frac{1/2+\gamma-it}{2})} \\ + Z_{-\delta,\beta,\gamma,-\alpha,a,b}(0) \left(\frac{q}{\pi} \right)^{-\alpha-\delta} \frac{\Gamma(\frac{1/2-\alpha-it}{2}) \Gamma(\frac{1/2-\delta+it}{2})}{\Gamma(\frac{1/2+\alpha+it}{2}) \Gamma(\frac{1/2+\delta-it}{2})} \\ + Z_{\alpha,-\gamma,-\beta,\delta,a,b}(0) \left(\frac{q}{\pi} \right)^{-\beta-\gamma} \frac{\Gamma(\frac{1/2-\beta-it}{2}) \Gamma(\frac{1/2-\gamma+it}{2})}{\Gamma(\frac{1/2+\beta+it}{2}) \Gamma(\frac{1/2+\gamma-it}{2})} \\ \left. + Z_{\alpha,-\delta,\gamma,-\beta,a,b}(0) \left(\frac{q}{\pi} \right)^{-\beta-\delta} \frac{\Gamma(\frac{1/2-\beta-it}{2}) \Gamma(\frac{1/2-\delta+it}{2})}{\Gamma(\frac{1/2+\beta+it}{2}) \Gamma(\frac{1/2+\delta-it}{2})} \right) dt + \mathcal{E} + O((qT)^{-\epsilon}), \quad (4.1)$$

where

$$\mathcal{E} \ll \min\{\mathcal{E}_1, \mathcal{E}_2\}$$

with

$$\mathcal{E}_1 \ll_\epsilon (qT)^{\frac{5\kappa}{2} - \frac{1}{4} + \epsilon} X^{\frac{1}{2}}$$

and

$$\mathcal{E}_2 \ll_{\epsilon} (qT)^{\epsilon} \left[q^{\kappa + \frac{11}{4}} T^{\kappa - \frac{1}{4}} X^{\frac{3}{2}} + q^{\kappa} T^{\kappa - \frac{1}{2}} X + (qT)^{2\kappa - \frac{1}{2}} \min\{q^{\frac{7}{64}} (qT)^{\frac{3\kappa}{2}}, q^{\frac{3}{2}}\} \right. \\ \left. + q^{\frac{77\kappa}{128} + \frac{7}{32}} T^{\frac{77\kappa}{128} - \frac{25}{64}} X^{\frac{43}{64}} + q^{\frac{25\kappa}{16} - \frac{9}{32}} T^{\frac{25\kappa}{16} - \frac{25}{64}} X^{\frac{41}{32}} \right]$$

for $X := \min\{q, (qT)^{\kappa}\}$.

Using arguments identical to those in Section 2.3.4, one may prove a similar result for

$$I_{\alpha, \beta, \gamma, \delta}^{-}(T) := \frac{1}{\phi^*(q)T} \int \sum_{\chi \pmod{q}}^{-} L\left(\frac{1}{2} + \alpha + it, \chi\right) L\left(\frac{1}{2} + \beta + it, \chi\right) \\ L\left(\frac{1}{2} + \gamma - it, \bar{\chi}\right) L\left(\frac{1}{2} + \delta - it, \bar{\chi}\right) \sum_{a, b \leq (qT)^{\kappa}} \frac{\alpha_a \alpha_b \chi(a) \bar{\chi}(b)}{a^{1/2+it} b^{1/2-it}} \psi_t dt,$$

but with each $\frac{1}{2}$ in the gamma functions in (4.1) replaced with $\frac{3}{2}$.

The error term is rather intricate, but as $\mathcal{E} \ll (qT)^{3\kappa - \frac{1}{4}} + (qT)^{-\epsilon}$, we see that the main term of Theorem 4.1 is larger than the error term for q and T in any range, as long as $T \gg q^{\epsilon}$ and $\kappa < \frac{1}{12}$. Also, as T grows larger and larger compared to q , κ may increase up to a limit of $\kappa < \frac{1}{4}$.

Part of the proof of this theorem relies on Proposition 1 of [50], which requires certain functions to be supported on square-free integers. It is as a consequence of this that the coefficients α_n must be supported on square-free integers and that q must either be a prime or a powerful number. Removing this square-free condition from Proposition 1 of [50] would have the immediate consequence that Theorem 4.1 would hold for any integer q and allow the coefficients of the twist to be supported on any integer. In our proof, one can bypass the use of Proposition 1 from [50] (at a cost to the error term) to obtain the following corollary.

Corollary 4.1. *Suppose that q is any positive integer, and α_n may be supported on integers that are not square-free, but the rest of the assumptions of Theorem 4.1 hold, then*

$$I_{\alpha, \beta, \gamma, \delta}^{+}(T) = \frac{1}{2} \sum_d \sum_{\substack{a, b \\ (a, b) = 1}} \frac{\alpha_{ad} \alpha_{bd}}{abd} \int \psi_t \left(Z_{\alpha, \beta, \gamma, \delta, a, b}(0) \right. \\ \left. + Z_{-\gamma - \delta, -\alpha, -\beta, a, b}(0) \left(\frac{q}{\pi} \right)^{-\alpha - \beta - \gamma - \delta} \frac{\Gamma(\frac{1/2 - \alpha - it}{2}) \Gamma(\frac{1/2 - \beta - it}{2}) \Gamma(\frac{1/2 - \gamma + it}{2}) \Gamma(\frac{1/2 - \delta + it}{2})}{\Gamma(\frac{1/2 + \alpha + it}{2}) \Gamma(\frac{1/2 + \beta + it}{2}) \Gamma(\frac{1/2 + \gamma - it}{2}) \Gamma(\frac{1/2 + \delta - it}{2})} \right)$$

$$\begin{aligned}
& + Z_{-\gamma, \beta, -\alpha, \delta, a, b}(0) \left(\frac{q}{\pi} \right)^{-\alpha-\gamma} \frac{\Gamma(\frac{1/2-\alpha-it}{2})\Gamma(\frac{1/2-\gamma+it}{2})}{\Gamma(\frac{1/2+\alpha+it}{2})\Gamma(\frac{1/2+\gamma-it}{2})} \\
& + Z_{-\delta, \beta, \gamma, -\alpha, a, b}(0) \left(\frac{q}{\pi} \right)^{-\alpha-\delta} \frac{\Gamma(\frac{1/2-\alpha-it}{2})\Gamma(\frac{1/2-\delta+it}{2})}{\Gamma(\frac{1/2+\alpha+it}{2})\Gamma(\frac{1/2+\delta-it}{2})} \\
& + Z_{\alpha, -\gamma, -\beta, \delta, a, b}(0) \left(\frac{q}{\pi} \right)^{-\beta-\gamma} \frac{\Gamma(\frac{1/2-\beta-it}{2})\Gamma(\frac{1/2-\gamma+it}{2})}{\Gamma(\frac{1/2+\beta+it}{2})\Gamma(\frac{1/2+\gamma-it}{2})} \\
& + Z_{\alpha, -\delta, \gamma, -\beta, a, b}(0) \left(\frac{q}{\pi} \right)^{-\beta-\delta} \frac{\Gamma(\frac{1/2-\beta-it}{2})\Gamma(\frac{1/2-\delta+it}{2})}{\Gamma(\frac{1/2+\beta+it}{2})\Gamma(\frac{1/2+\delta-it}{2})} \Big) dt \\
& + O\left((qT)^{-\epsilon} + (qT)^{\frac{5\kappa}{2}-\frac{1}{4}+\epsilon} X^{\frac{1}{2}}\right) \quad (4.2)
\end{aligned}$$

for $X = \min\{q, (qT)^\kappa\}$.

This follows directly from the proof of Theorem 4.1, except that we stop at (4.29) rather than continuing on to prove further bounds on the error term.

4.2 Initial manipulations

We may assume that $q \geq 2$ as the $q = 1$ case (i.e. the twisted fourth moment of the Riemann zeta-function) has already been proved in [6].

We begin by stating the relevant approximate functional equation for the fourth moment, which is similar to that stated in the second chapter.

Lemma 4.1 (Approximate Functional Equation). *Let χ be an even primitive character and let*

$$V_{\alpha, \beta, \gamma, \delta}(x, t) = \frac{1}{2\pi i} \int_{(\epsilon)} G(s) X_{\alpha, \beta, \gamma, \delta}(s, t) \left(\frac{\pi}{q} \right)^{-2s} x^{-s} \frac{ds}{s}$$

where

$$X_{\alpha, \beta, \gamma, \delta}(s, t) = \frac{\Gamma(\frac{1/2+\alpha+it+s}{2})\Gamma(\frac{1/2+\beta+it+s}{2})}{\Gamma(\frac{1/2+\alpha+it}{2})\Gamma(\frac{1/2+\beta+it}{2})} \frac{\Gamma(\frac{1/2+\gamma-it+s}{2})\Gamma(\frac{1/2+\delta-it+s}{2})}{\Gamma(\frac{1/2+\gamma-it}{2})\Gamma(\frac{1/2+\delta-it}{2})}$$

then set

$$Y_{\alpha, \beta, \gamma, \delta}(t) = \left(\frac{\pi}{q} \right)^{\alpha+\beta+\gamma+\delta} \frac{\Gamma(\frac{1/2-\alpha-it}{2})\Gamma(\frac{1/2-\beta-it}{2})}{\Gamma(\frac{1/2+\alpha+it}{2})\Gamma(\frac{1/2+\beta+it}{2})} \frac{\Gamma(\frac{1/2-\gamma+it}{2})\Gamma(\frac{1/2-\delta+it}{2})}{\Gamma(\frac{1/2+\gamma-it}{2})\Gamma(\frac{1/2+\delta-it}{2})}$$

and

$$\tilde{V}_{\alpha, \beta, \gamma, \delta}(x, t) = Y_{-\gamma, -\delta, -\alpha, -\beta}(t) V_{\alpha, \beta, \gamma, \delta}(x, t)$$

and $G(s)$ is a helpful function. For convenience, we assume that $G(s)$ is divisible by an even polynomial $P_{\alpha, \beta, \gamma, \delta}(s)$ such that $G(s)/P_{\alpha, \beta, \gamma, \delta}(s)$ is independent of $\alpha, \beta, \gamma, \delta$

and $P_{\alpha,\beta,\gamma,\delta}(s)$ is symmetric in the parameters $\alpha, \beta, \gamma, \delta$, invariant under the transformations sending each parameter to its negative i.e. $\alpha \rightarrow -\alpha$ etc. and has zeros at the points of symmetry $s = -(\alpha + \gamma)/2$ etc. Then

$$\begin{aligned} & L\left(\frac{1}{2} + \alpha + it, \chi\right) L\left(\frac{1}{2} + \beta + it, \chi\right) L\left(\frac{1}{2} + \gamma - it, \bar{\chi}\right) L\left(\frac{1}{2} + \delta - it, \bar{\chi}\right) \\ &= \sum_{m,n \geq 1} \frac{\chi(m_1 m_2) \bar{\chi}(n_1 n_2)}{m_1^{1/2+\alpha} m_2^{1/2+\beta} n_1^{1/2+\gamma} n_2^{1/2+\delta}} \left(\frac{n_1 n_2}{m_1 m_2}\right)^{it} V_{\alpha,\beta,\gamma,\delta}(m_1 m_2 n_1 n_2, t) \\ &+ \sum_{m,n \geq 1} \frac{\chi(m_1 m_2) \bar{\chi}(n_1 n_2)}{m_1^{1/2-\gamma} m_2^{1/2-\delta} n_1^{1/2-\alpha} n_2^{1/2-\beta}} \left(\frac{n_1 n_2}{m_1 m_2}\right)^{it} \tilde{V}_{-\gamma,-\delta,-\alpha,-\beta}(m_1 m_2 n_1 n_2, t). \end{aligned}$$

The proof of this is standard, see for example Proposition 2.1 of [61]. By Stirling's approximation

$$Y_{\alpha,\beta,\gamma,\delta}(t) = \left(\frac{qt}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} (1 + O((1+|t|)^{-1})), \quad (4.3)$$

$$X_{\alpha,\beta,\gamma,\delta}(s, t) = \left(\frac{t}{2}\right)^{2s} (1 + O(t^{-1}(1+|s|^2))), \quad (4.4)$$

and

$$x^j t^k \frac{\partial^{j+k}}{\partial x^j \partial t^k} V_{\alpha,\beta,\gamma,\delta}(x, t) \ll_{j,k,C} \left(1 + \left|\frac{x}{(qt)^2}\right|\right)^{-C} \quad (4.5)$$

for all $C > 0$. Applying Lemma 2.3 to the approximate functional equation gives

$$\begin{aligned} & \sum_{\chi \pmod{q}}^+ L\left(\frac{1}{2} + \alpha + it, \chi\right) L\left(\frac{1}{2} + \beta + it, \chi\right) L\left(\frac{1}{2} + \gamma - it, \bar{\chi}\right) L\left(\frac{1}{2} + \delta - it, \bar{\chi}\right) \\ & \quad \times \sum_{a,b \leq (qT)^\kappa} \frac{\alpha_a \alpha_b \chi(a) \bar{\chi}(b)}{a^{1/2+it} b^{1/2-it}} \\ &= \sum_{w|q} \mu\left(\frac{q}{w}\right) \frac{\phi(w)}{2} \sum_{\substack{a,b,m_1,m_2,n_1,n_2 \\ am_1 m_2 \equiv \pm bn_1 n_2 \pmod{w} \\ (abm_1 m_2 n_1 n_2, q)=1}} \frac{\alpha_a \alpha_b}{(ab)^{1/2} m_1^{1/2+\alpha} m_2^{1/2+\beta} n_1^{1/2+\gamma} n_2^{1/2+\delta}} \left(\frac{bn_1 n_2}{am_1 m_2}\right)^{it} \\ & \quad \times V_{\alpha,\beta,\gamma,\delta}(m_1 m_2 n_1 n_2, t) \\ &+ \sum_{w|q} \mu\left(\frac{q}{w}\right) \frac{\phi(w)}{2} \sum_{\substack{a,b,m_1,m_2,n_1,n_2 \\ am_1 m_2 \equiv \pm bn_1 n_2 \pmod{w} \\ (abm_1 m_2 n_1 n_2, q)=1}} \frac{\alpha_a \alpha_b}{(ab)^{1/2} m_1^{1/2-\gamma} m_2^{1/2-\delta} n_1^{1/2-\alpha} n_2^{1/2-\beta}} \left(\frac{bn_1 n_2}{am_1 m_2}\right)^{it} \\ & \quad \times \tilde{V}_{-\gamma,-\delta,-\alpha,-\beta}(m_1 m_2 n_1 n_2, t). \end{aligned}$$

We shall separate this into the diagonal terms and the off-diagonal terms to see that

$$I_{\alpha,\beta,\gamma,\delta}^+(T) = \mathcal{D}_{\alpha,\beta,\gamma,\delta}(T) + \mathcal{O}_{\alpha,\beta,\gamma,\delta}(T) + \tilde{\mathcal{D}}_{-\gamma,-\delta,-\alpha,-\beta}(T) + \tilde{\mathcal{O}}_{-\gamma,-\delta,-\alpha,-\beta}(T),$$

where

$$\mathcal{D}_{\alpha,\beta,\gamma,\delta}(T) = \frac{1}{2T} \int \sum_{\substack{a,b,m_1,m_2,n_1,n_2 \\ am_1m_2=bn_1n_2 \\ (abm_1m_2n_1n_2,q)=1}} \frac{\alpha_a\alpha_b}{(ab)^{1/2}m_1^{1/2+\alpha}m_2^{1/2+\beta}n_1^{1/2+\gamma}n_2^{1/2+\delta}} \times V_{\alpha,\beta,\gamma,\delta}(m_1m_2n_1n_2, t) \psi_t dt$$

and

$$\mathcal{O}_{\alpha,\beta,\gamma,\delta}(T) = \sum_{w|q} \mu\left(\frac{q}{w}\right) \frac{\phi(w)}{2\phi^*(q)T} \int \sum_{\substack{a,b,m_1,m_2,n_1,n_2 \\ am_1m_2 \equiv \pm bn_1n_2 \pmod{w} \\ am_1m_2 \neq bn_1n_2 \\ (abm_1m_2n_1n_2,q)=1}} \frac{\alpha_a\alpha_b}{(ab)^{1/2}} \frac{1}{m_1^{1/2+\alpha}m_2^{1/2+\beta}n_1^{1/2+\gamma}n_2^{1/2+\delta}} \left(\frac{bn_1n_2}{am_1m_2}\right)^{it} V_{\alpha,\beta,\gamma,\delta}(m_1m_2n_1n_2, t) \psi_t dt.$$

The definitions of $\tilde{\mathcal{D}}_{-\gamma,-\delta,-\alpha,-\beta}(T)$ and $\tilde{\mathcal{O}}_{-\gamma,-\delta,-\alpha,-\beta}(T)$ are similar.

4.3 The diagonals

We begin by opening up the integral in $V_{\alpha,\beta,\gamma,\delta}(x, t)$ so that

$$\mathcal{D}_{\alpha,\beta,\gamma,\delta}(T) = \frac{\phi^*(q)}{2T} \int \int_{(\epsilon)} \sum_{\substack{a,b,m,n \\ am=bn \\ (abmn,q)=1}} \frac{\alpha_a\alpha_b\tau_{\alpha,\beta}(m)\tau_{\gamma,\delta}(n)}{(ab)^{1/2}m^{1/2+s}n^{1/2+s}} G(s) X_{\alpha,\beta,\gamma,\delta}(s, t) \left(\frac{q}{\pi}\right)^{2s} \psi_t \frac{ds}{s} dt.$$

Writing $d = (a, b)$ and relabelling $a = ad, b = bd, n = an$, and $m = bn$, we may write the sum as

$$\sum_{\substack{a,b,m,n \\ am=bn \\ (abmn,q)=1}} \frac{\alpha_a\alpha_b\tau_{\alpha,\beta}(m)\tau_{\gamma,\delta}(n)}{(ab)^{1/2}m^{1/2+s}n^{1/2+s}} = \sum_d \sum_{\substack{a,b \\ (a,b)=1}} \frac{\alpha_{ad}\alpha_{bd}}{(ab)^{1+s}d} \sum_{\substack{n \\ (n,q)=1}} \frac{\tau_{\alpha,\beta}(bn)\tau_{\gamma,\delta}(an)}{n^{1+2s}}.$$

By the multiplicativity of $\tau_{\alpha,\beta}(n)$, we may write the inner sum as an Euler product

$$\sum_{\substack{n \\ (n,q)=1}} \frac{\tau_{\alpha,\beta}(bn)\tau_{\gamma,\delta}(an)}{n^{1+2s}} = \left(\prod_{p \nmid abq} \sum_{j=0}^{\infty} \frac{\tau_{\alpha,\beta}(p^j)\tau_{\gamma,\delta}(p^j)}{p^{j(1+2s)}} \right) \times \left(\prod_{p|a} \sum_{j=0}^{\infty} \frac{\tau_{\alpha,\beta}(p^j)\tau_{\gamma,\delta}(p^{p_a+j})}{p^{j(1+2s)}} \right) \left(\prod_{p|b} \sum_{j=0}^{\infty} \frac{\tau_{\alpha,\beta}(p^{p_b+j})\tau_{\gamma,\delta}(p^j)}{p^{j(1+2s)}} \right)$$

where $p^{p_a} \parallel a$. This Euler product is precisely $Z_{\alpha,\beta,\gamma,\delta,a,b}(2s)$ as

$$\sum_{\substack{n \\ (n,q)=1}} \frac{\tau_{\alpha,\beta}(n)\tau_{\gamma,\delta}(n)}{n^{1+2s}} = A_{\alpha,\beta,\gamma,\delta}(2s),$$

and therefore

$$\mathcal{D}_{\alpha,\beta,\gamma,\delta}(T) = \frac{1}{2T} \int \int_{(\epsilon)} \sum_d \sum_{\substack{a,b \\ (a,b)=1}} \frac{\alpha_{ad}\alpha_{bd}}{d(ab)^{1+s}} Z_{\alpha,\beta,\gamma,\delta,a,b}(2s) \\ G(s)X_{\alpha,\beta,\gamma,\delta}(s,t) \left(\frac{q}{\pi}\right)^{2s} \psi_t \frac{ds}{s} dt.$$

We move the contour of integration over s to $\Re(s) = -\frac{1}{4} + \epsilon$, crossing a simple pole at $s = 0$ however we do not pick up the poles at $s = -\frac{\alpha+\gamma}{2}$ etc. due to the specified zeros of $G(s)$ at these points.

To bound the integral on the new line, we shall need to bound $Z_{\alpha,\beta,\gamma,\delta,a,b}(2s)$, which we shall do using a bound of $\zeta_q(s)$ for s in the critical strip. Using a crude version of the proof of Lemma 1.3, we see that

$$\prod_{p|q} (1 - p^{-s}) \ll \exp \left(O \left(\sum_{p|q} p^{-\sigma} \right) \right),$$

and as

$$\sum_{p|q} p^{-\sigma} \leq \sum_{n \leq \log q} n^{-\sigma} + \sum_{\substack{p|q \\ p \geq \log q}} n^{-\sigma} \ll \frac{(\log q)^{1-\sigma}}{1-\sigma} + \frac{(\log q)^{1-\sigma}}{\log \log q}$$

we conclude that for $\sigma > 0$ we may bound

$$\prod_{p|q} (1 - p^{-s}) \ll_{\epsilon} q^{\epsilon}$$

uniformly for $\sigma > 0$. Then, using a simple bound arising from (1.8),

$$\zeta(s) \ll \frac{1}{|s-1|} + \frac{1}{2} + \frac{|s|}{\sigma}$$

for $\sigma > 0$, we may bound

$$\zeta_q(s) \ll_{\epsilon} q^{\epsilon} \frac{|s|}{\sigma} \quad (4.6)$$

for $\sigma > 0$. Therefore on the new contour

$$A_{\alpha,\beta,\gamma,\delta}(2s) \ll q^{\epsilon} |s|^4.$$

We may use the divisor bound to show that $B_{\alpha,\beta,\gamma,\delta,a,b}(s) \ll_{\epsilon} (qT)^{\epsilon}$, and so finally by (4.4), the new integral is $\ll_{\epsilon} (qT)^{\frac{\kappa-1}{2}+\epsilon}$ which is an acceptable error term.

Therefore

$$\mathcal{D}_{\alpha,\beta,\gamma,\delta}(T) = \frac{1}{2T} \int \sum_d \sum_{\substack{a,b \\ (a,b)=1}} \frac{\alpha_{ad}\alpha_{bd}}{abd} Z_{\alpha,\beta,\gamma,\delta,a,b}(0) \psi_t dt + O_{\epsilon} \left((qT)^{\frac{\kappa-1}{2}+\epsilon} \right).$$

In a similar fashion,

$$\begin{aligned} \tilde{\mathcal{D}}_{-\gamma-\delta,-\alpha,-\beta}(T) &= \frac{1}{2T} \int \sum_d \sum_{\substack{a,b \\ (a,b)=1}} \frac{\alpha_{ad}\alpha_{bd}}{abd} Z_{-\gamma-\delta,-\alpha,-\beta,a,b}(0) \left(\frac{q}{\pi}\right)^{-\alpha-\beta-\gamma-\delta} \\ &\quad \frac{\Gamma(\frac{1/2-\alpha-it}{2})\Gamma(\frac{1/2-\beta-it}{2})\Gamma(\frac{1/2-\gamma+it}{2})\Gamma(\frac{1/2-\delta+it}{2})}{\Gamma(\frac{1/2+\alpha+it}{2})\Gamma(\frac{1/2+\beta+it}{2})\Gamma(\frac{1/2+\gamma-it}{2})\Gamma(\frac{1/2+\delta-it}{2})} \psi_t dt + O_{\epsilon} \left((qT)^{\frac{\kappa-1}{2}+\epsilon} \right). \end{aligned}$$

4.4 The off-diagonals

In order to shorten some already lengthy equations, we shall suppress the subscripts in the function $\mathcal{O}_{\alpha,\beta,\gamma,\delta}(T)$ to become $\mathcal{O}(T)$, as well as in other similar functions such as $V_{\alpha,\beta,\gamma,\delta}(x, t)$.

To treat the off-diagonal terms, we re-write the following sums

$$\sum_{\substack{a,b,m_1,m_2,n_1,n_2 \\ am_1m_2 \equiv \pm bn_1n_2 \pmod{w} \\ am_1m_2 \neq bn_1n_2}} = \sum_{\substack{a,b,m_1,m_2,n_1,n_2,h \\ am_1m_2 = \pm bn_1n_2 + wh \\ 0 < |h|}}$$

and also introduce dyadic partition functions for m_1, m_2, n_1 and n_2 . Let W be a smooth non-negative function supported in $[1, 2]$ such that

$$\sum_M W\left(\frac{m}{M}\right) = 1,$$

where M runs over a sequence of real numbers with $|\{M : X^{-1} \leq M \leq X\}| \ll \log X$.

The coefficients α_n are also partitioned dyadically by writing $\alpha_a(A)$ to be supported on $a \in [A, 2A]$ and $\alpha_b(B)$ is supported on $[B, 2B]$. This means that $A, B \ll (qT)^{\kappa}$.

We also introduce a smooth function g such that

$$g(x) + g\left(\frac{1}{x}\right) = 1$$

for all real x with the condition that $g(x) \ll_r (1+x)^{-r}$ for any fixed $r > 0$ and $x > 1$, as well as $g^{(j)}(x) \ll_j (qT)^{\epsilon}$ for any non-negative integer j . For the sake of simplifying

the proof later on, it must also satisfy the condition

$$\hat{g}\left(\pm\frac{(\alpha-\beta)}{2}\right) = \hat{g}\left(\pm\frac{(\gamma-\delta)}{2}\right) = 0 \quad (4.7)$$

where $\hat{g}(u)$ is the Mellin transform

$$\hat{g}(u) = \int_0^\infty g(x)x^{-u}dx. \quad (4.8)$$

It will be useful to note that $\hat{g}(u)$ has a simple pole of residue 1 at $u = 1$, and by repeated integration by parts on (4.8) we see that $\hat{g}(u) \ll_j |u|^{-j}$ so decays rapidly in the vertical direction in any fixed strip $|\Re(u)| \leq K$. By multiplying $\mathcal{O}(T)$ by

$$\left(g\left(\frac{m_1}{m_2}\right) + g\left(\frac{m_2}{m_1}\right)\right) \left(g\left(\frac{n_1}{n_2}\right) + g\left(\frac{n_2}{n_1}\right)\right) = 1$$

we may separate $\mathcal{O}(T)$ into four sums. We shall focus on the sum including $g\left(\frac{m_1}{m_2}\right)g\left(\frac{n_1}{n_2}\right)$ which means that we may assume that

$$M_1 \ll M_2(qT)^\epsilon, \text{ and } N_1 \ll N_2(qT)^\epsilon$$

as otherwise there is a contribution of at most $O_K((qT)^{-K})$. The other three sums will each give their own similar contribution to the main term, obeying an obvious symmetry. We may now split $\mathcal{O}(T)$ into four sums of the form

$$\sum_{A,B,M_1,M_2,N_1,N_2} \sum_{w|q} \mu\left(\frac{q}{w}\right) \frac{\phi(w)}{2\phi^*(q)T} S_w(A, B, M_1, M_2, N_1, N_2),$$

where $S_w(A, B, M_1, M_2, N_1, N_2)$ is defined to be

$$\int \sum_{\substack{a,b,m_1,m_2,n_1,n_2,h \\ am_1m_2=\pm bn_1n_2+wh \\ 0<|h| \\ (abm_1m_2n_1n_2,q)=1}} \frac{\alpha_a(A)\alpha_b(B)}{(ab)^{\frac{1}{2}}m_1^{\frac{1}{2}+\alpha}m_2^{\frac{1}{2}+\beta}n_1^{\frac{1}{2}+\gamma}n_2^{\frac{1}{2}+\delta}} \left(\frac{bn_1n_2}{am_1m_2}\right)^{it} g\left(\frac{m_1}{m_2}\right) g\left(\frac{n_1}{n_2}\right) \\ W\left(\frac{m_1}{M_1}\right) W\left(\frac{m_2}{M_2}\right) W\left(\frac{n_1}{N_1}\right) W\left(\frac{n_2}{N_2}\right) V(m_1m_2n_1n_2, t) \psi_t dt.$$

Let $M = M_1M_2$ and $N = N_1N_2$. By the support of $V(x, t)$, we may assume that $MN \ll (qT)^{2+\epsilon}$. We also note that we may bound the terms when

$$|wh| \geq H := (qT)^\epsilon \frac{\sqrt{ABMN}}{T}$$

by $O_K((qT)^{-K})$. This also implies that $AM \asymp BN$. This can be proved in an almost identical manner to Lemma 2.4, by repeated integration by parts over t . Hence we may restrict the sum over h to $0 < |h| \leq \frac{H}{w}$ with negligible contribution to the error term. The following proposition allows us to isolate the main terms.

Proposition 4.1. *Suppose that $\frac{q}{w}$ is square free, then $S_w(A, B, M_1, M_2, N_1, N_2)$ is equal to*

$$\sum_{\substack{k|q \\ (k,w)=1}} \mu(k) \frac{\phi(q/k)}{2q} (\mathcal{M}_{w,k}^+(A, B, M_1, M_2, N_1, N_2) + \mathcal{M}_{w,k}^-(A, B, M_1, M_2, N_1, N_2)) \\ + \mathcal{E}_w + O(w^{-1}(qT)^{1-\epsilon}),$$

where

$$\mathcal{M}_{w,k}^+(A, B, M_1, M_2, N_1, N_2) = \frac{1}{2\pi i} \int \int \int \sum_{\substack{d \geq 1 \\ (d,q)=1}} \sum_{\substack{0 < |h| \\ (h,k_1)=1}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1,bn_1)=d \\ (abm_1n_1,q)=1}} \frac{\alpha_a(A)\alpha_b(B)}{a^{1+\delta+s}b^{1+\beta+s}} \\ \frac{d^{1+\beta+\delta+2s}}{m_1^{1+\alpha+\delta+2s}n_1^{1+\beta+\gamma+2s}} W\left(\frac{m_1}{M_1}\right) W\left(\frac{n_1}{N_1}\right) W\left(\frac{xbn_1}{dM_2}\right) W\left(\frac{abm_1n_1x - whd^2}{dbn_1N_2}\right) \\ g\left(\frac{m_1d}{xbn_1}\right) g\left(\frac{n_1^2bd}{abm_1n_1x - whd^2}\right) x^{-\frac{1}{2}-\beta-s-it} \left(x - \frac{whd^2}{abm_1n_2}\right)^{-\frac{1}{2}-\delta-s+it} \\ G(s)X(s,t) \frac{q^{2s}}{\pi^{2s}} \psi_t dx \frac{ds}{s} dt,$$

and

$$\mathcal{M}_{w,k}^-(A, B, M_1, M_2, N_1, N_2) = \frac{1}{2\pi i} \int \int \int \sum_{\substack{d \geq 1 \\ (d,q)=1}} \sum_{\substack{0 < |h| \\ (h,k_1)=1}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1,bn_1)=d \\ (abm_1n_1,q)=1}} \frac{\alpha_a(A)\alpha_b(B)}{a^{1+\delta+s}b^{1+\beta+s}} \\ \frac{d^{1+\beta+\delta+2s}}{m_1^{1+\alpha+\delta+2s}n_1^{1+\beta+\gamma+2s}} W\left(\frac{m_1}{M_1}\right) W\left(\frac{n_1}{N_1}\right) W\left(\frac{xbn_1}{dM_2}\right) W\left(\frac{whd^2 - abm_1n_1x}{dbn_1N_2}\right) \\ g\left(\frac{m_1d}{xbn_1}\right) g\left(\frac{n_1^2bd}{whd^2 - abm_1n_1x}\right) G(s)x^{-\frac{1}{2}-\beta-s-it} \left(\frac{whd^2}{abm_1n_2} - x\right)^{-\frac{1}{2}-\delta-s+it} \\ X(s,t) \frac{q^{2s}}{\pi^{2s}} \psi_t dx \frac{ds}{s} dt,$$

and

$$\mathcal{E}_w \ll \min\{\mathcal{E}_{w,1}, \mathcal{E}_{w,2}\}$$

with

$$\mathcal{E}_{w,1} \ll_{\epsilon} \frac{1}{w} (qT)^{\frac{5\kappa}{2} + \frac{3}{4} + \epsilon} X^{\frac{1}{2}}$$

and

$$\mathcal{E}_{w,2} \ll_{\epsilon} (qT)^{\epsilon} \frac{1}{w} \left[q^{\kappa + \frac{15}{4}} T^{\kappa + \frac{3}{4}} X^{\frac{3}{2}} + q^{\kappa+1} T^{\kappa + \frac{1}{2}} X + (qT)^{2\kappa + \frac{1}{2}} \min\{q^{\frac{7}{64}} (qT)^{\frac{3\kappa}{2}}, q^{\frac{3}{2}}\} \right]$$

$$+ q^{\frac{77\kappa}{128} + \frac{39}{32} T^{\frac{77\kappa}{128} + \frac{39}{64}} X^{\frac{43}{64}} + q^{\frac{25\kappa}{16} + \frac{23}{32} T^{\frac{25\kappa}{16} + \frac{39}{64}} X^{\frac{41}{32}} \Big]$$

for $X := \min\{q, (qT)^\kappa\}$.

We shall prove this proposition at the end of the chapter.

By the support of $W(z)$, the integral in the main terms is over $x \asymp \frac{dM_2}{BN_1} \asymp \frac{dN_2}{AM_1}$. Hence, by summing over the terms M_2 and N_2 such that $M_1 \leq M_2(qT)^\epsilon$ and $N_1 \leq N_2(qT)^\epsilon$, we introduce an error of size at most $O_K((qT)^{-K})$ due to (4.5). Also if $M_2 \leq M_1(qT)^{-\epsilon}$ or $N_2 \leq N_1(qT)^{-\epsilon}$ then by the rapid decay of $g(z)$, we have a contribution of at most $O_K((qT)^{-K})$. Hence, we may remove the dyadic partition over M_2 and N_2 with a negligible contribution to the error term. In a similar fashion, we may introduce all M_1 and N_1 and then all A and B that satisfy the condition $BN_1 \leq AM_1$. So for any A, B, M_1, N_1 satisfying this inequality, we may define $\mathcal{M}_{w,k}^+(A, B, M_1, N_1)$ to be

$$\begin{aligned} &= \sum_{M_2, N_2} \mathcal{M}_{w,k}^+(A, B, M_1, M_2, N_1, N_2) \\ &= \frac{1}{(2\pi i)^3} \int \int_{(2\epsilon)} \int \int_{(\epsilon)} \int_{(\epsilon)} \sum_{\substack{d \geq 1 \\ (d,q)=1}} \sum_{\substack{0 < |h| \\ (h,k)=1}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1, bn_1)=d \\ (abm_1n_1, q)=1}} \frac{\alpha_a(A) \alpha_b(B) d^{1+\beta+\delta+2s-u-v}}{a^{1+\delta+s-v} b^{1+\beta+s-u}} \\ &\quad \frac{1}{m_1^{1+\alpha+\delta+2s+u-v} n_1^{1+\beta+\gamma+2s+v-u}} W\left(\frac{m_1}{M_1}\right) W\left(\frac{n_1}{N_1}\right) \hat{g}(u) \hat{g}(v) G(s) \\ &\quad x^{-\frac{1}{2}-\beta-s-it+u} \left(x - \frac{whd^2}{abm_1n_2}\right)^{-\frac{1}{2}-\delta-s+it+v} X(s, t) \frac{q^{2s}}{\pi^{2s}} \psi_t du dv dx \frac{ds}{s} dt, \end{aligned}$$

where we have written $g(z)$ using its inverse Mellin transform,

$$g(x) = \frac{1}{2\pi i} \int_{(c)} x^{-u} \hat{g}(u) du.$$

Turning our attention to the x -integral,

$$\int_0^\infty x^{-\frac{1}{2}-\beta-s+u-it} \left(x - \frac{whd^2}{abm_1n_1}\right)^{-\frac{1}{2}-\delta-s+it+v} dx, \quad (4.9)$$

we note that similarly to in the proof of Proposition 2.1, this is restricted to $x > \frac{whd^2}{abM_1n_2}$ when $h > 0$, and to $x > 0$ when $h < 0$. For absolute convergence, if $h > 0$, we impose the conditions

$$\Re(2s + \beta + \delta - u - v) > 0, \quad \Re(s + \delta - v) < \frac{1}{2}$$

and if $h < 0$ we impose the conditions

$$\Re(2s + \beta + \delta - u - v) > 0, \quad \Re(s + \beta - u) < \frac{1}{2}.$$

Under these conditions, (4.9) is equal to to (see for example 17.43.21 and 17.43.22 of [43])

$$\left(\frac{w|h|d^2}{abm_1n_1}\right)^{-2s-\beta-\delta+u+v} \times \begin{cases} \frac{\Gamma(2s+\beta+\delta-u-v)\Gamma(\frac{1}{2}-\delta-s+it+v)}{\Gamma(\frac{1}{2}+\beta+s-u+it)} & \text{if } h > 0, \\ \frac{\Gamma(2s+\beta+\delta-u-v)\Gamma(\frac{1}{2}-\beta-s-it+u)}{\Gamma(\frac{1}{2}+\delta+s-v-it)} & \text{if } h < 0. \end{cases}$$

Therefore, by grouping $+h$ with $-h$, $\mathcal{M}_{w,k}^+(A, B, M_1, N_1)$ is equal to

$$\begin{aligned} & \frac{1}{(2\pi i)^3} \int \int_{(2\epsilon)} \int_{(\epsilon)} \int_{(\epsilon)} \sum_{\substack{d \geq 1 \\ (d,q)=1}} \sum_{\substack{0 < h \\ (h,k)=1}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1,bn_1)=d \\ (abm_1n_1,q)=1}} \frac{\alpha_a(A)\alpha_b(B)d^{1-\beta-\delta-2s+u+v}}{a^{1-\beta-s+u}b^{1-\delta-s+v}} \hat{g}(u) \hat{g}(v) \\ & \frac{(wh)^{-2s-\beta-\delta+u+v}}{m_1^{1+\alpha-\beta+2u}n_1^{1+\gamma-\delta+2v}} W\left(\frac{m_1}{M_1}\right) W\left(\frac{n_1}{N_1}\right) H_{u,v,t}^+(s) G(s) X(s, t) \frac{q^{2s}}{\pi^{2s}} \psi_t du dv \frac{ds}{s} dt, \end{aligned}$$

where

$$H_{u,v,t}^+(s) = \Gamma(2s + \beta + \delta - u - v) \left(\frac{\Gamma(\frac{1}{2} - \delta - s + it + v)}{\Gamma(\frac{1}{2} + \beta + s - u + it)} + \frac{\Gamma(\frac{1}{2} - \beta - s - it + u)}{\Gamma(\frac{1}{2} + \delta + s - v - it)} \right).$$

In the $\mathcal{M}^-$ case, due to the extra minus sign in the x -integral, we arrive at the same result but with $H_{u,v,t}^+(s)$ replaced by

$$H_{u,v,t}^-(s) := \frac{\Gamma(1/2 - \alpha - it - s - u)\Gamma(1/2 - \beta + it - s - v)}{\Gamma(1 - \alpha - \beta - 2s - u - v)}.$$

Writing $H_{u,v,t}(s) := H_{u,v,t}^+(s) + H_{u,v,t}^-(s)$ we may write

$$\begin{aligned} \mathcal{M}_{w,k}(A, B, M_1, N_1) &= \mathcal{M}_{w,k}^+(A, B, M_1, N_1) + \mathcal{M}_{w,k}^-(A, B, M_1, N_1) \\ &= \frac{1}{(2\pi i)^3} \int \int_{(2\epsilon)} \int_{(\epsilon)} \int_{(\epsilon)} \sum_{\substack{d \geq 1 \\ (d,q)=1}} \sum_{\substack{0 < h \\ (h,k)=1}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1,bn_1)=d \\ (abm_1n_1,q)=1}} \frac{\alpha_a\alpha_b d^{1-\beta-\delta-2s+u+v}}{a^{1-\beta-s+u}b^{1-\delta-s+v}} \hat{g}(u) \hat{g}(v) \\ & \frac{(wh)^{-2s-\beta-\delta+u+v}}{m_1^{1+\alpha-\beta+2u}n_1^{1+\gamma-\delta+2v}} W\left(\frac{m_1}{M_1}\right) W\left(\frac{n_1}{N_1}\right) H_{u,v,t}(s) G(s) X(s, t) \frac{q^{2s}}{\pi^{2s}} \psi_t du dv \frac{ds}{s} dt. \quad (4.10) \end{aligned}$$

Recall that we have only proved this in the case that $BN_1 \leq AM_1$. If $AM_1 \leq BN_1$ then the method above gives the same result as (4.10) but with a and b swapped, α and γ swapped, β and δ swapped, and u and v swapped, leaving (4.10) unchanged. Hence, we may remove the assumption that $BN_1 \leq AM_1$ and remove the final dyadic partitions to arrive at the equation

$$\mathcal{O}(T) = \sum_{w|q} \mu\left(\frac{q}{w}\right) \frac{\phi(w)}{2\phi^*(q)T} \mathcal{M}_w + \mathcal{E} + O((qT)^{-\epsilon})$$

with

$$\begin{aligned}\mathcal{M}_{w,k} &= \sum_{A,B,M_1,N_1} \mathcal{M}_{w,k}(A,B,M_1,N_1) \\ &= \frac{1}{(2\pi i)^3} \int \int_{(2\epsilon)} \int_{(\epsilon)} \int_{(\epsilon)} \sum_{\substack{d \geq 1 \\ (d,q)=1}} \sum_{\substack{1 \leq h \\ (h,k)=1}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1,bn_1)=d \\ (abm_1n_1,q)=1}} \frac{\alpha_a \alpha_b d^{1-\beta-\delta-2s+u+v}}{a^{1-\beta-s+u} b^{1-\delta-s+v}} \hat{g}(u) \hat{g}(v) \\ &\quad \frac{(wh)^{-2s-\beta-\delta+u+v}}{m_1^{1+\alpha-\beta+2u} n_1^{1+\gamma-\delta+2v}} H_{u,v,t}(s) G(s) X(s,t) \frac{q^{2s}}{\pi^{2s}} \psi_t du dv \frac{ds}{s} dt\end{aligned}$$

and $\mathcal{E} \ll \min\{\mathcal{E}_1, \mathcal{E}_2\}$ with

$$\mathcal{E}_1 \ll_{\epsilon} (qT)^{\frac{5\kappa}{2} - \frac{1}{4} + \epsilon} X^{\frac{1}{2}}$$

and

$$\begin{aligned}\mathcal{E}_2 \ll_{\epsilon} (qT)^{\epsilon} \left[q^{\kappa + \frac{11}{4}} T^{\kappa - \frac{1}{4}} X^{\frac{3}{2}} + q^{\kappa} T^{\kappa - \frac{1}{2}} X + (qT)^{2\kappa - \frac{1}{2}} \min\{q^{\frac{7}{64}} (qT)^{\frac{3\kappa}{2}}, q^{\frac{3}{2}}\} \right. \\ \left. + q^{\frac{77\kappa}{128} + \frac{7}{32}} T^{\frac{77\kappa}{128} - \frac{25}{64}} X^{\frac{43}{64}} + q^{\frac{25\kappa}{16} - \frac{9}{32}} T^{\frac{25\kappa}{16} - \frac{25}{64}} X^{\frac{41}{32}} \right]\end{aligned}$$

for $X := \min\{q, (qT)^{\kappa}\}$.

We turn our attention to the sum

$$\sum_{\substack{1 \leq d \\ (d,q)=1}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1,bn_1)=d}} \frac{\alpha_a \alpha_b d^{1-2s-\beta-\delta+u+v}}{a^{1-s-\beta+u} b^{1-s-\delta+v}} m_1^{-1-2u-\alpha+\beta} n_1^{-1-2v-\gamma+\delta}. \quad (4.11)$$

Writing $j = (a, b)$ so that $a = a'j$ and $b = b'j$ with $(a', b') = 1$ implies that $(a'm_1, b'n_1) = d'$ for $d = jd'$. Also, let $k_0 = (b', m_1)$ and $l_0 = (a', n_1)$, and relabel $m_1 = k_0 m'_1$ and $n_1 = l_0 n'_1$. Then as $(a', b') = 1$, we see that $(a'm_1, b'n_1) = k_0 l_0 (a'm'_1, b'n'_1)$ and therefore $d = j k_0 l_0 d''$ for some d'' . This means that we may re-write the above sum (removing all of the superscripts ') as

$$\sum_{\substack{1 \leq j \\ (j,q)=1}} \sum_{(a,b)=1} \frac{\alpha_{aj} \alpha_{bj}}{j a^{1-s-\beta+u} b^{1-s-\delta+v}} \sum_{\substack{k_0|b \\ l_0|a}} \sum_{\substack{m_1, n_1, d \\ (d,q)=1 \\ (m_1, n_1)=d \\ (m_1, qb/k_0)=1 \\ (n_1, qa/l_0)=1}} \frac{d^{1-2s-\beta-\delta+u+v} k_0^{-2s-u+v-\alpha-\delta} l_0^{-2s-v+u-\beta-\gamma}}{m_1^{1+2u+\alpha-\beta} n_1^{1+2v+\gamma-\delta}}. \quad (4.12)$$

To ease notation, we write

$$x = 2u + \alpha - \beta, y = 2v + \gamma - \delta \text{ and } z = -2s - \beta - \delta + u + v$$

and so

$$\sum_{\substack{m_1, n_1, d \\ (d, q)=1 \\ (m_1, n_1)=d \\ (m_1, qb/k_0)=1 \\ (n_1, qa/l_0)=1}} \frac{d^{1-2s-\beta-\delta+u+v}}{m_1^{1+2u+\alpha-\beta} n_1^{1+2v+\gamma-\delta}} = \sum_{\substack{m_1, n_1, d \\ (d, q)=1 \\ (m_1, n_1)=d \\ (m_1, qb/k_0)=1 \\ (n_1, qa/l_0)=1}} \frac{d^{1+z}}{m_1^{1+x} n_1^{1+y}}, \quad (4.13)$$

which factors into the Euler product

$$\prod_{p \nmid \frac{abq}{k_0 l_0}} \left(\sum_{d=0}^{\infty} \frac{p^{d(1+z)}}{p^{d(2+x+y)}} \sum_{\substack{m, n \geq 0 \\ \min(m, n)=0}} \frac{1}{p^{m(1+x)+n(1+y)}} \right) \prod_{p|a/l_0} \left(1 - \frac{1}{p^{1+x}} \right)^{-1} \\ \times \prod_{p|b/k_0} \left(1 - \frac{1}{p^{1+y}} \right)^{-1}.$$

Note that the product and sums converge as $\Re(u) = \Re(v) = \epsilon > 0$ and $\Re(s) = 2\epsilon$.

The first product factors into

$$\prod_{p \nmid \frac{abq}{k_0 l_0}} \left(1 - \frac{1}{p^{1+x+y-z}} \right)^{-1} \left(1 - \frac{1}{p^{1+y}} \right)^{-1} \left(1 - \frac{1}{p^{1+x}} \right)^{-1} \left(1 - \frac{1}{p^{2+x+y}} \right)^{-1}$$

and hence (4.13) is equal to

$$\frac{\zeta_q(1+x+y-z)\zeta_q(1+x)\zeta_q(1+y)}{\zeta_q(2+x+y)} \\ \times \prod_{p|a/l_0} c_p(x+y-z, y, x+y) \prod_{p|b/k_0} c_p(x+y-z, x, x+y),$$

where

$$c_p(a, b, c) = \left(1 - \frac{1}{p^{1+a}} \right) \left(1 - \frac{1}{p^{1+b}} \right) \left(1 - \frac{1}{p^{2+c}} \right)^{-1}.$$

Therefore (4.12) is equal to

$$\frac{\zeta_q(1+\alpha+\gamma+u+v+2s)\zeta_q(1+\alpha-\beta+2u)\zeta_q(1+\gamma-\delta+2v)}{\zeta_q(2+\alpha+\gamma-\beta-\delta+2u+2v)} \\ \times \sum_{\substack{1 \leq j \\ (j, q)=1}} \sum_{(a, b)=1} \frac{\alpha_{aj} \alpha_{bj}}{j a^{1-s-\beta+u} b^{1-s-\delta+v}} \eta_{\alpha, \beta, \gamma, \delta, a, b}(u, v, s),$$

where

$$\eta_{\alpha, \beta, \gamma, \delta, a, b}(u, v, s) := \eta_{\alpha, \beta, \gamma, \delta, a}(u, v, s) \eta_{\gamma, \delta, \alpha, \beta, b}(v, u, s)$$

with

$$\eta_{\alpha, \beta, \gamma, \delta, a}(u, v, s) := \sum_{l_0|a} l_0^{-2s-v+u-\beta-\gamma}$$

$$\times \prod_{p|a/l_0} c_p(u+v+\alpha+\gamma+2s, 2v+\gamma-\delta, \alpha+\gamma-\beta-\delta+2u+2v). \quad (4.14)$$

We also make the substitution

$$\sum_{\substack{1 \leq h \\ (h,k)=1}} h^{-2s-\beta-\delta+u+v} = \prod_{p|k} (1 - p^{-2s-\beta-\delta+u+v}) \sum_{\substack{1 \leq h \\ (h,k)=1}} h^{-2s-\beta-\delta+u+v}$$

to arrive at the equation (relabelling j as d)

$$\begin{aligned} \mathcal{M}_{w,k} &= \frac{1}{(2\pi i)^3} \int \int_{(2\epsilon)} \int_{(\epsilon)} \int_{(\epsilon)} \prod_{p|k} (1 - p^{-2s-\beta-\delta+u+v}) \sum_{\substack{1 \leq h \\ (h,k)=1}} (wh)^{-2s-\beta-\delta+u+v} \\ &\hat{g}(u) \hat{g}(v) \frac{\zeta_q(1+\alpha+\gamma+u+v+2s) \zeta_q(1+\alpha-\beta+2u) \zeta_q(1+\gamma-\delta+2v)}{\zeta_q(2+\alpha+\gamma-\beta-\delta+2u+2v)} H_{u,v,t}(s) \\ &X(s, t) \sum_{\substack{1 \leq d \\ (d,q)=1}} \sum_{(a,b)=1} \frac{\alpha_{ad} \alpha_{bd}}{da^{1-s-\beta+u} b^{1-s-\delta+v}} \eta_{\alpha,\beta,\gamma,\delta,a,b}(u, v, s) G(s) \frac{q^{2s}}{\pi^{2s}} \psi_t du dv \frac{ds}{s} dt. \end{aligned}$$

By Lemma (2.6),

$$\begin{aligned} \sum_{w|q} \left(\frac{q}{w}\right) \phi(w) \sum_{\substack{k|q \\ (k,w)=1}} \frac{\mu(k) \phi\left(\frac{q}{k}\right)}{k} w^{-2s-\beta-\delta+u+v} \prod_{p|k} (1 - p^{-2s-\beta-\delta+u+v}) \\ = \phi^*(q) \prod_{p|q} (1 - p^{2s+\beta+\delta-u-v-1}) q^{-2s-\beta-\delta+u+v}, \end{aligned}$$

and so

$$\frac{1}{2\phi^*(q)T} \sum_{w|q} \left(\frac{q}{w}\right) \phi(w) \sum_{\substack{k|q \\ (k,w)=1}} \frac{\mu(k) \phi\left(\frac{q}{k}\right)}{k} \mathcal{M}_{w,k} = \mathcal{M},$$

where $\mathcal{M}$ is equal to

$$\begin{aligned} &\frac{1}{2(2\pi i)^3 T} \int \int_{(2\epsilon)} \int_{(\epsilon)} \int_{(\epsilon)} \sum_{1 \leq h} h^{-2s-\beta-\delta+u+v} \prod_{p|q} (1 - p^{2s+\beta+\delta-u-v-1}) \hat{g}(u) \hat{g}(v) \\ &\frac{\zeta_q(1+\alpha+\gamma+u+v+2s) \zeta_q(1+\alpha-\beta+2u) \zeta_q(1+\gamma-\delta+2v)}{\zeta_q(2+\alpha+\gamma-\beta-\delta+2u+2v)} H_{u,v,t}(s) X(s, t) \\ &\sum_{\substack{1 \leq d \\ (d,q)=1}} \sum_{(a,b)=1} \frac{\alpha_{ad} \alpha_{bd}}{da^{1-s-\beta+u} b^{1-s-\delta+v}} \eta_{\alpha,\beta,\gamma,\delta,a,b}(u, v, s) \frac{q^{-\beta-\delta+u+v}}{\pi^{2s}} G(s) \psi_t du dv \frac{ds}{s} dt. \quad (4.15) \end{aligned}$$

By Lemma 2.5, $H_{u,v,t}(s)$ has simple poles at $s = \frac{1}{2} - \beta - it + u$ and $\frac{1}{2} - \delta + it + v$, both with residue 2, and a zero at $\frac{1-\beta-\delta+u+v}{2}$. We shall now endeavour to replace the sum over h in (4.15) with a zeta-function. We move the v -contour to the line $\Re(v) = \frac{\epsilon}{2}$

then move the s -contour to the line $\Re(s) = \frac{1}{2} + \frac{7\epsilon}{8}$, crossing a pole of $H_{u,v,t}(s)$ at $s = \frac{1}{2} - \delta + it + v$. We label the new contour $\mathcal{P}$ and the residue $\mathcal{R}$. In $\mathcal{R}$, we move the integral over u to $\Re(u) = \frac{\epsilon}{4}$ which does not cross any poles. In both $\mathcal{P}$ and $\mathcal{R}$ the sum over h now converges and we may write it as a zeta function, so that

$$\begin{aligned} \mathcal{P} = & \frac{1}{2(2\pi i)^3 T} \int \int_{\left(\frac{1}{2} + \frac{7\epsilon}{8}\right)} \int_{\left(\frac{\epsilon}{2}\right)} \int_{(\epsilon)} \zeta(2s + \beta + \delta - u - v) \prod_{p|q} (1 - p^{2s + \beta + \delta - u - v - 1}) \\ & \frac{\zeta_q(1 + \alpha + \gamma + u + v + 2s) \zeta_q(1 + \alpha - \beta + 2u) \zeta_q(1 + \gamma - \delta + 2v)}{\zeta_q(2 + \alpha + \gamma - \beta - \delta + 2u + 2v)} H_{u,v,t}(s) X(s, t) \hat{g}(u) \\ & \hat{g}(v) \sum_{\substack{1 \leq d \\ (d,q)=1}} \sum_{(a,b)=1} \frac{\alpha_{ad} \alpha_{bd}}{da^{1-s-\beta+u} b^{1-s-\delta+v}} \eta_{\alpha,\beta,\gamma,\delta,a,b}(u, v, s) \frac{q^{-\beta-\delta+u+v}}{\pi^{2s}} G(s) \psi_t du dv \frac{ds}{s} dt \end{aligned}$$

and

$$\begin{aligned} \mathcal{R} = & \frac{1}{(2\pi i)^2 T} \int \int_{\left(\frac{\epsilon}{2}\right)} \int_{\left(\frac{\epsilon}{4}\right)} \zeta(1 + \beta - \delta + 2it - u + v) \prod_{p|q} (1 - p^{\beta-\delta+2it-u+v}) \\ & \frac{\zeta_q(2 + \alpha + \gamma - 2\delta + 2it + u + 3v) \zeta_q(1 + \alpha - \beta + 2u) \zeta_q(1 + \gamma - \delta + 2v)}{\zeta_q(2 + \alpha + \gamma - \beta - \delta + 2u + 2v)} \\ & X\left(\frac{1}{2} - \delta + it + v, t\right) \hat{g}(u) \hat{g}(v) \sum_{\substack{1 \leq d \\ (d,q)=1}} \sum_{(a,b)=1} \frac{\alpha_{ad} \alpha_{bd}}{da^{\frac{1}{2}+\delta-\beta-it+u-v} b^{\frac{1}{2}-it}} \frac{q^{-\beta-\delta+u+v}}{\pi^{1-2\delta+2it+2v}} \\ & \eta_{\alpha,\beta,\gamma,\delta,a,b}\left(u, v, \frac{1}{2} - \delta + it + v\right) G\left(\frac{1}{2} - \delta + it + v\right) \psi_t \frac{du dv dt}{\frac{1}{2} - \delta + it + v}. \end{aligned}$$

In $\mathcal{P}$, we move the s -contour back to the line $\Re(s) = \frac{1}{4} + \frac{\epsilon}{2}$, crossing the same pole again at $s = \frac{1}{2} - \delta + it + v$. The pole at $\frac{1-\beta-\delta+u+v}{2}$ coming from $\zeta(2s + \beta + \delta - u - v)$ has residue zero due to the zero at that point from $H_{u,v,t}(s)$. Label the new integral as $\mathcal{P}'$ and the new residue as $\mathcal{R}'$ so that (4.15) is equal to

$$\mathcal{P}' + \mathcal{R} - \mathcal{R}'.$$

As the only difference between $\mathcal{R}$ and $\mathcal{R}'$ is in the u -contour, $\mathcal{R} - \mathcal{R}'$ is precisely the residue at $u = v + \beta - \delta + 2it$. However, at this point

$$\prod_{p|q} (1 - p^{\beta-\delta+2it-u+v}) = 0$$

and so the residue is zero. Hence

$$\mathcal{M} = \mathcal{P}'$$

$$\begin{aligned}
&= \frac{1}{2(2\pi i)^3 T} \int \int_{(\frac{1}{4}+\frac{\epsilon}{2})} \int_{(\frac{\epsilon}{2})} \int_{(\epsilon)} \zeta(2s + \beta + \delta - u - v) \prod_{p|q} (1 - p^{2s+\beta+\delta-u-v-1}) \\
&\quad \frac{\zeta_q(1 + \alpha + \gamma + u + v + 2s) \zeta_q(1 + \alpha - \beta + 2u) \zeta_q(1 + \gamma - \delta + 2v)}{\zeta_q(2 + \alpha + \gamma - \beta - \delta + 2u + 2v)} H_{u,v,t}(s) X(s, t) \hat{g}(u) \\
&\quad \hat{g}(v) \sum_{\substack{1 \leq d \\ (d,q)=1}} \sum_{(a,b)=1} \frac{\alpha_{ad} \alpha_{bd}}{d a^{1-s-\beta+u} b^{1-s-\delta+v}} \eta_{\alpha,\beta,\gamma,\delta,a,b}(u, v, s) \frac{q^{-\beta-\delta+u+v}}{\pi^{2s}} G(s) \psi_t du dv \frac{ds}{s} dt.
\end{aligned}$$

Like in the second moment, using Lemma 8.2 of [124] we may write

$$H_{u,v,t}(s) = \pi^{1/2} \frac{\Gamma(\frac{\beta-u+\delta-v+2s}{2}) \Gamma(\frac{1/2-\beta+u-it-s}{2}) \Gamma(\frac{1/2-\delta+v+it-s}{2})}{\Gamma(\frac{1-\beta+u-\delta+v-2s}{2}) \Gamma(\frac{1/2+\beta-u+it+s}{2}) \Gamma(\frac{1/2+\delta-v-it+s}{2})}.$$

Using Stirling's approximation we see that for $|u|, |v|, |s| \ll (qT)^\epsilon$,

$$H_{u,v,t}(s) \ll (qT)^\epsilon T^{\Re(-2s+u+v)}.$$

Equipped with this bound, we move the contours over u and v to $\Re(u) = \Re(v) = -\frac{1}{4} + \epsilon$, crossing simple poles at $u = 0$ and $v = 0$, but not collecting residues from $u = \frac{\beta-\alpha}{2}$ or $v = \frac{\delta-\gamma}{2}$ due to (4.7). We shall denote the contribution from the residue at $u = v = 0$ by $\mathcal{M}_0$. Recall that $\hat{g}(u)$ decays rapidly in the vertical direction for u in any fixed strip, as does $X(s, t)$ in s . On the new contours, we have either $\Re(u+v) = -\frac{1}{2} + 2\epsilon$ or $\Re(u+v) = -\frac{1}{4} + \epsilon$. In this region, using the bound

$$\zeta(1+s) = \frac{1}{s} + O(1)$$

for s in a neighbourhood of 1, and the bound (4.6), as well as the divisor bound to show that

$$\prod_{p|q} (1 - p^{2s+\beta+\delta-u-v-1}) \eta_{\alpha,\beta,\gamma,\delta,a,b}(u, v, s) \ll_\epsilon (qT)^\epsilon,$$

we may conclude that the new contours are of size at most

$$\ll_\epsilon (qT)^{\epsilon+u+v+\kappa(2s-u-v)} \ll_\epsilon (qT)^{\frac{3\kappa}{4}-\frac{1}{4}+\epsilon} + (qT)^{\kappa-\frac{1}{2}+\epsilon}$$

which is acceptable. This leaves the contribution from the residue at $u = v = 0$,

$$\begin{aligned}
\mathcal{M}_0 &= \frac{1}{4\pi i} \frac{\zeta_q(1 + \alpha - \beta) \zeta_q(1 + \gamma - \delta)}{\zeta_q(2 + \alpha + \gamma - \beta - \delta)} \int \int_{(\frac{1}{4}+\frac{\epsilon}{2})} \zeta(2s + \beta + \delta) \prod_{p|q} (1 - p^{2s+\beta+\delta-1}) \\
&\quad \zeta_q(1 + \alpha + \gamma + 2s) H_{0,0,t}(s) X(s, t) \sum_{\substack{1 \leq d \\ (d,q)=1}} \sum_{(a,b)=1} \frac{\alpha_{ad} \alpha_{bd}}{d a^{1-s-\beta} b^{1-s-\delta}} \eta_{\alpha,\beta,\gamma,\delta,a,b}(0, 0, s) \frac{q^{-\beta-\delta}}{\pi^{2s}}
\end{aligned}$$

$$G(s)\psi_t \frac{ds}{s} dt.$$

Applying the functional equation

$$\begin{aligned} \pi^{-(\beta+\delta+2s)/2} \Gamma\left(\frac{\beta+\delta+2s}{2}\right) \zeta(\beta+\delta+2s) \\ = \pi^{-(1-\beta-\delta-2s)/2} \Gamma\left(\frac{1-\beta-\delta-2s}{2}\right) \zeta(1-\beta-\delta-2s) \end{aligned} \quad (4.16)$$

gives that

$$\zeta(2s+\beta+\delta)H_{0,0,t}(s) = \zeta(1-2s-\beta-\delta)\pi^{\beta+\delta+2s} \frac{\Gamma(\frac{1/2-\beta-it-s}{2})\Gamma(\frac{1/2-\delta+it-s}{2})}{\Gamma(\frac{1/2+\beta+it+s}{2})\Gamma(\frac{1/2+\delta-it+s}{2})}.$$

We introduce the other three main terms arising from $g\left(\frac{m_2}{m_1}\right)g\left(\frac{n_1}{n_2}\right)$ etc. and so it shall now be necessary to reintroduce the subscripts $\alpha, \beta, \gamma, \delta$. We conclude that

$$\mathcal{O}_{\alpha,\beta,\gamma,\delta}(T) = \mathcal{M}_{\alpha,\beta,\gamma,\delta} + \mathcal{M}_{\beta,\alpha,\gamma,\delta} + \mathcal{M}_{\alpha,\beta,\delta,\gamma} + \mathcal{M}_{\beta,\alpha,\delta,\gamma} + \mathcal{E}$$

where,

$$\mathcal{M}_{\alpha,\beta,\gamma,\delta} = \frac{1}{4\pi i} \frac{\zeta_q(1+\alpha-\beta)\zeta_q(1+\gamma-\delta)}{\zeta_q(2+\alpha+\gamma-\beta-\delta)} \int \int_{(\epsilon)} \mathfrak{M}_{\alpha,\beta,\gamma,\delta}(s,t) G(s) \psi_t \frac{ds}{s} dt,$$

with

$$\begin{aligned} \mathfrak{M}_{\alpha,\beta,\gamma,\delta}(s,t) &= \zeta_q(1-2s-\beta-\delta)\zeta_q(1+\alpha+\gamma+2s) \\ &\quad \frac{\Gamma(\frac{1/2-\beta-it-s}{2})\Gamma(\frac{1/2-\delta+it-s}{2})\Gamma(\frac{1/2+\alpha+it+s}{2})\Gamma(\frac{1/2+\gamma-it+s}{2})}{\Gamma(\frac{1/2+\beta+it}{2})\Gamma(\frac{1/2+\delta-it}{2})\Gamma(\frac{1/2+\alpha+it}{2})\Gamma(\frac{1/2+\gamma-it}{2})} \\ &\quad \sum_{\substack{1 \leq d \\ (d,q)=1}} \sum_{(a,b)=1} \frac{\alpha_{ad}\alpha_{bd}}{da^{1-s-\beta}b^{1-s-\delta}} \eta_{\alpha,\beta,\gamma,\delta,a,b}(0,0,s) \left(\frac{q}{\pi}\right)^{-\beta-\delta}. \end{aligned}$$

Proceeding identically, one can show that

$$\tilde{\mathcal{O}}_{-\gamma,-\delta,-\alpha,-\beta}(T) = \tilde{\mathcal{M}}_{-\gamma,-\delta,-\alpha,-\beta} + \tilde{\mathcal{M}}_{-\delta,-\gamma,-\alpha,-\beta} + \tilde{\mathcal{M}}_{-\gamma,-\delta,-\beta,-\alpha} + \tilde{\mathcal{M}}_{-\delta,-\gamma,-\beta,-\alpha} + \tilde{\mathcal{E}}$$

where $\tilde{\mathcal{E}}$ satisfies the same bounds as $\mathcal{E}$, and, for example,

$$\begin{aligned} \tilde{\mathcal{M}}_{-\delta,-\gamma,-\beta,-\alpha} &= \frac{\phi^*(q)}{2\pi i} \frac{\zeta_q(1+\alpha-\beta)\zeta_q(1+\gamma-\delta)}{\zeta_q(2+\alpha+\gamma-\beta-\delta)} \int \int_{(\epsilon)} \mathfrak{M}_{-\delta,-\gamma,-\beta,-\alpha}(s,t) G(s) \\ &\quad Y_{\beta,\alpha,\delta,\gamma}(t) \psi_t \frac{ds}{s} dt. \end{aligned}$$

One can check that

$$\mathfrak{M}_{\alpha,\beta,\gamma,\delta}(s, t) = \mathfrak{M}_{-\delta,-\gamma,-\beta,-\alpha}(-s, t)Y_{\beta,\alpha,\delta,\gamma}(t)$$

and as $G(s)$ vanishes at $-\frac{\alpha+\gamma}{2}$ and $-\frac{\beta+\delta}{2}$, we see that by a change of variables $s \rightarrow -s$,

$$\mathcal{M}_{\alpha,\beta,\gamma,\delta} + \tilde{\mathcal{M}}_{-\delta,-\gamma,-\beta,-\alpha} = \frac{1}{2} \frac{\zeta_q(1+\alpha-\beta)\zeta_q(1+\gamma-\delta)}{\zeta_q(2+\alpha+\gamma-\beta-\delta)} \int \mathfrak{M}_{\alpha,\beta,\gamma,\delta}(0, t)\psi_t dt,$$

coming from the residue at $s = 0$. The other terms pair off in a similar fashion. To conclude the proof, we must show that

$$\begin{aligned} & \frac{\zeta_q(1+\alpha-\beta)\zeta_q(1+\gamma-\delta)\zeta_q(1+\alpha+\gamma)\zeta_q(1-\beta-\delta)}{\zeta_q(2+\alpha+\gamma-\beta-\delta)} \sum_{1 \leq d} \sum_{\substack{a,b \\ (a,b)=1}} \frac{\alpha_{ad}\alpha_{bd}}{da^{1-\beta}b^{1-\delta}} \\ & \times \eta_{\alpha,\beta,\gamma,\delta,a,b}(0, 0, 0) = \sum_{1 \leq d} \sum_{\substack{a,b \\ (a,b)=1}} \frac{\alpha_{ad}\alpha_{bd}}{dab} Z_{\alpha,-\delta,\gamma,-\beta,a,b}(0). \end{aligned} \quad (4.17)$$

The proof of (4.17) is practically identical to the final part of the proof of Theorem 1.1 in [6], but we shall recreate it here for the sake of completeness. To prove (4.17), we need only show that for any prime p and positive integer v ,

$$p^{v\beta}\eta_{\alpha,\beta,\gamma,\delta,p^v}(0, 0, 0) = B_{\alpha,-\delta,\gamma,-\beta,p^v}(0), \quad (4.18)$$

as then by multiplicativity

$$a^\beta b^\delta \eta_{\alpha,\beta,\gamma,\delta,a,b}(0, 0, 0) = B_{\alpha,-\delta,\gamma,-\beta,a}(0)B_{\gamma,-\beta,\alpha,-\delta,b}(0)$$

and (4.17) follows immediately. By (4.14),

$$\eta_{\alpha,\beta,\gamma,\delta,p^v}(0, 0, 0) = p^{-v(\beta+\gamma)} + \sum_{j=0}^{v-1} p^{-j(\beta+\gamma)} c_p(\alpha + \gamma, \gamma - \delta, \alpha + \gamma - \beta - \delta)$$

and so $p^{v\beta}\eta_{\alpha,\beta,\gamma,\delta,p^v}(0, 0, 0)$ is equal to

$$p^{-v\gamma} + \frac{p^{-v\gamma} - p^{v\beta}}{p^{-(\beta+\gamma)} - 1} \left(1 - \frac{1}{p^{1+\alpha+\gamma}}\right) \left(1 - \frac{1}{p^{1+\gamma-\delta}}\right) \left(1 - \frac{1}{p^{2+\alpha-\beta+\gamma-\delta}}\right)^{-1}. \quad (4.19)$$

Turning to $B_{\alpha,-\delta,\gamma,-\beta,p^v}(0)$, Hughes and Young proved in Lemma 6.9 of [61] that

$$B_{\alpha,-\delta,\gamma,-\beta,p^v}(0) = \left(1 - \frac{1}{p^{2+\alpha-\beta+\gamma-\delta}}\right)^{-1} \frac{p^{-\beta}}{p^{-\beta-\gamma} - 1} (B_1 - p^{-1}B_1 + p^{-2}B_2), \quad (4.20)$$

where

$$B_0 = p^{-\gamma-v\gamma} - p^{\beta+v\beta},$$

$$B_1 = p^{\beta-\gamma}(p^{-\alpha} + p^{\delta})(p^{-v\gamma} - p^{v\beta}),$$

$$B_2 = p^{-\alpha+\beta-\gamma+\delta}(p^{\beta-v\gamma} - p^{v\beta-\gamma}).$$

By comparing the coefficients of p^0, p^{-1} , and p^{-2} in (4.19) and (4.20), one can see that (4.18) holds. Therefore

$$\begin{aligned} \mathcal{M}_{\alpha,\beta,\gamma,\delta} + \tilde{\mathcal{M}}_{-\delta,-\gamma,-\beta,-\alpha} &= \frac{1}{2} \sum_{1 \leq d} \sum_{\substack{a,b \\ (a,b)=1}} \frac{\alpha_{ad}\alpha_{bd}}{dab} Z_{\alpha,-\delta,\gamma,-\beta,a,b}(0) \\ &\quad \times \left(\frac{q}{\pi}\right)^{-\beta-\delta} \int \frac{\Gamma(\frac{1/2-\beta-it}{2})\Gamma(\frac{1/2-\delta+it}{2})}{\Gamma(\frac{1/2+\beta+it}{2})\Gamma(\frac{1/2+\delta-it}{2})} \psi_t dt \end{aligned}$$

and Theorem 4.1 follows.

4.5 Proof of Proposition 4.1

This proof has a similar structure to that of Proposition 2.1, as we use Poisson summation to isolate the main term in much the same way. However, the context of the fourth moment makes the details of manipulating the main term and bounding the error terms much more complicated.

We begin by removing the condition $(m_2 n_2, q) = 1$ using Möbius inversion. For the sake of brevity, define $F(a, b, m_1, m_2, n_1, n_2)$ to be

$$\begin{aligned} &\int \frac{\alpha_a \alpha_b}{(ab)^{\frac{1}{2}} m_1^{\frac{1}{2}+\alpha} m_2^{\frac{1}{2}+\beta} n_1^{\frac{1}{2}+\gamma} n_2^{\frac{1}{2}+\delta}} \left(\frac{bn_1 n_2}{am_1 m_2}\right)^{it} g\left(\frac{m_1}{m_2}\right) g\left(\frac{n_1}{n_2}\right) \\ &\quad W\left(\frac{m_1}{M_1}\right) W\left(\frac{m_2}{M_2}\right) W\left(\frac{n_1}{N_1}\right) W\left(\frac{n_2}{N_2}\right) V(m_1 m_2 n_1 n_2, t) \psi_t dt, \end{aligned}$$

so that

$$\begin{aligned} S_w(A, B, M_1, M_2, N_1, N_2) &= \sum_{\substack{a,b,m_1,m_2,n_1,n_2,h \\ am_1 m_2 = \pm bn_1 n_2 + wh \\ 0 < |h| \leq \frac{H}{w} \\ (abm_1 m_2 n_1 n_2, q) = 1}} F(a, b, m_1, m_2, n_1, n_2) \\ &= \sum_{\substack{k_1 | q \\ (k_1, w) = 1}} \mu(k_1) \sum_{\substack{a,b,m_1,m_2,n_1,n_2,h \\ am_1 m_2 = \pm bn_1 n_2 k_1 + wh \\ 0 < |h| \leq \frac{H}{w} \\ (h, k_1) = 1 \\ (abm_1 m_2 n_1, q) = 1}} F(a, b, m_1, m_2, n_1, k_1 n_2). \end{aligned}$$

Note that if $(k_1, wh) > 1$ then $(k_1, am_1 m_2) > 1$ and so $(q, am_1 m_2) > 1$ which implies that the inner sum is empty. Applying Möbius inversion again to m_2 we see that

$$\begin{aligned}
& S_w(A, B, M_1, M_2, N_1, N_2) \\
&= \sum_{\substack{k_1|q \\ (k_1, w)=1}} \sum_{\substack{k_2|q \\ (k_1, k_2)=1}} \mu(k_1 k_2) \sum_{\substack{a, b, m_1, m_2, n_1, n_2, h \\ am_1 m_2 k_2 = \pm b n_1 n_2 k_1 + wh \\ 0 < |h| \leq \frac{H}{w} \\ (h, k_1)=1 \\ (abm_1 n_1, q)=1}} F(a, b, m_1, k_2 m_2, n_1, k_1 n_2).
\end{aligned}$$

We may ignore the terms with $(k_1, k_2) > 1$, as in this case $(k_1, k_2) | wh$ which implies that $(k_2, wh) > 1$. Finally we employ Möbius inversion another two times for m_1 and n_1 to show that

$$\begin{aligned}
& S_w(A, B, M_1, M_2, N_1, N_2) \\
&= \sum_{\substack{k_1|q \\ (k_1, w)=1}} \sum_{\substack{k_2|q \\ (k_1, k_2)=1}} \sum_{\substack{k_3|q \\ k_4|q}} \mu(k_1 k_2) \mu(k_3) \mu(k_4) S_{w, k_1, k_2, k_3, k_4}(A, B, M_1, M_2, N_1, N_2)
\end{aligned}$$

where

$$\begin{aligned}
& S_{w, k_1, k_2, k_3, k_4}(A, B, M_1, M_2, N_1, N_2) := \\
& \sum_{\substack{a, b, m_1, m_2, n_1, n_2, h \\ am_1 m_2 k_2 k_4 = \pm b n_1 n_2 k_1 k_3 + wh \\ 0 < |h| \leq \frac{H}{w} \\ (h, k_1)=1 \\ (abm_1 n_1, q)=1}} F(a, b, k_4 m_1, k_2 m_2, k_3 n_1, k_1 n_2).
\end{aligned}$$

It will be useful to prove a trivial bound on $S_{w, k_1, k_2, k_3, k_4}(A, B, M_1, M_2, N_1, N_2)$. Suppose without loss of generality that $k_2 k_4 \geq k_1 k_3$. To satisfy the condition $am_1 m_2 k_2 k_4 = \pm b n_1 n_2 k_1 k_3 + wh$, for each choice of a, m_1 , and m_2 , the integer h must be congruent to $am_1 m_2 k_2 k_4 \pmod{k_1 k_3}$ and then b, n_1 and n_2 must be divisors of $am_1 m_2 k_2 k_4 - wh$. Therefore

$$\begin{aligned}
S_{w, k_1, k_2, k_3, k_4}(A, B, M_1, M_2, N_1, N_2) &\ll_{\epsilon} \frac{T(qT)^{\epsilon}}{\sqrt{ABMN}} \frac{AM_1 M_2}{k_2 k_4} \left(1 + \frac{H}{w k_1 k_3}\right) \\
&\ll T(qT)^{\epsilon} \left(\frac{1}{\sqrt{k_1 k_2 k_3 k_4}} + \frac{H}{w k_1 k_2 k_3 k_4}\right) \quad (4.21)
\end{aligned}$$

by the divisor bound. Hence if $(qT)^{\kappa+\epsilon} \leq k_1 k_2 k_3 k_4$ then

$$S_{w, k_1, k_2, k_3, k_4}(A, B, M_1, M_2, N_1, N_2) \ll_{\epsilon} (w^{-1}(qT)^{1-\epsilon}),$$

and we may absorb it into the error term. Consequently we may assume that

$$k_1 k_2 k_3 k_4 \leq (qT)^{\kappa+\epsilon}. \quad (4.22)$$

Let $d = (am_1k_2k_4, bn_1k_1k_3)$, meaning that $d|wh$ which allows us to write the condition $am_1m_2k_2k_4 = \pm bn_1n_2k_1k_3 + wh$ as

$$m_2 \equiv \frac{\overline{am_1k_2k_4}}{d} \frac{wh}{d} \pmod{\frac{bn_1k_1k_3}{d}}.$$

Then by Poisson summation $S_{w,k_1,k_2,k_3,k_4}(A, B, M_1, M_2, N_1, N_2)$ is

$$\begin{aligned} &= \sum_{d \geq 1} \sum_{\substack{0 < |h| \leq \frac{H}{w} \\ (h,k_1)=1 \\ d|wh}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1k_2k_4, bn_1k_1k_3)=d \\ (ab,q)=1}} \sum_{m_2 \equiv \frac{\overline{am_1k_2k_4}}{d} \frac{wh}{d} \pmod{\frac{bn_1k_1k_3}{d}}} \\ &\quad F\left(a, b, k_4m_1, k_2m_2, k_3n_1, \pm \frac{am_1m_2k_2k_4 - wh}{bn_1k_3}\right) \\ &= \sum_{d \geq 1} \sum_{\substack{0 < |h| \leq \frac{H}{w} \\ (h,k_1)=1 \\ d|wh}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1k_2k_4, bn_1k_1k_3)=d \\ (ab,q)=1}} \sum_{l \in \mathbb{Z}} e\left(-\frac{l(wh/d)\overline{am_1k_2k_4/d}}{bn_1k_1k_3/d}\right) \\ &\quad \frac{1}{k_1k_2} \int F\left(a, b, k_4m_1, \frac{xbn_1k_3}{d}, k_3n_1, \pm \frac{xabm_1n_1k_3k_4 - whd}{bn_1k_3d}\right) e\left(\frac{lx}{k_1k_2}\right) dx. \end{aligned}$$

To write this out in full, we have arrived at the conclusion that

$$\begin{aligned} &S_{w,k_1,k_2,k_3,k_4}(A, B, M_1, M_2, N_1, N_2) \\ &= S_{w,k_1,k_2,k_3,k_4}^+(A, B, M_1, M_2, N_1, N_2) + S_{w,k_1,k_2,k_3,k_4}^-(A, B, M_1, M_2, N_1, N_2), \end{aligned}$$

where $S_{w,k_1,k_2,k_3,k_4}^+(A, B, M_1, M_2, N_1, N_2)$ is equal to

$$\begin{aligned} &\frac{1}{k_1k_2} \frac{1}{2\pi i} \int \int \int \sum_{d \geq 1} \sum_{\substack{0 < |h| \leq \frac{H}{w} \\ (h,k_1)=1 \\ d|wh}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1k_2k_4, bn_1k_1k_3)=d \\ (ab,q)=1}} \sum_{l \in \mathbb{Z}} \frac{\alpha_a \alpha_b}{(ab)^{\frac{1}{2}} (m_1k_4)^{\frac{1}{2}+\alpha+s} (n_1k_3)^{\frac{1}{2}+\gamma+s}} \\ &\quad \left(\frac{bn_1k_3}{d}\right)^{-\frac{1}{2}-\beta-s} \left(\frac{am_1k_4}{d}\right)^{-\frac{1}{2}-\delta-s} e\left(-\frac{l(wh/d)\overline{am_1k_2k_4/d}}{bn_1k_1k_3/d}\right) W\left(\frac{m_1k_4}{M_1}\right) W\left(\frac{n_1k_3}{N_1}\right) \\ &\quad W\left(\frac{xbn_1k_3}{dM_2}\right) W\left(\frac{abm_1n_1k_3k_4x - whd}{dbn_1k_3N_2}\right) x^{-\frac{1}{2}-\beta-s-it} \left(x - \frac{whd}{abm_1n_2k_3k_4}\right)^{-\frac{1}{2}-\delta-s+it} \\ &\quad g\left(\frac{m_1k_4d}{xbn_1k_3}\right) g\left(\frac{n_1^2bdk_3^2}{abm_1n_1k_3k_4x - whd}\right) e\left(\frac{lx}{k_1k_2}\right) G(s)X(s, t) \frac{q^{2s}}{\pi^{2s}} \psi_t dx \frac{ds}{s} dt \quad (4.23) \end{aligned}$$

and $S_{w,k_1,k_2,k_3,k_4}^-(A, B, M_1, M_2, N_1, N_2)$ is the same as (4.23) but substituting

$$W\left(\frac{abm_1n_1k_3k_4x - whd}{dbn_1k_3N_2}\right) \left(x - \frac{whd}{abm_1n_2k_3k_4}\right)^{-\frac{1}{2}-\delta-s+it} g\left(\frac{n_1^2bdk_3^2}{abm_1n_1k_3k_4x - whd}\right)$$

for

$$W\left(\frac{whd - abm_1n_1k_3k_4x}{dbn_1k_3N_2}\right) \left(\frac{whd}{abm_1n_2k_3k_4} - x\right)^{-\frac{1}{2}-\delta-s+it} g\left(\frac{n_1^2bdk_3^2}{whd - abm_1n_1k_3k_4x}\right).$$

4.5.1 The main terms

The main terms come from the term $l = 0$. We shall separate the $l = 0$ term by writing

$$\begin{aligned} S_{w,k_1,k_2,k_3,k_4}^+ (A, B, M_1, M_2, N_1, N_2) \\ = S_{w,k_1,k_2,k_3,k_4}^{+,l=0} (A, B, M_1, M_2, N_1, N_2) + S_{w,k_1,k_2,k_3,k_4}^{+,l \neq 0} (A, B, M_1, M_2, N_1, N_2), \end{aligned}$$

where $S_{w,k_1,k_2,k_3,k_4}^{+,l=0} (A, B, M_1, M_2, N_1, N_2)$ is

$$\begin{aligned} \frac{1}{k_1 k_2 2\pi i} \int \int \int \sum_{d \geq 1} \sum_{\substack{0 < |h| \leq \frac{H}{w} \\ (h,k_1)=1 \\ d|wh}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1 k_2 k_4, bn_1 k_1 k_3)=d \\ (ab,q)=1}} \frac{\alpha_a \alpha_b}{(ab)^{\frac{1}{2}} (m_1 k_4)^{\frac{1}{2}+\alpha+s} (n_1 k_3)^{\frac{1}{2}+\gamma+s}} \\ \left(\frac{bn_1 k_3}{d} \right)^{-\frac{1}{2}-\beta-s} \left(\frac{am_1 k_4}{d} \right)^{-\frac{1}{2}-\delta-s} W \left(\frac{m_1 k_4}{M_1} \right) W \left(\frac{n_1 k_3}{N_1} \right) W \left(\frac{xbn_1 k_3}{dM_2} \right) \\ W \left(\frac{abm_1 n_1 k_3 k_4 x - whd}{dbn_1 k_3 N_2} \right) g \left(\frac{m_1 k_4 d}{xbn_1 k_3} \right) g \left(\frac{n_1^2 k_3^2 bd}{abm_1 n_1 k_3 k_4 x - whd} \right) \\ x^{-\frac{1}{2}-\beta-s-it} \left(x - \frac{whd}{abm_1 n_2 k_3 k_4} \right)^{-\frac{1}{2}-\delta-s+it} G(s) X(s, t) \frac{q^{2s}}{\pi^{2s}} \psi_t dx \frac{ds}{s} dt. \end{aligned}$$

By bounding everything trivially, we see that

$$S_{w,k_1,k_2,k_3,k_4}^{+,l=0} (A, B, M_1, M_2, N_1, N_2) \ll_{\epsilon} (qT)^{\epsilon} \frac{TH}{wk_1 k_2 k_3 k_4},$$

and therefore we may re-introduce the terms where $k_1 k_2 k_3 k_4 \geq (qT)^{\kappa+\epsilon}$ with no change to the order of the error term. Then we may extend the sum over h to infinity, contributing an error term of size at most $O_K((qT)^{-K})$ due to the t -integral.

By Möbius inversion for any function $f(m, n)$,

$$\sum_{\substack{k_3|q \\ k_4|q}} \mu(k_3) \mu(k_4) \sum_{m_1, n_1} f(m_1 k_4, n_1 k_3) = \sum_{(m_1 n_1, q)=1} f(m_1, n_1),$$

and so we may remove the sums over k_3 and k_4 by introducing the condition that $(m_1 n_1, q) = 1$. Turning our attention to the sum over d , given that $(k_1, k_2) = 1$, $k_1 k_2 | q$, $(abm_1 n_1, q) = 1$ and $(am_1 k_2, bn_1 k_1) = d$, we can conclude that $(d, q) = 1$. Therefore $d|h$. Hence up to an error of size at most $O(w^{-1}(qT)^{1-\epsilon})$,

$$\sum_{\substack{k_1|q \\ (k_1, w)=1}} \sum_{\substack{k_2|q \\ (k_1, k_2)=1}} \sum_{\substack{k_3|q \\ k_4|q}} \mu(k_1 k_2) \mu(k_3) \mu(k_4) S_{w,k_1,k_2,k_3,k_4}^{+,l=0} (A, B, M_1, M_2, N_1, N_2)$$

$$\begin{aligned}
&= \sum_{\substack{k_1|q \\ (k_1,w)=1}} \sum_{\substack{k_2|q \\ (k_1,k_2)=1}} \frac{\mu(k_1 k_2)}{2\pi i k_1 k_2} \int \int \int \sum_{\substack{d \geq 1 \\ (d,q)=1}} \sum_{\substack{0 < |h| \\ (h,k_1)=1}} \sum_{\substack{a,b,m_1,n_1 \\ (am_1,bn_1)=d \\ (abm_1n_1,q)=1}} \frac{\alpha_a \alpha_b d^{1+\beta+\delta+2s}}{a^{1+\delta+s} b^{1+\beta+s}} \\
&\quad \frac{1}{m_1^{1+\alpha+\delta+2s} n_1^{1+\beta+\gamma+2s}} W\left(\frac{m_1}{M_1}\right) W\left(\frac{n_1}{N_1}\right) W\left(\frac{xbn_1}{dM_2}\right) W\left(\frac{abm_1n_1x - whd^2}{dbn_1N_2}\right) \\
&\quad g\left(\frac{m_1d}{xbn_1}\right) g\left(\frac{n_1^2bd}{abm_1n_1x - whd^2}\right) x^{-\frac{1}{2}-\beta-s-it} \left(x - \frac{whd^2}{abm_1n_1}\right)^{-\frac{1}{2}-\delta-s+it} X(s,t) \frac{q^{2s}}{\pi^{2s}} \\
&\quad G(s) \psi_t dx \frac{ds}{s} dt. \quad (4.24)
\end{aligned}$$

By the same reasoning as (2.17),

$$\sum_{\substack{k_1|q \\ (k_1,w)=1}} \sum_{\substack{k_2|q \\ (k_1,k_2)=1}} \frac{\mu(k_1 k_2)}{k_1 k_2} = \sum_{\substack{k_1|q \\ (k_1,w)=1}} \mu(k_1) \frac{\phi(q/k_1)}{q}.$$

This gives the first main term of the proposition, with the second term coming from the same procedure applied to $S_{w,k_1,k_2,k_3,k_4}^{-,l=0}(A, B, M_1, M_2, N_1, N_2)$.

4.5.2 The error term

We shall demonstrate how to bound $S_{w,k_1,k_2,k_3,k_4}^{+,l \neq 0}(A, B, M_1, M_2, N_1, N_2)$, and the S^- case is similar. First, we write

$$\begin{aligned}
f_d(x, a, b, m_1, n_1, h) &= W\left(\frac{xbn_1k_3}{dM_2}\right) W\left(\frac{abm_1n_1k_3k_4x - whd}{dbn_1k_3N_2}\right) \\
&\quad x^{-\frac{1}{2}-\beta-s-it} \left(x - \frac{whd}{abm_1n_2k_3k_4}\right)^{-\frac{1}{2}-\delta-s+it} g\left(\frac{m_1k_4d}{xbn_1k_3}\right) g\left(\frac{n_1^2bdk_3^2}{abm_1n_1k_3k_4x - whd}\right).
\end{aligned}$$

By repeated integration by parts j times, noting that the integral is over $x \asymp \frac{dM_2}{BN_1} \asymp d\sqrt{\frac{M_2N_2}{ABM_1N_1}}$, we see that

$$\int f_d(x, a, b, m_1, n_1, h) e\left(\frac{lx}{k_1k_2}\right) dx \ll_j (qT)^\epsilon \left(\frac{k_1k_2}{d|l|} \sqrt{\frac{ABM_1N_1}{M_2N_2}}\right)^j,$$

so we may assume that $0 < |l| \leq L := \frac{k_1k_2}{d} \sqrt{\frac{ABM_1N_1}{M_2N_2}} (qT)^\epsilon$, as otherwise there is a negligible contribution to the error term. We then write $f_d(x, a, b, m_1, n_1, h)$ in terms of its Mellin transform,

$$\begin{aligned}
f_d(x, a, b, m_1, n_1, h) &= \frac{1}{(2\pi i)^6} \int_{(\epsilon)} \dots \int_{(\epsilon)} \hat{f}_d(u_1, u_2, u_3, u_4, u_5, u_6) \\
&\quad \times x^{-u_1} a^{-u_2} b^{-u_3} m^{-u_4} n^{-u_5} h^{-u_6} du_1 \dots du_6.
\end{aligned}$$

By repeated integration by parts j times on each variable $u_1, \dots, u_6$ in the expression

$$\begin{aligned} \hat{f}_d(u_1, u_2, u_3, u_4, u_5, u_6) &= \int_0^\infty \dots \int_0^\infty x^{u_1-1} a^{u_2-1} b^{u_3-1} m^{u_4-1} n^{u_5-1} h^{u_6-1} \\ &\quad \times f_d(x, a, b, m_1, n_1, h) da \dots dh, \end{aligned}$$

we can see that

$$\hat{f}_d(u_1, u_2, u_3, u_4, u_5, u_6) \ll_{\epsilon_1, \epsilon_2, j} (qT)^{\epsilon_1} \left((qT)^{-\epsilon_2} \prod_{i=1}^6 (1 + |u_i|) \right)^{-j} \frac{1}{d} \sqrt{\frac{ABM_1N_1}{M_2N_2}} (qT)^\epsilon$$

for any fixed $j \geq 0$. This implies that $S_{w, k_1, k_2, k_3, k_4}^{+, l \neq 0}(A, B, M_1, M_2, N_1, N_2)$ is

$$\begin{aligned} \ll_\epsilon \sum_{d \geq 1} \frac{d}{k_1 k_2 A B M_1 N_1} \int \int \int_{x \asymp d \sqrt{\frac{M_2 N_2}{A B M_1 N_1}}} \int_{(\epsilon)} \dots \int_{(\epsilon)} |\hat{f}_d(u_1, u_2, u_3, u_4, u_5, u_6)| |Z_d| \\ |G(s)| \psi_t du_1 \dots du_6 dx \frac{ds}{|s|} dt \quad (4.25) \end{aligned}$$

where Z_d is a sum of the form

$$\begin{aligned} \sum_{\substack{0 < h \\ (h, k_1)=1 \\ d|wh}} \sum_{\substack{a, b, m_1, n_1 \\ (am_1 k_2 k_4, bn_1 k_1 k_3)=d \\ (ab, q)=1}} \sum_{0 < l \leq L} \alpha'_a \beta'_b W_1 \left(\frac{m_1 k_4}{M_1} \right) W_2 \left(\frac{n_1 k_3}{N_1} \right) \\ W_3 \left(\frac{hw}{H} \right) W_4 \left(\frac{l}{L} \right) e \left(-\frac{l(wh/d) \overline{am_1 k_2 k_4/d}}{bn_1 k_1 k_3/d} \right), \end{aligned}$$

with $\alpha'_a, \beta'_b \ll (qT)^\epsilon$, supported on square-free integers co-prime to q on $[A, 2A]$ and $[B, 2B]$ respectively, and $W_i(x)$ are smooth functions with $W_i^{(j)}(x) \ll_j (qT)^\epsilon$ for $j \geq 0$, with support on $[0, 2]$.

We write the sum

$$\sum_{(h, k_1)=1} f(h) = \sum_{x|k_1} \mu(x) \sum_h f(xh)$$

and then in the second sum relabel wx as w , so it is sufficient to find bounds for

$$\begin{aligned} \sum_{\substack{0 < h \\ d|wh}} \sum_{\substack{a, b, m_1, n_1 \\ (am_1 k_2 k_4, bn_1 k_1 k_3)=d \\ (ab, q)=1}} \sum_{0 < l \leq L} \alpha'_a \beta'_b W_1 \left(\frac{m_1 k_4}{M_1} \right) W_2 \left(\frac{n_1 k_3}{N_1} \right) \\ W_3 \left(\frac{hw}{H} \right) W_4 \left(\frac{l}{L} \right) e \left(-\frac{l(wh/d) \overline{am_1 k_2 k_4/d}}{bn_1 k_1 k_3/d} \right). \end{aligned}$$

We write $d = d' d_w$, where $d_w = (d, w)$, $h = d' h'$, and $w' = \frac{w}{d_w}$ so that

$$Z_d = Z_{d'd_w} = \sum_{0 < |h'|} \sum_{\substack{a,b,m_1,n_1 \\ (am_1k_2k_4, bn_1k_1k_3)=d'd_w \\ (ab,q)=1}} \sum_{0 < l \leq L} \alpha'_a \beta'_b W_1 \left(\frac{m_1 k_4}{M_1} \right) W_2 \left(\frac{n_1 k_3}{N_1} \right) \\ W_3 \left(\frac{h' w d'}{H} \right) W_4 \left(\frac{l}{L} \right) e \left(-\frac{lw' h' \overline{am_1 k_2 k_4 / d' d_w}}{bn_1 k_1 k_3 / d' d_w} \right).$$

If w' is not equal to 1, we shall require that the ak_2k_4 and bk_1k_3 terms in the exponential are square-free and co-prime to w' , and we shall show how we can reduce Z_d to this. Suppose that $p^2 | ak_2k_4$, then as α_n is supported over square-free integers co-prime to q , and (and therefore k_2k_4), it must be the case that $p^2 | k_2k_4$. Recall that by the presence of the Möbius functions $\mu(k_1k_2)\mu(k_3)\mu(k_4)$ in the definition of $S_w(A, B, M_1, M_2, N_1, N_2)$, we may assume that k_2k_4 is cube-free. As w' is not equal to 1, then as q is either a prime or a powerful number and $\frac{q}{w}$ is square-free, it must be the case that $p | w$. Therefore, $p | d_w$ or $p | w'$. If $p | d_w$ (and so $p | n_1k_1k_3$), then we write $d_w = d'_w p$, $k_2k_4 = k'_2k'_4 p$ and $n_1k_1k_3 = n'_1k'_1k'_3 p$. This allows us to write

$$Z_d = \sum_{0 < |h'|} \sum_{\substack{a,b,m_1,n_1 \\ (am_1k'_2k'_4, bn'_1k'_1k'_3)=d'd'_w \\ (ab,q)=1}} \sum_{0 < l \leq L} \alpha'_a \beta'_b W_1 \left(\frac{m_1 k_4}{M_1} \right) W_2 \left(\frac{n_1 k_3}{N_1} \right) \\ W_3 \left(\frac{h' w d'}{H} \right) W_4 \left(\frac{l}{L} \right) e \left(-\frac{lw' h' \overline{am_1 k'_2 k'_4 / d' d'_w}}{bn'_1 k'_1 k'_3 / d' d'_w} \right).$$

In this new sum $p^2 \nmid k'_2k'_4$. Alternatively, if $p | w'$, then writing $w' = w''p$ and $k_2k_4 = k'_2k'_4 p$, the sum becomes

$$Z_d = \sum_{0 < |h'|} \sum_{\substack{a,b,m_1,n_1 \\ (am_1k_2k_4, bn_1k_1k_3)=d'd_w \\ (ab,q)=1}} \sum_{0 < l \leq L} \alpha'_a \beta'_b W_1 \left(\frac{m_1 k_4}{M_1} \right) W_2 \left(\frac{n_1 k_3}{N_1} \right) \\ W_3 \left(\frac{h' w d'}{H} \right) W_4 \left(\frac{l}{L} \right) e \left(-\frac{lw'' h' \overline{am_1 k_2 k_4 / d' d_w}}{bn_1 k_1 k_3 / d' d_w} \right).$$

And so again $p^2 \nmid ak'_2k'_4$.

This means that we may reduce the problem to finding bounds for the following types of sum:

$$\mathcal{Z}_d = \sum_{0 < h} \sum_{\substack{a,b,m_1,n_1 \\ (am_1, bn_1)=d}} \sum_{0 < l \leq L} \alpha_a^{(k'_2k'_4)} \beta_b^{(k'_1k'_3)} W_1 \left(\frac{m_1 k_4}{M_1} \right) W_2 \left(\frac{n_1 k_3}{N_1} \right) \\ W_3 \left(\frac{h w d}{H} \right) W_4 \left(\frac{l}{L} \right) e \left(-\frac{lw' h \overline{am_1 / d}}{bn_1 / d} \right) \quad (4.26)$$

where $w'|w$, $\alpha_a^x := \alpha'_{a/x} \mathbb{1}_{x|a}$ and, in the case that $w' \neq 1$, is supported on square-free integers co-prime to w' . Recall that we may assume that $k_1 k_2 k_3 k_4 \leq (qT)^{\kappa+\epsilon}$. We also have that $k'_i | k_i$ for $i = 1, \dots, 4$ and $k'_1 k'_2 k'_3 k'_4 w | q$, and $k_1 k_2 | q$.

By (4.25),

$$S_{w,k_1,k_2,k_3,k_4}^{+,l \neq 0}(A, B, M_1, M_2, N_1, N_2) \ll \frac{T}{k_1 k_2 A B_1 M_1 N_1} \sum_{d \ll H} d |\mathcal{Z}_d|. \quad (4.27)$$

We shall bound this in several ways, starting with the Weil bound. We write

$$\begin{aligned} \mathcal{Z}_d = \sum_{d=d_1 d_2} \sum_{0 < h} \sum_{\substack{a,b,m_1,n_1 \\ (am_1, bn_1/d)=1 \\ d|bn_1 \\ (a,d_2)=1}} \sum_{0 < l \leq L} \alpha_{ad_1}^{(k'_2 k'_4)} \beta_b^{(k'_1 k'_3)} W_1 \left(\frac{m_1 d_2 k_4}{M_1} \right) W_2 \left(\frac{n_1 k_3}{N_1} \right) \\ W_3 \left(\frac{hwd}{H} \right) W_4 \left(\frac{l}{L} \right) e \left(-\frac{lw'h \overline{am_1}}{bn_1/d} \right), \end{aligned} \quad (4.28)$$

and then by partial summation and Weil's bound

$$\begin{aligned} \sum_{\substack{m_1 \\ (m_1, bn_1/d)=1}} W_1 \left(\frac{m_1 d_2 k_4}{M_1} \right) e \left(-\frac{lw'h \overline{am_1}}{bn_1/d} \right) \\ \ll_{\epsilon} (qT)^{\epsilon} \left(w'lh, \frac{bn_1}{d} \right)^{\frac{1}{2}} \left(\frac{bn_1}{d} \right)^{\frac{1}{2}+\epsilon} \left(1 + \frac{d_1 M_1}{k_4 bn_1} \right). \end{aligned}$$

Due to the fact that

$$\left(w'lh, \frac{bn_1}{d} \right) \leq (1 + w' \mathbb{1}_{w'|n_1}) \left(lh, \frac{bn_1}{d} \right)$$

we can obtain

$$\begin{aligned} \mathcal{Z}_d &\ll_{\epsilon} (qT)^{\epsilon} \sum_{d_1 d_2 = d} \sum_{\substack{a,l,h \\ l \leq L \\ h \leq \frac{H}{wd}}} |\alpha_{ad_1}^{(k'_2 k'_4)}| \sum_{\substack{b,n_1 \\ n_1 \leq \frac{N_1}{k_3} \\ d|bn_1}} |\beta_b^{(k'_1 k'_3)}| \left(lh, \frac{bn_1}{d} \right)^{\frac{1}{2}} (1 + w' \mathbb{1}_{w'|n_1})^{\frac{1}{2}} \\ &\quad \times \left(\frac{bn_1}{d} \right)^{\frac{1}{2}+\epsilon} \left(1 + \frac{d_1 M_1}{k_4 bn_1} \right) \\ &\ll_{\epsilon} (qT)^{\epsilon} \frac{ALHB^{\frac{3}{2}} (k'_1 k'_3)^{\frac{1}{2}} N_1^{\frac{3}{2}}}{wd_1 d^{\frac{5}{2}} k_3^{\frac{3}{2}}} \left(1 + \frac{d_1 M_1 k_3}{BN_1 k'_1 k'_3 k_4} \right) \\ &\ll_{\epsilon} (qT)^{\epsilon} \frac{ABM_1 N_1 k_1 k_2}{dT} \frac{AB^{\frac{1}{2}} N_1^{\frac{1}{2}} k_1^{\frac{1}{2}}}{d^{\frac{5}{2}}} \left(BN_1 + \frac{M_1}{k'_2 k'_4 k_4} \right). \end{aligned}$$

So

$$S_{w,k_1,k_2,k_3,k_4}^{+,l \neq 0}(A, B, M_1, M_2, N_1, N_2) \ll_{\epsilon} \frac{AB^{\frac{1}{2}} N_1^{\frac{1}{2}} k_1^{\frac{1}{2}}}{w} \left(BN_1 + \frac{M_1}{k'_2 k'_4 k_4} \right) \sum_d \frac{1}{d^{\frac{5}{2}}}$$

$$\ll_{\epsilon} \frac{1}{w} (qT)^{\frac{5\kappa}{2} + \frac{3}{4} + \epsilon} k_1'^{\frac{1}{2}}.$$

Writing $X := \min\{q, (qT)^{\kappa}\}$, we see that

$$S_{w, k_1, k_2, k_3, k_4}^{+, l \neq 0}(A, B, M_1, M_2, N_1, N_2) \ll_{\epsilon} \frac{1}{w} (qT)^{\frac{5\kappa}{2} + \frac{3}{4} + \epsilon} X^{\frac{1}{2}}. \quad (4.29)$$

This is sufficient to prove Corollary 4.1.

Now we split up the sum over d into two sections:

$$1 \leq d \leq (qT)^{-10\epsilon} \frac{(AB)^{\frac{1}{2}}}{k_1 k_2} \left(\frac{k_1' k_2' k_3' k_4' w'}{q} \right)^2$$

(if this range exists), and

$$d \geq (qT)^{-10\epsilon} \frac{(AB)^{\frac{1}{2}}}{k_1 k_2} \left(\frac{k_1' k_2' k_3' k_4' w'}{q} \right)^2.$$

Using Weil's bound as before on the second section of the sum we find that its contribution is

$$\begin{aligned} & \ll_{\epsilon} \frac{AB^{\frac{1}{2}} N_1^{\frac{1}{2}} k_1'^{\frac{1}{2}}}{w} \left(BN_1 + \frac{M_1}{k_2' k_4' k_4} \right) \sum_{d \geq 1 + (qT)^{-10\epsilon} \frac{(AB)^{\frac{1}{2}}}{k_1 k_2} \left(\frac{k_1' k_2' k_3' k_4' w'}{q} \right)^2} \frac{1}{d^{\frac{5}{2}}} \\ & \ll_{\epsilon} \frac{(qT)^{\epsilon}}{w} \min \left((qT)^{\frac{5\kappa}{2} + \frac{3}{4}} k_1'^{\frac{1}{2}}, (qT)^{\kappa + \frac{3}{4}} (k_1 k_2)^{\frac{3}{2}} q^3 \right) \\ & \ll_{\epsilon} \frac{(qT)^{\epsilon}}{w} \min \left((qT)^{\frac{5\kappa}{2} + \frac{3}{4}} X^{\frac{1}{2}}, q^{\kappa + \frac{15}{4}} T^{\kappa + \frac{3}{4}} X^{\frac{3}{2}} \right). \end{aligned}$$

Therefore

$$\frac{T}{k_1 k_2 AB_1 M_1 N_1} \sum_{d \geq (qT)^{-10\epsilon} \frac{(AB)^{\frac{1}{2}}}{k_1 k_2} \left(\frac{k_1' k_2' k_3' k_4' w'}{q} \right)^2} d |Z_d|$$

is at most

$$\ll_{\epsilon} \frac{(qT)^{\epsilon}}{w} \min \left((qT)^{\frac{5\kappa}{2} + \frac{3}{4}} X^{\frac{1}{2}}, q^{\kappa + \frac{15}{4}} T^{\kappa + \frac{3}{4}} X^{\frac{3}{2}} \right). \quad (4.30)$$

For the first section of the sum, we shall use the following proposition. This proposition is an updated version of Proposition 3 in [8] by Bettin, Chandee, and Radziwiłł, which is fundamentally based on Watt's Proposition 4.1 in [120]. Bettin, Chandee, and Radziwiłł's version is precisely what is needed for the $w' = 1$ case, as demonstrated in [6]. Harman, Watt, and Wong then proved a different version of Watt's bound (Proposition 1 of [50]) which when substituted in the place of Watt's bound into the proof of Proposition 3 in [8] may be used to prove the $w' > 1$ case under the slightly

stricter assumptions that α_a and β_b are supported on square-free integers co-prime to w' . We draw the reader's attention to the remark directly after Proposition 1 in [50] as this improves the error term of Bettin, Chandee, and Radziwiłł's result. The outcome of substituting Harman, Watt, and Wong's bound in the place of Watt's is as follows.

Lemma 4.2 (Harman, Watt, and Wong). *Let $H, C, P, V, R, S \geq 1$, $q \in \mathbb{N}$ and $0 < \delta < 1$. Assume that for some $\epsilon > 0$ we have*

$$X_q := \left(\frac{RVSP}{HCq} \right)^{1/2} \geq (RSPV)^\epsilon$$

and

$$(RS)^2 \geq \max \left(H^2C, \frac{SP}{V} (RSPV)^\epsilon \right).$$

Let $\alpha(x), \beta(x)$ be smooth complex-valued functions, supported on $[H, 2H]$ and $[C, 2C]$ respectively with the condition that

$$\alpha^{(j)}(x), \beta^{(j)}(x) \ll_j (\delta x)^{-j}$$

for all $j \geq 0$. Let a_r, b_r be sequences of complex numbers supported on $[R, 2R]$, $[S, 2S]$ respectively with the condition that $a_r \ll r^\epsilon, b_s \ll s^\epsilon$. Suppose that $\gamma_{r,s}(x, y)$ is supported on $[V, 2V] \times [P, 2P]$ with the assumption that for all $i, j \geq 0$, $\gamma_{r,s}^{(i,j)}(x, y) \ll_{i,j} x^{-i}y^{-j}$ and $\gamma_{r,s} = 0$ if $(rs, q) > 1$ or $\mu(rs) = 0$. Then

$$\begin{aligned} \sum_{s \asymp S} \sum_{r \asymp R} \sum_{\substack{h, c, p, v \\ (rv, sp)=1}} \alpha(h) \beta(c) \gamma_{r,s}(v, p) a_r b_s e \left(\pm \frac{hcq\overline{rv}}{sp} \right) &\ll \delta^{-\frac{7}{2}} HC \left(RV + q^{\frac{7}{64}} RSX_1 \right) \\ &\times \left(1 + \frac{HC}{RS} \right)^{\frac{1}{2}} \left(1 + \frac{P}{VR} \right)^{\frac{1}{2}} \left(1 + \frac{H^2CPX_q^2}{VS^3R^4} \right)^{\frac{7}{64}} (HCRVPS)^{20\epsilon}. \end{aligned}$$

If $q = 1$, then the condition that $\gamma_{r,s} = 0$ if $(rs, q) > 1$ or $\mu(rs) = 0$ may be ignored.

This lemma is precisely the reason that we require q to be a prime or powerful number as if it were not then it would be possible that $w' > 1$ with either $ak'_2k'_4$ or $bk'_1k'_3$ not square-free. For example, consider the case when $q = 12, w = 3, k_2k_4 = 1$ and $k_1k_3 = 4$.

We substitute in

- $h \leftrightarrow h \leq \frac{H}{wd} \leq (qT)^\epsilon \frac{\sqrt{ABMN}}{wdT}$,
- $c \leftrightarrow l \leq L = (qT)^\epsilon k_1k_2 \frac{\sqrt{ABMN}}{dM_2N_2}$,

- $r \leftrightarrow a \leq \frac{Ak'_2k'_4}{d_1}$,
- $s \leftrightarrow b \leq \frac{Bk'_1k'_3}{d_3}$,
- $V \leftrightarrow \frac{M_1}{d_2k_4}$,
- $P \leftrightarrow \frac{N_1}{d_4k_3}$,
- $\delta \leftrightarrow (qT)^{-\epsilon}$,
- $X_q \leftrightarrow (qT)^{-\epsilon} \left(\frac{ABM_1N_1k'_1k'_2k'_3k'_4w}{dk_3k_4qHL} \right)^{\frac{1}{2}}$,

to

$$\mathcal{Z}_d = \sum_{d=d_1d_2=d_3d_4} \sum_{0 < h} \sum_{\substack{a,b,m_1,n_1 \\ (am_1,bn_1)=1 \\ (a,d_2)=(b,d_4)=1}} \sum_{0 < l \leq L} \alpha_{ad_1}^{(k'_2k'_4)} \beta_{bd_3}^{(k'_1k'_3)} W_1 \left(\frac{m_1d_2k_4}{M_1} \right) W_2 \left(\frac{n_1d_4k_3}{N_1} \right) \\ W_3 \left(\frac{hwd}{H} \right) W_4 \left(\frac{l}{L} \right) e \left(-\frac{lw'h\overline{am_1}}{bn_1} \right). \quad (4.31)$$

We satisfy the required conditions as long as

$$\left(\frac{ABM_1N_1k'_1k'_2k'_3k'_4}{dk_3k_4HL} \right) \gg (qT)^\epsilon$$

and

$$\left(\frac{ABk'_1k'_2k'_3k'_4}{d_1d_3} \right)^2 \geq (qT)^\epsilon \max \left\{ \frac{k_1k_2(ABMN)^{\frac{3}{2}}}{w^2T^2d^3M_2N_2}, \frac{BN_1k'_1k'_3k_4}{d_1M_1k_3} \right\}. \quad (4.32)$$

The first condition is satisfied by the limits of H and L . The second condition is satisfied as

$$d \ll (qT)^{-10\epsilon} (AB)^{\frac{1}{2}} \left(\frac{k'_1k'_2k'_3k'_4w}{qk_1k_2} \right)^2,$$

$MN \ll (qT)^2$, and $BN_1 \leq AM_1$. Hence

$$\mathcal{Z}_d(z) \ll_\epsilon (qT)^\epsilon \frac{HL}{wd} \left(\frac{AM_1k'_2k'_4}{d_1d_2k_4} + w'^{\frac{7}{64}} \frac{ABk'_1k'_2k'_3k'_4X_1}{d_1d_3} \right) \left(1 + \frac{HLd_1d_3}{dABwk'_1k'_2k'_3k'_4} \right)^{\frac{1}{2}} \\ \left(1 + \frac{N_1d_3k_4}{AM_1k'_2k'_4k_3} \right)^{\frac{1}{2}} \left(1 + \frac{H^2LN_1X_w^2d_1^4d_2d_3^3k_4}{w^2d^2d_4M_1A^4B^3k_3(k'_2k'_4)^4(k'_1k'_3)^3} \right)^{\frac{7}{64}} \\ \ll_\epsilon (qT)^\epsilon \frac{k_1k_2ABM_1N_1}{wTd^2} \left(\frac{AM_1k'_2k'_4}{d_1d_2k_4} + w'^{\frac{7}{64}} w^{\frac{1}{2}} \frac{AB(k'_1k'_2k'_3k'_4)^{\frac{3}{2}}T^{\frac{1}{2}}}{d_1d_3(k_1k_2k_3k_4)^{\frac{1}{2}}} \right) \\ \left(1 + \frac{M_1N_1d_1d_3k_1k_2}{d^2Twk'_1k'_2k'_3k'_4} \right)^{\frac{1}{2}} \left(1 + \frac{N_1d_3k_4}{AM_1k'_2k'_4k_3} \right)^{\frac{1}{2}}$$

$$\left(1 + \frac{(MN)^{\frac{1}{2}} N_1^2 d_1^3 d_3^3}{(k'_1 k'_2 k'_3 k'_4)^2 w' w A^{\frac{5}{2}} B^{\frac{3}{2}} T d^2 d_4 k_3^2 k'_2 k'_4}\right)^{\frac{7}{64}}.$$

Note that as $d_3 \leq B k'_1 k'_3$,

$$\left(1 + \frac{N_1 d_3 k_4}{A M_1 k'_2 k'_4 k_3}\right) \leq \left(1 + \frac{k'_1 k'_3 k_4}{k'_2 k'_4 k_3}\right).$$

By redistributing and cancelling the various d_i terms, the bound becomes

$$\begin{aligned} &\ll_{\epsilon} (qT)^{\epsilon} \frac{k_1 k_2 A B M_1 N_1}{w T d^2} \left(\frac{A M_1 k'_2 k'_4}{k_4} + w'^{\frac{7}{64}} w^{\frac{1}{2}} \frac{A B (k'_1 k'_2 k'_3 k'_4)^{\frac{3}{2}} T^{\frac{1}{2}}}{(k_1 k_2 k_3 k_4)^{\frac{1}{2}}} \right) \\ &\left(1 + \frac{M_1 N_1 k_1 k_2}{T w k'_1 k'_2 k'_3 k'_4}\right)^{\frac{1}{2}} \left(1 + \frac{k'_1 k'_3 k_4}{k'_2 k'_4 k_3}\right)^{\frac{1}{2}} \left(1 + \frac{(MN)^{\frac{1}{2}} N_1^2}{(k'_1 k'_2 k'_3 k'_4)^2 w w' A^{\frac{5}{2}} B^{\frac{3}{2}} T k_3^2 k'_2 k'_4}\right)^{\frac{7}{64}}. \end{aligned}$$

We also use the bound $M_1 N_1 \ll (qT)^{1+\epsilon}$ so that

$$\left(1 + \frac{M_1 N_1 k_1 k_2}{T w k'_1 k'_2 k'_3 k'_4}\right) \ll (qT)^{\epsilon} \frac{q k_1 k_2}{w k'_1 k'_2 k'_3 k'_4}.$$

This means that

$$\frac{T}{k_1 k_2 A B_1 M_1 N_1} \sum_{d \leq (qT)^{-10\epsilon} \frac{(AB)^{\frac{1}{2}}}{k_1 k_2} \left(\frac{k'_1 k'_2 k'_3 k'_4 w'}{q}\right)^2} d |Z_d|$$

is at most

$$\begin{aligned} &\ll_{\epsilon} (qT)^{\epsilon} \frac{1}{w} \left(\frac{A M_1 k'_2 k'_4}{k_4} \left(\frac{q k_1 k_2}{w k'_1 k'_2 k'_3 k'_4} \right)^{\frac{1}{2}} + \frac{w'^{\frac{7}{64}} A B k'_1 k'_2 k'_3 k'_4 (qT)^{\frac{1}{2}}}{(k_3 k_4)^{\frac{1}{2}}} \right) \left(1 + \frac{k'_1 k'_3 k_4}{k'_2 k'_4 k_3}\right)^{\frac{1}{2}} \\ &\left(1 + \frac{(MN)^{\frac{1}{2}} N_1^2}{(k'_1 k'_2 k'_3 k'_4)^2 w w' A^{\frac{5}{2}} B^{\frac{3}{2}} T k_3^2 k'_2 k'_4}\right)^{\frac{7}{64}}. \end{aligned}$$

By multiplying out the first two sets of brackets and then the third, this is

$$\begin{aligned} &\ll_{\epsilon} (qT)^{\epsilon} \frac{1}{w} \left[\frac{A M_1}{k_4} \left(\frac{q k_1 k_2 k'_2 k'_4}{k'_1 k'_3} \right)^{\frac{1}{2}} + \frac{w'^{\frac{7}{64}} A B k'_1 k'_2 k'_3 k'_4 (qT)^{\frac{1}{2}}}{(k_3 k_4)^{\frac{1}{2}}} + A M_1 \left(\frac{q k_1 k_2}{w k_3 k_4} \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \frac{w'^{\frac{7}{64}} A B (k'_1 k'_3)^{\frac{3}{2}} (k'_2 k'_4)^{\frac{1}{2}} (qT)^{\frac{1}{2}}}{k_3} \right] \left(1 + \frac{(MN)^{\frac{1}{2}} N_1^2}{(k'_1 k'_2 k'_3 k'_4)^2 w w' A^{\frac{5}{2}} B^{\frac{3}{2}} T k_3^2 k'_2 k'_4}\right)^{\frac{7}{64}} \\ &\ll_{\epsilon} (qT)^{\epsilon} \frac{1}{w} \left[\frac{A M_1}{k_4} \left(\frac{q k_1 k_2 k'_2 k'_4}{k'_1 k'_3} \right)^{\frac{1}{2}} + \frac{w'^{\frac{7}{64}} A B k'_1 k'_2 k'_3 k'_4 (qT)^{\frac{1}{2}}}{(k_3 k_4)^{\frac{1}{2}}} + A M_1 \left(\frac{q k_1 k_2}{w k_3 k_4} \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \frac{w'^{\frac{7}{64}} A B (k'_1 k'_3)^{\frac{3}{2}} (k'_2 k'_4)^{\frac{1}{2}} (qT)^{\frac{1}{2}}}{k_3} + \frac{A^{\frac{98}{128}} B^{-\frac{21}{128}} M^{\frac{71}{128}} N^{\frac{21}{128}} q^{\frac{1}{2}}}{T^{\frac{7}{64}}} \left((k_1 k_2)^{\frac{1}{2}} k_2^{\frac{11}{64}} + (k_1 k_2)^{\frac{1}{2}} \right) \right. \\ &\quad \left. + A^{\frac{93}{128}} B^{\frac{107}{128}} M^{\frac{7}{128}} N^{\frac{21}{128}} q^{\frac{1}{2}} T^{\frac{25}{64}} \left((k'_1 k'_2 k'_3 k'_4)^{\frac{25}{32}} + (k'_1 k'_2 k'_3 k'_4)^{\frac{41}{32}} \right) \right]. \end{aligned}$$

This is maximised when $A, B \asymp (qT)^\kappa$, $M_1 \asymp (qT)^{\frac{1}{2}+\epsilon}$, and $M, N \asymp (qT)^{1+\epsilon}$, giving the bound

$$\begin{aligned} &\ll_\epsilon (qT)^\epsilon \frac{1}{w} \left[q^{\kappa+1} T^{\kappa+\frac{1}{2}} (k_1 k_2 k'_2)^{\frac{1}{2}} + q^{2\kappa+\frac{1}{2}} T^{2\kappa+\frac{1}{2}} w'^{\frac{7}{64}} (k'_1 k'_2 k'_3 k'_4)^{\frac{3}{2}} \right. \\ &\quad \left. + q^{\frac{77\kappa}{128} + \frac{39}{32}} T^{\frac{77\kappa}{128} + \frac{39}{64}} (k_1 k_2)^{\frac{1}{2}} k'_2{}^{\frac{11}{64}} + q^{\frac{25\kappa}{16} + \frac{23}{32}} T^{\frac{25\kappa}{16} + \frac{39}{64}} (k'_1 k'_2 k'_3 k'_4)^{\frac{41}{32}} \right] \\ &\ll_\epsilon (qT)^\epsilon \frac{1}{w} \left[q^{\kappa+1} T^{\kappa+\frac{1}{2}} X + (qT)^{2\kappa+\frac{1}{2}} \min\{q^{\frac{7}{64}} (qT)^{\frac{3\kappa}{2}}, q^{\frac{3}{2}}\} \right. \\ &\quad \left. + q^{\frac{77\kappa}{128} + \frac{39}{32}} T^{\frac{77\kappa}{128} + \frac{39}{64}} X^{\frac{43}{64}} + q^{\frac{25\kappa}{16} + \frac{23}{32}} T^{\frac{25\kappa}{16} + \frac{39}{64}} X^{\frac{41}{32}} \right]. \end{aligned}$$

This, along with (4.30) and (4.29) shows that

$$S_{w, k_1, k_2, k_3, k_4}^{+, l \neq 0}(A, B, M_1, M_2, N_1, N_2) \ll \min\{\mathcal{E}_1, \mathcal{E}_2\}$$

where

$$\mathcal{E}_1 \ll_\epsilon \frac{1}{w} (qT)^{\frac{5\kappa}{2} + \frac{3}{4} + \epsilon} X^{\frac{1}{2}}$$

and

$$\begin{aligned} \mathcal{E}_2 \ll_\epsilon (qT)^\epsilon \frac{1}{w} &\left[q^{\kappa+\frac{15}{4}} T^{\kappa+\frac{3}{4}} X^{\frac{3}{2}} + q^{\kappa+1} T^{\kappa+\frac{1}{2}} X + (qT)^{2\kappa+\frac{1}{2}} \min\{q^{\frac{7}{64}} (qT)^{\frac{3\kappa}{2}}, q^{\frac{3}{2}}\} \right. \\ &\quad \left. + q^{\frac{77\kappa}{128} + \frac{39}{32}} T^{\frac{77\kappa}{128} + \frac{39}{64}} X^{\frac{43}{64}} + q^{\frac{25\kappa}{16} + \frac{23}{32}} T^{\frac{25\kappa}{16} + \frac{39}{64}} X^{\frac{41}{32}} \right]. \end{aligned}$$

This concludes the proof of Proposition 4.1.

Chapter 5

Moments of $\zeta(s)$ in vertical arithmetic progressions

5.1 Introduction

In an extremely interesting paper [84], Li and Radziwiłł demonstrated a remarkable correspondence between the second moment of $\zeta(s)$, averaged over an arithmetic progression, and the standard continuous second moment. Specifically, they showed that, for $\alpha > 0, \beta$ real, and as $T \rightarrow \infty$

$$\sum_{1 \leq l \leq T} \left| \zeta \left(\frac{1}{2} + i(\alpha l + \beta) \right) \right|^2 = (1 + f(\alpha, \beta) + o_\alpha(1)) \int_0^T \left| \zeta \left(\frac{1}{2} + i(\alpha t + \beta) \right) \right|^2 dt$$

where

$$f(\alpha, \beta) = \begin{cases} \frac{2 \cos\left(\beta \log \frac{m}{n}\right) \sqrt{mn} - 2}{mn + 1 - 2\sqrt{mn} \cos\left(\beta \log \frac{m}{n}\right)} & \text{if } e^{\frac{2\pi g}{\alpha}} \text{ is rational for some } g \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

with $\frac{m}{n} \neq 1$ being the smallest reduced fraction such that $e^{\frac{2\pi g}{\alpha}} = \frac{m}{n}$ for some $g > 0$.

One marvellous consequence of this result is that the average behaviour of $\zeta(s)$ on arithmetic progressions up the critical line is that of a unitary family for most *but not all* progressions. There is a set of Lebesgue measure zero of exceptional vertical arithmetic progressions on which the mean square of $\zeta(s)$ acquires an extra main term. This extra main term depends on the rationality of $e^{\frac{2\pi g}{\alpha}}$ for integers g , and interestingly the “more rational” $e^{\frac{2\pi g}{\alpha}}$ is (by which we mean the smaller the numerator

and denominator of the corresponding reduced fraction), the larger this extra main term is.

In [84], Li and Radziwiłł continue on to prove that at least $\frac{1}{3}$ of points on an arithmetic progression up the critical line are not zeros of $\zeta(s)$, as well as a couple of results on the extreme absolute values of $\zeta(s)$ in progressions. They can prove these for all vertical arithmetic progressions, as they show that the abnormal behaviour on the small set of exceptional vertical progressions can be smoothed out by an appropriate mollifier. It would be of interest to explore the consequences of using an amplifier rather than a mollifier and exploring how this amplified extra main term could be exploited in applications, such as the search for extreme absolute values of $\zeta(s)$.

In this chapter, we will prove a result on the fourth moment of $\zeta(s)$ on arithmetic progressions. The fourth moment is seen as being much more complicated than the second moment, as is evidenced by its asymptotic formula being significantly more complex. An important tool that we shall use is the asymptotic formula for the twisted fourth moment by Hughes and Young in [61]. We could use the asymptotic formula by Bettin, Bui, Li, and Radziwiłł [6] that is stronger with a smaller error term, however this improvement comes from averaging over the coefficients of the twist and it is actually clearer for our purposes to use the result of Hughes and Young. Also, we do not need the smaller error term provided in [6]. To state Hughes and Young's result, we need some definitions.

For $T \geq 2$, let ϕ_t be a smooth, real-valued, bounded function with support on $[T, 2T]$ and with $\phi_t^{(j)} \ll_j (\log T)^{\frac{1}{2}}$. Let $a, b \in \mathbb{N}$ and for complex $w, x, y, z \ll (\log T)^{-1}$ define

$$J_{w,x,y,z}(a, b; T) = \int_{-\infty}^{\infty} \left(\frac{a}{b}\right)^{-it} \zeta\left(\frac{1}{2} + w + it\right) \zeta\left(\frac{1}{2} + x + it\right) \zeta\left(\frac{1}{2} + y - it\right) \zeta\left(\frac{1}{2} + z - it\right) \phi_t dt.$$

Define

$$A_{w,x,y,z}(s) := \frac{\zeta(1+s+w+y)\zeta(1+s+w+z)\zeta(1+s+x+y)\zeta(1+s+x+z)}{\zeta(2+2s+w+x+y+z)}$$

and

$$\tau_{x,y}(n) = \sum_{n_1 n_2 = n} n_1^{-x} n_2^{-y}.$$

For integers a, b and for each prime p define n_p so that $p^{n_p} \parallel n$, and define

$$B_{w,x,y,z,a,b}(s) = \prod_{p|a} \left(\frac{\sum_{j=0}^{\infty} \tau_{w,x}(p^j) \tau_{y,z}(p^{j+a_p}) p^{-j(s+1)}}{\sum_{j=0}^{\infty} \tau_{w,x}(p^j) \tau_{y,z}(p^j) p^{-j(s+1)}} \right) \\ \times \prod_{p|b} \left(\frac{\sum_{j=0}^{\infty} \tau_{w,x}(p^{j+b_p}) \tau_{y,z}(p^j) p^{-j(s+1)}}{\sum_{j=0}^{\infty} \tau_{w,x}(p^j) \tau_{y,z}(p^j) p^{-j(s+1)}} \right).$$

Let

$$Z_{w,x,y,z,a,b}(s) = A_{w,x,y,z}(s) B_{w,x,y,z,a,b}(s).$$

Theorem 5.1 (Hughes and Young). *Suppose that $(a, b) = 1$ and $ab \leq T^{\frac{2}{11}-\epsilon}$, then*

$$J_{w,x,y,z}(a, b; T) = \frac{1}{\sqrt{ab}} \int_{-\infty}^{\infty} \phi_t \left(Z_{w,x,y,z,a,b}(0) + \left(\frac{t}{2\pi} \right)^{-w-x-y-z} Z_{-y,-z,-w,-x,a,b}(0) \right. \\ \left. + \left(\frac{t}{2\pi} \right)^{-w-y} Z_{-y,x,-w,z,a,b}(0) + \left(\frac{t}{2\pi} \right)^{-w-z} Z_{-z,x,y,-w,a,b}(0) + \left(\frac{t}{2\pi} \right)^{-x-y} Z_{w,-y,-x,z,a,b}(0) \right. \\ \left. + \left(\frac{t}{2\pi} \right)^{-x-z} Z_{w,-z,y,-x,a,b}(0) \right) dt + o(T). \quad (5.1)$$

We shall use this theorem to find the main term of the fourth moment in a vertical arithmetic progression. For the fixed real constants $\alpha > 0$ and β , define for $T > 2$

$$I_{\alpha,\beta}(T) := \sum_{l \in \mathbb{Z}} \left| \zeta \left(\frac{1}{2} + i(\alpha l + \beta) \right) \right|^4 \phi_l. \quad (5.2)$$

The main result of this chapter is the following theorem.

Theorem 5.2. *Suppose that $\alpha > 0$ is a constant such that $e^{\frac{2\pi}{\alpha}} = \frac{a}{b}$ with $a, b \in \mathbb{N}$ and $(a, b) = 1$, and β is a real constant. Then as $T \rightarrow \infty$,*

$$I_{\alpha,\beta}(T) = (1 + \delta_{\alpha,\beta} + o_{\alpha}(1)) \int \left| \zeta \left(\frac{1}{2} + i(\alpha t + \beta) \right) \right|^4 \phi_t dt$$

where

$$\delta_{\alpha,\beta} := 2 \sum_{g=1}^{\infty} \frac{\cos(g \frac{2\pi\beta}{\alpha})}{(ab)^{\frac{g}{2}}} \prod_{p^{e_p} \parallel ab} \left(1 + g e_p \frac{p-1}{p+1} \right).$$

This demonstrates that a surprising extra main term is present in the fourth moment too, for at least some progressions. If the pattern from Theorem 2 in [84] continues, and it seems natural that it should, then we would expect Theorem 5.2 to hold for all real $\alpha > 0, \beta$, with

$$\delta_{\alpha,\beta} = \begin{cases} 2 \sum_{g=1}^{\infty} \frac{\cos(gk \frac{2\pi\beta}{\alpha})}{(ab)^{\frac{g}{2}}} \prod_{p^{e_p} \parallel ab} \left(1 + g e_p \frac{p-1}{p+1} \right) & \text{if } e^{\frac{2\pi k}{\alpha}} \text{ is rational for some } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\frac{a}{b}$ is the smallest reduced fraction such that $e^{\frac{2\pi k}{\alpha}} = \frac{a}{b}$ for some $k > 0$. Discrete moments of this form have been studied by Good [42], but to the right of the critical line. In general, moments to the right of the critical line and moments on the critical line are quite different creatures, and the area to the right of the critical line is of less interest than the line itself.

Despite seeming like only a small step further than the result of Li and Radziwiłł, showing that the same pattern holds in the fourth moment actually contains a number of significant challenges. Perhaps this should not be surprising, as the jump from the second continuous moment to the fourth continuous moment is quite large in difficulty. In the context of moments on arithmetic progressions, in the second moment one encounters sums of the form

$$\sum_{\substack{mn \ll T^{1+\epsilon} \\ \left| \frac{m}{n} - \theta \right| \leq \theta T^{-1}}} \frac{1}{\sqrt{mn}}$$

for some real θ . In the fourth moment these sums are of the form

$$\sum_{\substack{mn \ll T^{2+\epsilon} \\ \left| \frac{m}{n} - \theta \right| \leq \theta T^{-1}}} \frac{\tau(m)\tau(n)}{\sqrt{mn}} \tag{5.3}$$

which require a more sophisticated treatment due to the extra length and the divisor functions. In [84], the authors find an asymptotic formula for the twisted second moment but the length of the twist is just a little short to be enough to find the asymptotic formula for the fourth moment, and increasing the length of the twist would involve breaking the half-barrier. In the case that θ is rational (with not too large a numerator or denominator relative to T), there are already techniques to find the asymptotic behaviour of sums such as (5.3), see for example [6, 18, 61]. However, when θ is irrational, none of these techniques are applicable. Even finding bounds of the expected size on sums such as (5.3) would be enough to rigorously prove the asymptotic formula for the fourth moment for all progressions. While finding these bounds for small irrational θ has been elusive, we demonstrate in Section 5.2.3 how this can be done for larger $\theta \gg T^\delta$ for some positive δ .

5.2 Proof of Theorem 5.2

After some initial set-up, we apply Poisson summation to the sum over l in (5.2), moving the sum into the frequency domain. We show the 0 frequency is equal to the continuous fourth moment of $\zeta(s)$, then divide the remaining frequencies into two groups: the smaller frequencies and the larger frequencies. We show the smaller frequencies make up the extra main term, then bound the larger frequencies into an error term.

5.2.1 Set-up

Lemma 5.1 (Approximate functional equation). *We let $G(s)$ be a helpful function and let*

$$g(s, t) = \frac{\left| \Gamma\left(\frac{1/2+s+it}{2}\right) \right|^4}{\left| \Gamma\left(\frac{1/2+it}{2}\right) \right|^4}.$$

Set

$$V(x, t) = \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} g(s, t) \pi^{-2s} x^{-s} ds$$

then

$$\left| \zeta\left(\frac{1}{2} + it\right) \right|^4 = 2 \sum_{m, n} \frac{\tau(m)\tau(n)}{(mn)^{\frac{1}{2}}} \left(\frac{n}{m}\right)^{it} V(mn, t) + O_A((1 + |t|)^{-A}).$$

The proof of this is standard, see for example Proposition 2.1 of [61].

Note that by Stirling's approximation (1.16)

$$g(s, t) = \left(\frac{t}{2}\right)^{2s} (1 + O(t^{-1}(1 + |s|^2))) \quad (5.4)$$

and for any fixed $A > 0$

$$t^j \frac{\partial^j}{\partial t^j} V(x, t) \ll_{A,j} (1 + |x|/t^2)^{-A}. \quad (5.5)$$

By Lemma 5.1

$$I_{\alpha, \beta}(T) = 2 \sum_{l \in \mathbb{Z}} \sum_{mn \leq T^{2+\epsilon}} \frac{\tau(m)\tau(n)}{(mn)^{1/2}} \left(\frac{n}{m}\right)^{i(\alpha l + \beta)} V(mn, \alpha l + \beta) \phi_l + O_A(T^{-A}).$$

Taking Poisson summation over l we see this is

$$2 \sum_{g \in \mathbb{Z}} \sum_{mn \leq T^{2+\epsilon}} \frac{\tau(m)\tau(n)}{(mn)^{1/2}} \int_{-\infty}^{\infty} \left(\frac{n}{m}\right)^{i(\alpha t + \beta)} V(mn, \alpha t + \beta) e(-gt) \phi_t dt + O_A(T^{-A}).$$

We write

$$I_{\alpha,\beta}(T) = \sum_{g \in \mathbb{Z}} I_{\alpha,\beta,g}(T) + O_A(T^{-A})$$

where

$$I_{\alpha,\beta,g}(T) := 2 \sum_{mn \leq T^{2+\epsilon}} \frac{\tau(m)\tau(n)}{(mn)^{1/2}} \left(\frac{n}{m}\right)^{i\beta} \int_{-\infty}^{\infty} \left(\frac{nb^g}{ma^g}\right)^{i\alpha t} V(mn, \alpha t + \beta) \phi_t dt$$

so that we may treat the different frequencies g separately. We split into the following cases: $|g| \leq \frac{\log T}{10 \log(ab)}$ and $|g| > \frac{\log T}{10 \log(ab)}$.

5.2.2 Case 1: the small frequencies

By Lemma 5.1

$$I_{\alpha,\beta,g}(T) = \int \left(\frac{a^g}{b^g}\right)^{-i\alpha t} \left| \zeta\left(\frac{1}{2} + i(\alpha t + \beta)\right) \right|^4 \phi_t dt + O_A(T^{-A}).$$

If $g = 0$ then this is precisely the continuous fourth moment

$$\int \left| \zeta\left(\frac{1}{2} + i(\alpha t + \beta)\right) \right|^4 \phi_t dt.$$

If $1 \leq |g| \leq \frac{\log T}{10 \log(ab)}$ then $(ab)^{|g|} \leq T^{\frac{1}{10}}$ and so we can apply Theorem 5.1 to show

$$\begin{aligned} I_{\alpha,\beta,g}(T) = & \lim_{w,x,y,z \rightarrow 0} \left(\frac{a}{b}\right)^{ig\beta} (ab)^{-\frac{|g|}{2}} \int_{-\infty}^{\infty} \phi_t \left(Z_{w,x,y,z,(ab)^g}(0) \right. \\ & + \left(\frac{\alpha t + \beta}{2\pi}\right)^{-w-x-y-z} Z_{-y,-z,-w,-x,(ab)^g}(0) + \left(\frac{\alpha t + \beta}{2\pi}\right)^{-w-y} Z_{-y,x,-w,z,(ab)^g}(0) \\ & + \left(\frac{\alpha t + \beta}{2\pi}\right)^{-w-z} Z_{-z,x,y,-w,(ab)^g}(0) + \left(\frac{\alpha t + \beta}{2\pi}\right)^{-x-y} Z_{w,-y,-x,z,(ab)^g}(0) \\ & \left. + \left(\frac{\alpha t + \beta}{2\pi}\right)^{-x-z} Z_{w,-z,y,-x,(ab)^g}(0) \right) dt + o(T). \quad (5.6) \end{aligned}$$

Finding the limit as $w, x, y, z \rightarrow 0$ seems like it might result in a long and tedious calculation with a complicated result comprised of many terms. This would be the case if we were interested in lower order terms, however we shall only focus on the leading term of the limit, vastly simplifying the calculation.

Lemma 5.2. *For large enough t ,*

$$\lim_{w,x,y,z \rightarrow 0} \left(Z_{w,x,y,z,a,b}(0) + t^{-w-x-y-z} Z_{-y,-z,-w,-x,a,b}(0) + t^{-w-y} Z_{-y,x,-w,z,a,b}(0) \right)$$

$$\begin{aligned}
& + t^{-w-z} Z_{-z,x,y,-w,a,b}(0) + t^{-x-y} Z_{w,-y,-x,z,a,b}(0) + t^{-x-z} Z_{w,-z,y,-x,a,b}(0) \Big) dt \\
& = \frac{B_{0,0,0,0,a,b}(0)(\log t)^4}{2\pi^2} + O_{a,b}((\log t)^3). \quad (5.7)
\end{aligned}$$

Proof. We begin by writing $w = y = s$ and $x = z = 2s$, to reduce the limit to being in only one variable, while avoiding the various poles at $x = w, y = z$ etc. By the definition of $Z_{w,x,y,z,a,b}(0)$ this makes the limit equal to

$$\begin{aligned}
L(t) := \lim_{s \rightarrow 0} & \frac{\zeta(1+2s)\zeta(1+3s)^2\zeta(1+4s)}{\zeta(2+6s)} B_{s,2s,s,2s,a,b}(0) \\
& + \frac{\zeta(1-2s)\zeta(1-3s)^2\zeta(1-4s)}{\zeta(2-6s)} B_{-s,-2s,-s,-2s,a,b}(0) t^{-6s} \\
& + \frac{\zeta(1-2s)\zeta(1+s)^2\zeta(1+4s)}{\zeta(2+6s)} B_{-s,2s,-s,2s,a,b}(0) t^{-2s} \\
& + \frac{\zeta(1-s)\zeta(1-3s)\zeta(1+3s)\zeta(1+s)}{\zeta(2)} B_{-2s,2s,s,-s,a,b}(0) t^{-3s} \\
& + \frac{\zeta(1-s)\zeta(1-3s)\zeta(1+3s)\zeta(1+s)}{\zeta(2)} B_{s,-s,2s,-2s,a,b}(0) t^{-3s} \\
& + \frac{\zeta(1+2s)\zeta(1-s)^2\zeta(1-4s)}{\zeta(2-2s)} B_{s,-2s,s,-2s,a,b}(0) t^{-4s}.
\end{aligned}$$

For each zeta-function in the numerator of each fraction, we expand out its Laurent series $\zeta(1+s) = \frac{1}{s} + \gamma + \dots$ and note that the largest terms arising (after using L'Hôpital's rule to find the limit) will come from the $\frac{1}{s}$ terms in the Laurent series. Specifically

$$\begin{aligned}
L(t) = \lim_{s \rightarrow 0} & \frac{B_{s,2s,s,2s,a,b}(0)}{72s^4\zeta(2+6s)} + \frac{B_{-s,-2s,-s,-2s,a,b}(0)}{72s^4\zeta(2-6s)} t^{-6s} - \frac{B_{-s,2s,-s,2s,a,b}(0)}{8s^4\zeta(2+6s)} t^{-2s} \\
& + \frac{B_{-2s,2s,s,-s,a,b}(0)t^{-3s}}{9s^4\zeta(2)} + \frac{B_{s,-s,2s,-2s,a,b}(0)}{9s^4\zeta(2)} t^{-3s} - \frac{B_{s,-2s,s,-2s,a,b}(0)}{8s^4\zeta(2-2s)} t^{-4s} + O_{a,b}((\log t)^3).
\end{aligned}$$

Using L'Hôpital's rule, we see the leading term will be of size $\asymp (\log t)^4$, and we discard all other terms to find

$$\begin{aligned}
L(t) & = \frac{B_{0,0,0,0,a,b}(0)(\log t)^4}{4!\zeta(2)} \left(\frac{6^4}{72} - \frac{2^4}{8} + \frac{3^4}{9} + \frac{3^4}{9} - \frac{4^4}{8} \right) + O_{a,b}((\log t)^3) \\
& = \frac{B_{0,0,0,0,a,b}(0)(\log t)^4}{2\pi^2} + O_{a,b}((\log t)^3).
\end{aligned}$$

□

By Lemma 5.2, and grouping each g with $-g$,

$$\begin{aligned}
I_{\alpha,\beta}(T) &= \int \left| \zeta \left(\frac{1}{2} + i(\alpha t + \beta) \right) \right|^4 \phi_t dt \\
&+ \sum_{1 \leq g \leq \frac{\log T}{10 \log(ab)}} 2 \frac{\cos(g\beta \log \frac{a}{b})}{(ab)^{\frac{g}{2}}} B_{0,0,0,0,a^g,b^g}(0) \int \frac{(\log \frac{\alpha t + \beta}{2\pi})^4}{2\pi^2} \phi_t dt \\
&+ \sum_{|g| > \frac{\log T}{10 \log(ab)}} I_{\alpha,\beta,g}(T) + O_{a,b}(T(\log T)^3). \quad (5.8)
\end{aligned}$$

By Lemma 5.2, and Theorem 5.1

$$\begin{aligned}
\int \frac{(\log \frac{\alpha t + \beta}{2\pi})^4}{2\pi^2} \phi_t dt &= \int \left| \zeta \left(\frac{1}{2} + i(\alpha t + \beta) \right) \right|^4 \phi_t dt + O_{\alpha}(T(\log T)^3) \\
&= (1 + o_{\alpha}(1)) \int \left| \zeta \left(\frac{1}{2} + i(\alpha t + \beta) \right) \right|^4 \phi_t dt.
\end{aligned}$$

Now we turn our attention to $B_{0,0,0,0,a^g,b^g}(0)$.

$$B_{0,0,0,0,a^g,b^g}(0) = \prod_{p^{e_p} \parallel ab} \left(\frac{\sum_{j=0}^{\infty} (j+1)(j + ge_p + 1)p^{-j}}{\sum_{j=0}^{\infty} (j+1)^2 p^{-j}} \right).$$

Using the well known identities for $\Re(s) > 1$

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}, \text{ and } \frac{\zeta(s)^4}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^s}$$

then comparing Euler products we find

$$\sum_{j=0}^{\infty} (j+1)p^{-js} = (1 - p^{-s})^{-2} \quad (5.9)$$

and

$$\sum_{j=0}^{\infty} (j+1)^2 p^{-js} = \frac{(1 - p^{-2s})}{(1 - p^{-s})^4}. \quad (5.10)$$

The identities (5.9) and (5.10) converge at $s = 1$. Therefore

$$\frac{\sum_{j=0}^{\infty} (j+1)(j + ge_p + 1)p^{-j}}{\sum_{j=0}^{\infty} (j+1)^2 p^{-j}} = 1 + ge_p \frac{\sum_{j=0}^{\infty} (j+1)p^{-j}}{\sum_{j=0}^{\infty} (j+1)^2 p^{-j}} = 1 + ge_p \frac{1 - p^{-1}}{1 + p^{-1}}.$$

This implies

$$B_{0,0,0,0,a^g,b^g}(0) = \prod_{p^{e_p} \parallel ab} \left(1 + ge_p \frac{p-1}{p+1} \right) \quad (5.11)$$

and hence

$$B_{0,0,0,0,a^g,b^g}(0) \leq \tau((ab)^g) \ll_{\epsilon} (ab)^{g\epsilon}$$

by the divisor bound. Therefore

$$\begin{aligned}
\sum_{g > \frac{\log T}{10 \log(ab)}} \frac{\cos\left(g\beta \log \frac{a}{b}\right)}{(ab)^{\frac{g}{2}}} B_{0,0,0,0,a^g,b^g}(0) &\ll_{\epsilon} \sum_{g > \frac{\log T}{10 \log(ab)}} (ab)^{(\epsilon - \frac{1}{2})g} \\
&\ll_{\epsilon} (ab)^{\frac{\log T}{10 \log(ab)}(\epsilon - \frac{1}{2})} \\
&\ll_{\epsilon} T^{\epsilon - \frac{1}{20}}
\end{aligned}$$

and so the $g > \frac{\log T}{10 \log(ab)}$ terms may be included in (5.8) as their contribution is smaller than the error term and so

$$\begin{aligned}
I_{\alpha,\beta}(T) &= (1 + \delta_{\alpha,\beta}) \int \left| \zeta \left(\frac{1}{2} + i(\alpha t + \beta) \right) \right|^4 \phi_t dt + \sum_{|g| > \frac{\log T}{10 \log(ab)}} I_{\alpha,\beta,g}(T) \\
&\quad + O_{a,b}(T(\log T)^3). \quad (5.12)
\end{aligned}$$

5.2.3 Case 2: the large frequencies

To bound the contributions when $|g| > \frac{\log T}{10 \log(ab)}$, we need some initial lemmas.

Lemma 5.3 (*t*-integral). *For any integer $j \geq 0$*

$$\int_{-\infty}^{\infty} \left(\frac{nb^g}{ma^g} \right)^{i\alpha t} V(mn, \alpha t + \beta) \phi_t dt \ll_j \min \left(T, \frac{T(\log T)^{\frac{1}{2}}}{(T \log \frac{nb^g}{ma^g})^j} \right)$$

This is proved in a similar fashion to Lemma 2.4, by integration by parts j times along with (5.5), and by trivially bounding the integrand.

As a result of this lemma, it is clear that the only values of m and n that contribute significantly are the values such that

$$\left| \frac{n}{m} - \frac{a^g}{b^g} \right| \leq \frac{a^g}{b^g} T^{-1+\eta} \quad (5.13)$$

for some small fixed $\eta > 0$. We write $(n, m) \in \mathcal{C}_g$ to mean $\frac{n}{m}$ satisfies the condition (5.13).

Lemma 5.4 (Shiu). *For any $\epsilon > 0$, we have as $x \rightarrow \infty$*

$$\sum_{x-y \leq n < x} \tau(n) \ll_{\epsilon} y \log x,$$

provided that $x^{\epsilon} < y \leq x$.

This is a direct result of Theorem 2 in [111].

To bound the contribution from the large frequencies, we first introduce a dyadic decomposition. Let $W(x)$ be a smooth, non-negative function with support on $[1, 2]$ and M and N run over a sequence of real numbers with $\#\{M : M \leq X\} \ll \log X$ and similarly for N , with

$$\sum_M W\left(\frac{x}{M}\right) = \sum_N W\left(\frac{x}{N}\right) = 1$$

for all x . This means that we can write

$$I_{\alpha,\beta,g}(T) := \sum_M \sum_N I_{\alpha,\beta,g}(T; M, N)$$

where $I_{\alpha,\beta,g}(T; M, N)$ is defined as

$$2 \sum_{mn \leq T^{2+\epsilon}} \frac{\tau(m)\tau(n)}{(mn)^{1/2}} \left(\frac{n}{m}\right)^{i\beta} \int_{-\infty}^{\infty} \left(\frac{nb^g}{ma^g}\right)^{i\alpha t} V(mn, \alpha t + \beta) \phi_t W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right) dt.$$

We may assume that $g > \frac{\log T}{10 \log(ab)}$, as the $g < -\frac{\log T}{10 \log(ab)}$ is treated identically by relabelling a and b . By Lemma 5.3, we can see that unless $b^g N \asymp a^g M$, we have the bound $I_{\alpha,\beta,g}(T; M, N) \ll_A T^{-A}$, hence we may assume that $b^g N \asymp a^g M$. This means that for each M and g , there are at most $O(1)$ values of N such that $I_{\alpha,\beta,g}(T; M, N)$ that are not immediately negligibly small. Due to the rapid decay of $V(x, t)$ in x , we may assume that $MN \ll T^{2+\epsilon}$. This means that if $\frac{a^g}{b^g} > T^{2+\epsilon}$ and $MN \ll T^{2+\epsilon}$ then $b^g N \not\asymp a^g M$ and hence $I_{\alpha,\beta,g}(T; M, N) \ll_A T^{-A}$. This implies that we may assume that $g \ll \log T$. To summarise, we may restrict ourselves to considering the cases when

$$MN \ll T^{2+\epsilon}, \quad a^g M \asymp b^g N, \quad \text{and} \quad \frac{\log T}{10 \log(ab)} < g \ll \log T.$$

If $g > \frac{\log T}{10 \log(ab)}$, then

$$\frac{a^g}{b^g} \geq T^{\frac{\log \frac{a}{b}}{10 \log(ab)}}.$$

Let $\delta := \frac{\log \frac{a}{b}}{10 \log(ab)}$, and for each g we partition the $I_{\alpha,\beta,g}(T; M, N)$ into the following cases

- Case 2i: $M > T^{1-\frac{\delta}{4}} \left(\frac{a}{b}\right)^{-\frac{g}{2}}$, $MN \leq T^2$;
- Case 2ii: $M > T^{1-\frac{\delta}{4}} \left(\frac{a}{b}\right)^{-\frac{g}{2}}$, $MN > T^2$;
- Case 2iii: $M \leq T^{1-\frac{\delta}{4}} \left(\frac{a}{b}\right)^{-\frac{g}{2}}$.

We shall show that for each g , the total contribution from each case to the error term is $\ll_{\alpha, \epsilon} T(\log T)^{\frac{5}{2}+\epsilon}$. Therefore, summing over $g \ll \log T$, the error term in Theorem 5.2 is $\ll_{\alpha, \epsilon} T(\log T)^{\frac{7}{2}+\epsilon}$.

5.2.3.1 Case 2i

If $0 \leq \left| \frac{nb^g}{ma^g} - 1 \right| < T^{-1}$ then by Lemma 5.3

$$\int_{-\infty}^{\infty} \left(\frac{nb^g}{ma^g} \right)^{i\alpha t} V(mn, \alpha t + \beta) \phi_t dt \ll T.$$

For $1 \leq r \leq \frac{T}{2}$, if $rT^{-1} \leq \left| \frac{nb^g}{ma^g} - 1 \right| < (r+1)T^{-1}$ then

$$\log \frac{nb^g}{ma^g} \gg rT^{-1}$$

and by Lemma 5.3,

$$\int_{-\infty}^{\infty} \left(\frac{nb^g}{ma^g} \right)^{i\alpha t} V(mn, \alpha t + \beta) \phi_t dt \ll \frac{T(\log T)^{\frac{1}{2}}}{r^2}.$$

Finally, if $r \geq \frac{T}{2}$ and $rT^{-1} \leq \left| \frac{nb^g}{ma^g} - 1 \right| < (r+1)T^{-1}$ then

$$\log \frac{nb^g}{ma^g} \gg 1$$

and so

$$\int_{-\infty}^{\infty} \left(\frac{nb^g}{ma^g} \right)^{i\alpha t} V(mn, \alpha t + \beta) \phi_t dt \ll_A T^{-A}.$$

Also, as

$$NT^{-1} \asymp \frac{a^g}{b^g} MT^{-1} \gg T^{\frac{\delta}{4}}$$

we may use Lemma 5.4 to see that for $N \asymp \frac{a^g}{b^g} M$, we have the bound

$$\sum_{\substack{n \asymp N \\ rT^{-1} \leq \left| \frac{nb^g}{ma^g} - 1 \right| < (r+1)T^{-1}}} \tau(n) \ll NT^{-1} \log(N).$$

Therefore,

$$\begin{aligned} I_{\alpha, \beta, g}(T; M, N) &\ll \frac{T(\log T)^{\frac{1}{2}}}{\sqrt{MN}} \sum_{1 \leq r} \frac{1}{r^2} \sum_{m \asymp M} \tau(m) \sum_{\substack{n \asymp N \\ rT^{-1} \leq \left| \frac{nb^g}{ma^g} - 1 \right| < (r+1)T^{-1}}} \tau(n) \\ &\ll_{\delta} \frac{T(\log T)^{\frac{1}{2}}}{\sqrt{MN}} \sum_{1 \leq r} \frac{1}{r^2} \sum_{m \asymp M} \tau(m) NT^{-1} \log N \end{aligned}$$

$$\begin{aligned} &\ll_{\delta} \frac{T(\log T)^{\frac{1}{2}}}{\sqrt{MN}} \sum_{1 \leq r} \frac{1}{r^2} MNT^{-1}(\log M)(\log N) \\ &\ll_{\delta} \sqrt{MN}(\log T)^{\frac{5}{2}}. \end{aligned}$$

Note that δ is entirely dependent on a and b , which in turn are entirely dependent on α . Summing over the $M \leq T\left(\frac{a}{b}\right)^{-\frac{g}{2}}$ (equivalently, summing over all pairs $MN \leq T^2$) we see that for each g the contribution to the error term from $I_{\alpha,\beta,g}(T; M, N)$ in Case 2i is at most $\ll_{\alpha} T(\log T)^{\frac{5}{2}}$.

5.2.3.2 Case 2ii

We open up the integral in $V(mn, \alpha t + \beta)$, and use Lemma 5.3 to find

$$\begin{aligned} I_{\alpha,\beta,g}(T; M, N) &= \frac{1}{\pi i} \int_{-\infty}^{\infty} \int_{(1)} \sum_m \sum_{1 \leq r} \sum_{\substack{n \\ rT^{-1} \leq \left| \frac{nb^g}{ma^g} - 1 \right| < (r+1)T^{-1}}} \frac{\tau(m)\tau(n)}{(mn)^{1/2+s}} \left(\frac{n}{m}\right)^{i\beta} \\ &\quad \frac{G(s)}{s\pi^{2s}} g(s, \alpha t + \beta) \left(\frac{nb^g}{ma^g}\right)^{i\alpha t} \phi_t W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right) ds dt + O_A(T^{-A}) \end{aligned}$$

which by (5.4) and Lemma 5.4 is equal to

$$\begin{aligned} &\frac{1}{\pi i} \int_{-\infty}^{\infty} \int_{(1)} \sum_m \sum_{1 \leq r} \sum_{\substack{n \lesssim N \\ rT^{-1} \leq \left| \frac{nb^g}{ma^g} - 1 \right| < (r+1)T^{-1}}} \frac{\tau(m)\tau(n)}{(mn)^{1/2+s}} \left(\frac{n}{m}\right)^{i\beta} \frac{G(s)}{s\pi^{2s}} t^{2s} \left(\frac{nb^g}{ma^g}\right)^{i\alpha t} \\ &\quad \phi_t W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right) ds dt \\ &\quad + O\left(T^{1+\epsilon} \frac{(\log M)(\log N)}{\sqrt{MN}} \int_{(1)} \frac{|G(s)(1+s^2)|}{|s\pi^{2s}|} ds\right) + O_A(T^{-A}). \end{aligned}$$

Due to the rapid decay of $G(s)$,

$$T^{1+\epsilon} \frac{(\log M)(\log N)}{\sqrt{MN}} \int_{(1)} \frac{|G(s)(1+s^2)|}{|s\pi^{2s}|} ds \ll_{\epsilon} T^{1+\epsilon} (\log T)^{2+\epsilon} (MN)^{-\frac{1}{2}}$$

which is small enough as $MN \gg T^2$. To bound the first term, we integrate by parts twice with respect to t and use the rapid decay of $G(s)$ to show that the result is

$$\begin{aligned} &\ll_{\epsilon} \frac{T(\log T)^{\frac{1}{2}+\epsilon}}{(MN)^{3/2}} \sum_{m \lesssim M} \tau(m) \sum_{1 \leq r} \frac{1}{r^2} \sum_{\substack{n \lesssim N \\ rT^{-1} \leq \left| \frac{nb^g}{ma^g} - 1 \right| < (r+1)T^{-1}}} \tau(n) T^2 \\ &\ll_{\epsilon} \frac{T^2 (\log T)^{\frac{5}{2}+\epsilon}}{(MN)^{1/2}} \end{aligned}$$

and summing over all the possible $MN > T^2$ shows that for each g the contribution from the $I_{\alpha,\beta,g}(T; M, N)$ in Case 2ii is at most $O(T(\log T)^{\frac{5}{2}+\epsilon})$.

5.2.3.3 Case 2iii

In this final case, we have reduced the problem to a scenario that is similar to that of the twisted second moment as done by Li and Radziwiłł, and so we may use similar methods to those found in [84].

For each g , we find that the net contribution of all the $I_{\alpha,\beta,g}(M, N)$ in case 2iii is at most $\ll S_g$, where

$$S_g := T \sum_{m \leq X_1} \sum_{\substack{n \\ (n,m) \in \mathcal{C}_g}} \frac{\tau(m)\tau(n)}{\sqrt{mn}}$$

and where $X_1 := T^{1-\frac{\delta}{4}} \left(\frac{a}{b}\right)^{-\frac{g}{2}}$.

By the divisor bound,

$$S_g \ll_{\gamma} T^{1+\gamma} \sum_{m \leq X_1} \sum_{\substack{n \\ (n,m) \in \mathcal{C}_g}} \frac{1}{\sqrt{mn}}$$

for arbitrarily small $\gamma > 0$, and then by writing $d = (m, n)$

$$S_g \ll_{\gamma} T^{1+\gamma} \sum_{\substack{m,n \\ m \leq X_1 \\ (n,m) \in \mathcal{C}_g \\ (m,n)=1}} \sum_{d \leq X_1} \frac{1}{d\sqrt{mn}}.$$

The sum over d is $\ll \log X_1 \ll_{\epsilon} T^{\epsilon}$ and so (by relabelling γ)

$$S_g \ll_{\gamma} T^{1+\gamma} \sum_{\substack{m,n \\ m \leq X_1 \\ (n,m) \in \mathcal{C}_g \\ (m,n)=1}} \frac{1}{\sqrt{mn}}.$$

We introduce a dyadic decomposition over m , allowing us to write

$$S_g \ll_{\gamma} T^{1+\gamma} \sum_{M \ll X_1} S_g(M) \tag{5.14}$$

with

$$S_g(M) := \sum_{\substack{m,n \\ m \asymp M \\ (n,m) \in \mathcal{C}_g \\ (m,n)=1}} \frac{1}{\sqrt{mn}}.$$

If $(n, m) \in \mathcal{C}_g$ and $m \asymp M$ then

$$\frac{1}{\sqrt{mn}} \asymp M^{-1} \left(\frac{a}{b}\right)^{-\frac{g}{2}}.$$

Also, the number of pairs of co-prime integers m, n with $(n, m) \in \mathcal{C}_g$ and with $m \asymp M$ is at most

$$\ll \left(\frac{a}{b}\right)^g M^2 T^{\eta-1} + 1$$

as Farey fractions of denominator $\asymp M$ are each separated by a distance of at least M^{-2} . See chapter 3 of [49] for definitions and details of Farey fractions. Hence

$$S_g(M) \ll \left(\frac{a}{b}\right)^{\frac{g}{2}} M T^{\eta-1} + M^{-1} \left(\frac{a}{b}\right)^{-\frac{g}{2}}. \quad (5.15)$$

Let $X_0 := T^{\frac{1}{2}-\eta} \left(\frac{a}{b}\right)^{-\frac{g}{2}}$, and split (5.14) into two sums:

$$S_g \ll_\gamma T^{1+\gamma} \sum_{M \ll X_0} S_g(M) + T^{1+\gamma} \sum_{X_0 \ll M \ll X_1} S_g(M). \quad (5.16)$$

We bound the second sum using (5.15),

$$T^{1+\gamma} \sum_{X_0 \ll M \ll X_1} S_g(M) \ll_\gamma \left(\frac{a}{b}\right)^{\frac{g}{2}} X_1 T^{\eta+\gamma} + X_0^{-1} \left(\frac{a}{b}\right)^{-\frac{g}{2}} T^{1+\gamma} \ll_\gamma T^{1+\gamma+\eta-\frac{\delta}{4}} + T^{\gamma+\eta+\frac{1}{2}} \quad (5.17)$$

which is $\ll T$ for small enough γ and η relative to δ .

To bound the first sum in (5.16), notice that if both n, m and n', m' are co-prime pairs with $m, m' \ll X_0$, then either

$$\frac{n}{m} = \frac{n'}{m'}$$

or

$$\left| \frac{n}{m} - \frac{n'}{m'} \right| \geq \frac{1}{mm'} \gg T^{2\eta-1} \left(\frac{a}{b}\right)^g$$

and so both pairs cannot satisfy (5.13). Therefore, there can be at most one co-prime pair $(n, m) \in \mathcal{C}_g$ with $m \ll X_0$. Suppose that m, n is such a pair, so

$$\frac{n}{m} \asymp \left(\frac{a}{b}\right)^g \Rightarrow n \gg \left(\frac{a}{b}\right)^g \gg T^\delta$$

and hence

$$\frac{1}{\sqrt{mn}} \ll T^{-\frac{\delta}{2}}.$$

This implies that

$$T^{1+\gamma} \sum_{M \ll X_0} S_g(M) \ll T^{1+\gamma-\frac{\delta}{2}} \quad (5.18)$$

which is $\ll T$ for γ sufficiently small relative to δ .

Therefore, by (5.16), (5.17) and (5.18), for each g the total contribution from all of the $I_{\alpha, \beta, g}(M, N)$ in Case 2iii is $\ll_\delta T$. This concludes the proof of Theorem 5.2.

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Appendix A

Some adapted results

Here we shall lay out the proofs for the various adapted results called upon in subsections 2.4.2 and 2.4.3 where we incorporate the extra h and f terms into the exponential sums. The proofs in question are all similar to the original authors', and mostly are just a matter of tracking the h and f through the method, but as it is not immediately obvious how the h and f shall manifest themselves in the results we shall include the complete proof or (in the more simple cases) a guideline.

Theorem A.1 (Bettin and Chandee). *Define for $A, M, N \geq 1, \theta \in \mathbb{Z} \setminus \{0\}$, and co-prime natural numbers f, h ,*

$$\mathcal{B}_{f,h,\theta}(M, N, A) := \sum_{\substack{a \asymp A \\ m \asymp M \\ n \asymp N \\ (m,nf)=1}} \alpha_m \beta_n \nu_a e\left(\theta \frac{\overline{amh}}{fn}\right)$$

then

$$\begin{aligned} \mathcal{B}_{f,h,\theta}(M, N, A) &\ll \|\alpha\| \|\beta\| \|\nu\| \left(1 + \frac{|\theta|A}{fhMN}\right)^{1/4} \left(f^{1/4} h^{1/2} AMN^{3/4} \right. \\ &\quad + f^{3/4} h^{1/2} AM^{1/2} N^{5/4} + f^{-2/5} h^{6/5} AM^{6/5} N^{1/10} + f^{1/5} h^{6/5} A^{2/5} M^{6/5} N^{7/10} \\ &\quad \left. + f^{1/2} h^{3/5} A^{7/10} M^{3/5} N^{13/10} + f^{1/2} AN^{7/4}\right)^{1/2}. \end{aligned}$$

Proof. As the proof is essentially the same as Bettin and Chandee's work in [7] but now with the extra h, f in the exponential sum, we only loosely sketch the proof. By the Cauchy–Schwarz inequality,

$$\mathcal{B}_{f,h,\theta}(M, N, A) \ll \|\alpha\| \mathcal{C}_{f,h,\theta}(M, N, A; \beta, \nu)^{1/2}$$

where

$$\mathcal{C}_{f,h,\theta}(M, N, A; \beta, \nu) := \sum_{\substack{m \asymp M \\ (m,f)=1}} \left| \sum_{\substack{a \asymp A \\ n \asymp N \\ (m,n)=1}} \beta_n \nu_a e\left(\theta \frac{amh}{fn}\right) \right|^2.$$

Following Section 6 of [7], we see that by grouping ah as one variable,

$$\begin{aligned} \mathcal{C}_{f,h,\theta}(M, N, A; \beta, \nu) &\ll M^\epsilon \sum_{\substack{b \leq N, (b,\theta)=1 \\ b \text{ powerful}}} b^{1/2} \mathcal{C}_{bf,h,\theta}(M, N/b, A; \beta_b, \nu) \\ &\ll \|\beta\|^2 \|\nu\|^2 M^\epsilon \left(1 + \frac{|\theta|A}{fhMN}\right)^{1/2} (f^{1/2} h A M N^{1/2} B^{1/2} + f^{3/4} h^{1/2} A M^{1/2} N^{5/4} \\ &\quad + f^{-2/5} h^{6/5} A M^{6/5} N^{1/10} + f^{1/5} h^{6/5} A^{2/5} M^{6/5} N^{7/10} + f^{1/2} h^{3/5} A^{7/10} M^{3/5} N^{13/10} \\ &\quad + f^{1/2} A N^{7/4})^{1/2} + M^\epsilon \sum_{\substack{B \leq b \leq N, (b,\theta)=1 \\ b \text{ powerful}}} b^{1/2} \mathcal{C}_{bf,h,\theta}(M, N/b, A; \beta_b, \nu) \end{aligned}$$

for some B of our choice. Bounding the contribution from the $B \leq b \leq N$ trivially we see that

$$\sum_{\substack{B \leq b \leq N, (b,\theta)=1 \\ b \text{ powerful}}} b^{1/2} \mathcal{C}_{bf,h,\theta}(M, N/b, A; \beta_b, \nu) \ll \|\beta\|^2 \|\nu\|^2 \frac{A N M^{1+\epsilon}}{B^{1/2}}$$

and so the optimal value for B is $B = \frac{N^{1/2}}{f^{1/2}h}$. Hence

$$\begin{aligned} \mathcal{C}_{f,h,\theta}(M, N, A; \beta, \nu) &\ll \|\beta\|^2 \|\nu\|^2 M^\epsilon \left(1 + \frac{|\theta|A}{fhMN}\right)^{1/2} (f^{1/4} h^{1/2} A M N^{3/4} B^{1/2} \\ &\quad + f^{3/4} h^{1/2} A M^{1/2} N^{5/4} + f^{-2/5} h^{6/5} A M^{6/5} N^{1/10} + f^{1/5} h^{6/5} A^{2/5} M^{6/5} N^{7/10} \\ &\quad + f^{1/2} h^{3/5} A^{7/10} M^{3/5} N^{13/10} + f^{1/2} A N^{7/4})^{1/2}. \end{aligned}$$

This completes the proof. $\square$

The following lemma is proved almost identically to Heath-Brown's Lemma 6 of [59] (essentially the extra f, h just need to be inserted at the appropriate place), but we include it for completeness.

Lemma A.1 (Heath-Brown). *Suppose that $1 \leq U, K, S, T \leq x$, q, h, f are pairwise co-prime positive integers, and $\alpha(s), \beta(t)$ are sequences with $|\alpha(s)|, |\beta(t)| \leq 1$ and support on $[S, 2S]$ and $[T, 2T]$ respectively. Define*

$$\Omega_4 := \sum_{\substack{u \asymp U \\ (u,qh)=1}} \sum_{|k| \asymp K} \left| \sum_{(s,uf)=1} \sum_{(t,uf)=1} \alpha(s) \beta(t) e\left(\frac{qksth}{uf}\right) \right|$$

then

$$\Omega_4 \ll_{\epsilon} x^{\epsilon} UKST \left(\left(\frac{f}{KS} \right)^{1/4} + \left(\frac{Uf}{KST^2} \right)^{1/4} + \frac{1}{U^{1/4}} + \frac{1}{T^{1/2}} \right).$$

Proof. Define

$$N(u, l) := \#\{|k| \asymp K, s \in [S, 2S] : qk\overline{sh} \equiv l \pmod{uf}\}$$

and

$$\sigma(u, l) = \sum_t \beta(t) e\left(\frac{\overline{lt}}{uf}\right)$$

so that

$$\begin{aligned} \Omega_4 &\leq \sum_{\substack{u \asymp U \\ (u, qh)=1}} \sum_{|k| \asymp K} \sum_{(s, uf)=1} \left| \sum_{(t, uf)=1} \beta(t) e\left(\frac{qk\overline{sth}}{uf}\right) \right| \\ &= \sum_{\substack{u \asymp U \\ (u, qh)=1}} \sum_l N(u, l) |\sigma(u, l)|. \end{aligned}$$

By applying Hölder's inequality

$$\Omega_4^4 \ll \left(\sum_{u, l} N(u, l) \right)^2 \left(\sum_{u, l} N^2(u, l) \right) \left(\sum_{u, l} |\sigma(u, l)|^4 \right).$$

Trivially

$$\sum_l N(u, l) \ll KS$$

and hence

$$\sum_{u, l} N(u, l) \ll UKS.$$

Also

$$\begin{aligned} \sum_{u, l} N^2(u, l) &= \sum_{\substack{u \asymp U \\ (u, qh)=1}} \#\{k_1, k_2, s_1, s_2 : qk_1\overline{s_1h} \equiv qk_2\overline{s_2h} \pmod{uf}\} \\ &\leq \sum_{u \asymp U} \#\{k_1, k_2, s_1, s_2 : uf | k_1s_2 - k_2s_1\} \\ &\ll_{\epsilon} x^{\epsilon} \left(UKS + \frac{K^2 S^2}{f} \right) \end{aligned}$$

with the first term coming from the diagonal case (i.e. $k_1s_2 = k_2s_1$) and the second term from the off-diagonals and the divisor bound. Finally,

$$\sum_{u, l} |\sigma(u, l)|^4 \leq \sum_u Uf \sum_{t_1, t_2, t_3, t_4 \asymp T} \#\{u : u | t_2t_3t_4 + t_1t_3t_4 - t_1t_2t_4 - t_1t_2t_3\}$$

$$\ll_{\epsilon} U^2 f \mathcal{N} + x^{\epsilon} U f T^4$$

where

$$\begin{aligned} \mathcal{N} &= \#\{t_1, t_2, t_3, t_4 : t_2 t_3 t_4 + t_1 t_3 t_4 = t_1 t_2 t_4 + t_1 t_2 t_3\} \\ &= \#\{t_1, t_2, t_3, t_4 : t_1^{-1} + t_2^{-1} = t_3^{-1} + t_4^{-1}\} \\ &\ll_{\epsilon} T^{2+\epsilon} \end{aligned}$$

by Lemma 5 of [59]. Therefore

$$\Omega_4^4 \ll_{\epsilon} x^{\epsilon} (UKS)^2 \left(UKS + \frac{K^2 S^2}{f} \right) (fUT^4 + fU^2 T^2).$$

□

Lemma A.2 (Matomäki). *Let $X_d = \sqrt{\frac{dK}{MCRhf}}$ and suppose that a bounded sequence $a(k, r)$ is supported on square-free r with $(r, d) = 1$. Then,*

$$\begin{aligned} U &= \sum_{\substack{c \lesssim C \\ (c, h)=1}} \sum_{\substack{m \lesssim M \\ (m, cf)=1}} \left| \sum_{k \lesssim K} \sum_{\substack{r \lesssim R \\ (r, cf)=1}} a(k, r) e\left(\pm \frac{dkmrh}{cf}\right) \right| \\ &\ll_{\epsilon} (CMKR)^{\frac{1}{2}+\epsilon} \left[CM + MRK^{\frac{1}{2}}(K+R)^{\frac{1}{2}} + d^{\frac{7}{64}}(MChf)^{\frac{1}{2}} R^{\frac{3}{2}}(K+R)^{\frac{1}{2}} \right. \\ &\quad \times \left. \frac{(1 + X_d^{-1})^{\frac{7}{32}}}{1 + X_d} \left(1 + X_d + \sqrt{\frac{C}{hR^2M}} \right) \left(1 + X_d + \sqrt{\frac{K}{hfr}} \right) \right]^{\frac{1}{2}}. \end{aligned}$$

The proof of this lemma is identical to the proof of Lemma 11 of [88], but when appealing to Lemma 10 of the same paper, we use the substitutions $r = r_1 r_2 h$, $s = f$ and $X_d = \sqrt{\frac{dN}{MCr_1 r_2 h f}}$.

Lemma A.3 (Deshouillers and Iwaniec). *Let $C, D, U, V \geq 1$, h, f, w pairwise co-prime positive integers, and $|c(u, v)| \leq 1$. Then*

$$\begin{aligned} H(C, D, U, V) &:= \sum_{\substack{c \lesssim C \\ (c, h)=1}} \sum_{\substack{d \lesssim D \\ (d, cf)=1}} \left| \sum_{u \lesssim U} \sum_{v \lesssim (v, cf)=1} c(u, v) e\left(\frac{uwvdh}{cf}\right) \right| \\ &\ll_{\epsilon} (CDUV)^{\frac{1}{2}+\epsilon} \left((CD)^{\frac{1}{2}} + (U+V)^{\frac{1}{4}} [fCD(wU+hV)(fC+hV^2) \right. \\ &\quad \left. + h w D^2 U V^2]^{\frac{1}{4}} \right). \end{aligned}$$

Again, this follows very similarly to the original proof in [30], but we shall describe the differences.

Proof. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
H^2(C, D, U, V) &\leq CD \sum_{\substack{c \asymp fC \\ (c, h)=1}} \sum_{\substack{d \asymp D \\ (d, c)=1}} g(c, d) \left| \sum_{u \asymp U} \sum_{v \asymp (v, c)=1} c(u, v) e\left(\frac{uwv\overline{d}h}{c}\right) \right|^2 \\
&= CD \sum_{\substack{u_1, u_2 \asymp U \\ v_1, v_2 \asymp V}} c(u_1, v_1) \overline{c(u_2, v_2)} \\
&\quad \times \sum_{\substack{c \\ (c, hv_1v_2d)=1}} \sum_{\substack{d \\ (d, c)=1}} g(c, d) e\left(w(u_1v_2 - u_2v_1) \frac{\overline{v_1v_2d}h}{c}\right)
\end{aligned}$$

where $g(c, d)$ is any smooth function that majorizes the characteristic function on $[fC, 2fC] \times [D, 2D]$. By writing

$$b(n, r) = \sum_{\substack{v_1, v_2 \asymp V \\ hv_1v_2=r}} \sum_{\substack{u_1, u_2 \asymp U \\ w(u_1v_2 - u_2v_1)=n}} c(u_1, v_1) \overline{c(u_2, v_2)}$$

then

$$H^2(C, D, U, V) \leq CD \sum_{r \asymp R} \sum_{|n| \leq N} b(n, r) \sum_{\substack{c \\ (c, hr)=1}} \sum_{\substack{d \\ (d, c)=1}} g(c, d) e\left(n \frac{\overline{rd}}{c}\right)$$

with $R = hV^2$ and $N = 2wUV$. The $n = 0$ terms contribute at most

$$C^2 D^2 \sum_r |b(0, r)| \ll C^2 D^2 \sum_{1 \leq q \leq UV} \tau^2(q) \ll_{\epsilon} C^2 D^2 (UV)^{1+\epsilon}$$

by the divisor bound. Also

$$\begin{aligned}
\sum_{n, r} |b(n, r)|^2 &\ll V^{\epsilon} \sum_{v_1, v_2 \asymp V} \sum_{\substack{u_1, u_2, u_3, u_4 \asymp U \\ w(u_1 - u_3)v_2 = w(u_2 - u_4)v_1}} 1 \\
&\ll_{\epsilon} (UV)^{\epsilon} U^2 V (U + V)
\end{aligned}$$

so bounding the contribution to H^2 when $n \neq 0$ using Theorem 12 of [29] to get

$$\begin{aligned}
&(CDUV)^{\epsilon} CD (U^2 V (U + V))^{\frac{1}{2}} \\
&\quad \times \left[fC(hV^2 + wUV)(fC + hDV^2) + f^2 C^2 D(hV^2 + (wUV)^{\frac{1}{2}}) + whD^2 UV^3 \right]^{\frac{1}{2}}
\end{aligned}$$

and combining this with the $n = 0$ term gives the result. $\square$